A new dominance intensity method to deal with ordinal information about a DM's preferences within MAVT

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Abstract

Dominance measuring methods are an approach to deal with complex decision-making problems with imprecise information. These methods are based on the computation of pairwise dominance values and exploit the information in the dominance matrix in different ways to derive measures of dominance intensity and rank the alternatives under consideration. In this paper we propose a new dominance measuring method to deal with ordinal information about decision-maker preferences in both weights and component utilities. It takes advantage of the centroid of the polytope delimited by ordinal information and builds triangular fuzzy numbers whose distances to the crisp value 0 constitute the basis for the definition of a dominance intensity measure. Monte Carlo simulation techniques have been used to compare the performance of this method with other existing approaches.

1. Introduction

The additive model is widely used within *multi-attribute value theory* (MAVT) to rank alternatives in complex decision-making problems and it is considered a valid approach in many practical situations for the reasons described in <20; 29>. The functional form of the additive model is

$$v(A_i) = \sum_{j=1}^{n} w_j v_j(x_{ij}), \tag{1}$$

where x_{ij} is the performance over the attribute (or criterion) X_j, $j = 1, \ldots, n$, for the alternative A_i, $i = 1, \ldots, m$; and v_j and w_j are the value function and the weight for the attribute X_j, respectively. Note that $\sum_{j=1}^{n} w_j = 1$ and $w_j \geq 0$.

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The information available in most real complex decision-making problems is not precise. Inputs are often described within prescribed bounds or just satisfying certain relations. Different authors refer to this situation as *decision-making with imprecise information, incomplete information* or *partial information* <21; 22>.

Several reasons are given in the literature that justify why a *decision-maker* (DM) may wish to provide imprecise information <27; 33>. For example, performances that reflect social or environmental impacts may be intangible or non-monetary, and performances may be taken from statistics or measurements, which are not absolutely precise. Alternatively, DM might prefer not to reveal his/her preferences in public or not feel confident about giving precise information for parameters that change during the process. Besides, DMs could feel more comfortable providing a scale to represent the importance of the attributes, and might also have different more or less reliable sources of information. Moreover, the decision could be taken in a group decision-making situation, where a negotiation process usually outputs imprecise information <6; 13; 17; 32>

Many papers on MAVT have dealt with imprecise information. Sarabando and Dias <26> provided a brief overview of approaches proposed by different authors within the *multi-attribute utility theory* (MAUT) and MAVT framework to deal with imprecise information.

As attribute weights are usually the hardest parameters to elicit in *multi-attribute decision making* (MADM) problems <23>, works in the literature have mainly centered on the case in which the information regarding weights is imprecise, which is often represented by ordinal information.

Surrogate weighting (SW) methods can be used when the DM provides ordinal relations regarding attribute weights. These methods select a weight vector from a set of admissible weights to represent the set <3; 30>. The best SW method is the *rank-order centroid weights* (ROC) method <26>: $w_j = \frac{1/j}{\sum_{k=1}^{n} 1/k}, j = 1, ..., n$, n being he number of attributes.

The *stochastic multicriteria acceptability analysis* ($SMAA$) method was proposed for support in discrete group decision-making problems where the weight information is missing <8>. The *SMAA-2* method <10> extends the analysis to the sets of weight vectors for any rank from best to worst for each decision alternative and can be used to identify good compromise alternatives. SMAA-O <9> is a variant of SMAA for problems in which criteria are measured on ordinal scales.

The TOPSIS method has been extended to uncertain linguistic environments <35; 36> or used for determining DM weights with interval numbers <41>.

Sage and White <24> proposed the model of *imprecisely specified multi-attribute utility theory* (ISMAUT), where preference information about both weights and utilities is assumed not to be precise. Malakooti <11> suggested an efficient algorithm for ranking alternatives when there is imprecise information about preferences and alternative values. Ahn <1> extended Malakooti's work.

Another possibility described in the literature for dealing with imprecision is

based on the concepts of *pairwise* and *absolute dominance*. The use of absolute dominance values is exemplified by the modification of four classical decision rules to encompass an imprecise decision context concerning weights and component values/utilities <19; 25>, the *maximax or optimist*, the *maximin or pessimist*, the *minimax regret* and the *central value* rules.

A recent approach for dealing with imprecise information is to compute different measures of dominance to derive a ranking of alternatives <2>, known as *dominance measuring methods (DMMs)*. DMMs are based on the computation of a dominance matrix including pairwise dominance values, which are exploited in different ways to derive measures of dominance to rank the alternatives under consideration.

In this paper we propose a new *DMM* based on a dominance intensity measure to deal with ordinal information about the DM's preferences. Specifically, the DM will provide a ranking of attribute importance. Besides, the method takes into account a ranking of the alternatives in each attribute and also a ranking of the difference of values between consecutive alternatives.

As mentioned above, many methods accounting for ordinal information on weights and alternative values/utilities within MAVT/MAUT can be found in the literature. However, the ranking of the difference between the values of consecutive alternatives used to represent DM preferences is not so commonplace in the literature. Sarabando and Dias <27> propose new decision rules within MAVT to deal with such rankings on the basis of an additive model, whereas Salo and Hamalainen <25> transform them into linear constraints in the *preference ratios in multiattribute evaluation* PRIME method. In PRIME, preference elicitation and synthesis is based on 1) the conversion of possibly imprecise ratio judgments into an imprecisely specified preference model, 2) the use of dominance structures and decision rules in deriving decision recommendations, and 3) the sequencing of the elicitation process into a series of elicitation tasks.

Ordinal information has also been used in other disciplines apart from MAVT/ MAUT, for instance in fuzzy preference relations. Xu et al. <39> propose the ordinal consistency index to measure the degree of ordinal consistency of a fuzzy preference relation, which is to count the unreasonable 3-cycles in a directed graph that represents the fuzzy preference relation. The method can be used for a strict and non-strict fuzzy preference relation. Xu et al. <37; 40; 38> adapt the algorithms for incomplete reciprocal, inter-valued fuzzy and incomplete 2-tuple fuzzy linguistic preference relations, respectively.

The proposed dominance intensity measure takes advantage of the centroid of the polytope delimited by ordinal information, builds triangular fuzzy numbers on the basis of this centroid and incorporates a distance notion to derive dominance intensities to rank the alternatives under consideration.

We have also conducted a simulation study to analyze the performance of the proposed method regarding other dominance measuring methods proposed in the literature and Sarabando and Dias ranking method.

In Section 2, we review dominance measuring methods reported in the literature and the ranking method proposed by Sarabando and Dias. In Section 3, we propose the new dominance measuring method. In Section 4, we describe a

technique to find all the endpoints from a polytope delimited by constraints representing ordinal information. We average the endpoints to derive the centroid of the polytope, which is used in the proposed dominance measuring method. In Section 5, a simulation study is carried out to compare the proposed method with the dominance measuring methods reviewed in Section 2 and the method proposed by Sarabando and Dias. Finally, some conclusions are discussed in Section 6.

2. Review of dominance measuring methods and Sarabando and Dias's method

DMMs are based on the computation of a *dominance matrix*, D, including pairwise dominance values:

$$D = \begin{pmatrix} - & D_{12} & \cdots & D_{1(m-1)} & D_{1m} \\ D_{21} & - & \cdots & D_{2(m-1)} & D_{2m} \\ D_{31} & D_{32} & \cdots & D_{3(m-1)} & D_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{m1} & D_{m2} & \cdots & D_{m(m-1)} & - \end{pmatrix},$$

where

$$\begin{array}{rl} D_{kl} = & min\{v(A_k) - v(A_l) = \sum_{j=1}^{n} w_j v_k(x_{kj}) - \sum_{j=1}^{n} w_j v_l(x_{lj})\} \\ s.t. & \\ & \mathbf{v}_k = (v_{k1}, \ldots, v_{kn}), \mathbf{v}_l = (v_{l1}, \ldots, v_{ln}) \in V_{kl} \\ & \mathbf{w} = (w_1, ..., w_n) \in W \end{array} \tag{2}$$

where W and V_{kl} define the feasible region for weights and values associated with the alternatives A_k and A_l over each attribute, respectively, which represent imprecise information.

Note that given two alternatives A_k and A_l, alternative A_k dominates A_l if $D_{kl} \geq 0$, and there exists at least one $\mathbf{w}$, $\mathbf{v}_k$ and $\mathbf{v}_l$ such that the overall value of A_k is strictly greater than that of A_l. This concept of dominance is called *pairwise dominance*.

The *DMMs* exploit the information in D in different ways to derive measures of dominance to rank the alternatives under consideration. For instance, Ahn and Park <2> compute a dominating measure $\phi_k^+ = \sum_{\substack{l=1 \\ l \neq k}}^{m} D_{kl}$ and a dominated measure $\phi_k^- = \sum_{\substack{l=1 \\ l \neq k}}^{m} D_{lk}$ for each alternative A_k, and then derive a *net dominance* as $\phi_k = \phi_k^+ - \phi_k^-$. Ahn and Park proposed two ranking methods for these measures: ranking the alternatives according to either ϕ_k^+ or ϕ_k values (denoted as the *AP1* and *AP2* methods, respectively).

However, the results of simulation experiments when the DM weight preferences are represented by ordinal information suggest that *surrogate weighting*

methods, specifically the *ROC* method, are better than *AP1* and *AP2* at selecting the best alternative and ranking alternatives. The simulation study also showed *AP1* to be better than *AP2*. The reason is that *AP*2 uses duplicate information (row and column values).

Two *DMMs* were proposed in <14; 15>. The first one, *DME1*, was based on the same idea as implemented by Ahn and Park. It also computes dominating and dominated measures but they are combined into a *dominance intensity* rather than a net dominance index, which is used as a measure of the strength of preference.

DME1 is implemented as follows:

1. Compute the dominating indices DI_{k+}^{row} and DI_{k-}^{row} for each alternative A_k (by row):

$$DI_{k+}^{row} = \sum_{l=1,l\neq k,D_{kl}>0}^{m} D_{kl} \text{ and } DI_{k-}^{row} = \sum_{l=1,l\neq k,D_{kl}<0}^{m} D_{kl}.$$

2. Compute the dominating intensity DI_{k}^{row} for each alternative A_k:

$$DI_{k}^{row} = \frac{DI_{k+}^{row}}{DI_{k+}^{row} - DI_{k-}^{row}}$$

3. Compute dominated indices DI_{k+}^{col} and DI_{k-}^{col} for each alternative A_k (by column):

$$DI_{k+}^{col} = \sum_{l=1,l\neq k,D_{lk}>0}^{m} D_{lk} \text{ and } DI_{k-}^{col} = \sum_{l=1,l\neq k,D_{lk}<0}^{m} D_{lk}.$$

4. Compute the dominated intensity DI_{k}^{col} for each alternative A_k:

$$DI_{k}^{col} = \frac{DI_{k+}^{col}}{DI_{k+}^{col} - DI_{k-}^{col}}$$

5. Calculate a global dominance intensity (GDI) for each alternative A_k:

$$GDI_k = DI_{k}^{row} - DI_{k}^{col}, k = 1, ..., m$$

and rank the alternatives according to the GDI_k values, where the alternative with the maximum GDI_k is the best alternative.

DME1 improves *AP2* by reducing the duplicate information involved in the computations.

The second method, *DME2*, derives a *global dominance intensity index* to rank alternatives on the basis that

$$D_{kl} \leq \mathbf{w}^T (\mathbf{v}_k - \mathbf{v}_l) \leq -D_{lk}, \forall \mathbf{w} \in W, \mathbf{v}_k, \mathbf{v}_l \in V_{kl}.$$

DME2 is implemented as follows:

1. If $D_{kl} \geq 0$, then alternative A_k dominates A_l, and the dominance intensity of A_k over A_l (DI_{kl}) is 1, i.e., $DI_{kl} = 1$.
   Else ($D_{kl} < 0$):
   - If $D_{lk} \geq 0$, then alternative A_l dominates A_k, and $DI_{kl} = 0$.
   - Else ($D_{lk} < 0$), the dominance intensity of A_k over A_l, is defined as

$$DI_{kl} = \frac{-D_{lk}}{-D_{lk} - D_{kl}}. \tag{3}$$

2. Calculate a global dominance intensity (GDI) for each alternative A_k, i.e.,

$$GDI_k = \sum_{l=1,\ l \neq k}^{m} DI_{kl},$$

   and rank the alternatives according to the GDI_k values, where the alternative with the maximum GDI_k is the best alternative.

Another simulation study <16> was carried out to compare the *DME1* and *DME2* methods with modified decision rules (maximax, maximin and minimax regret and the central value rules) and *AP1* and *AP2*. Two measures of efficacy were considered, the proportion of all cases in which the method selects the same best alternative as in the TRUE ranking (hit ratio), where the TRUE ranking is determined beforehand, and how similar the overall alternative-ranking structures are in the TRUE and the method-driven rankings (rank-order correlation). The results show that *DME2* outperforms the other methods. The drawback of the *DME1* method is that when the dominance matrix D contains all negative elements, that is, when all the alternatives are non-dominated, the algorithm is unable to rank the alternatives.

These methods in <16> were adapted to account for imprecision concerning the inputs represented by value intervals, in alternative performances, component utilities and weights. The results of simulation studies <16> showed that *DME2* performs better than the *AP1* method and the adaptation of classical decision rules and comes quite close to the *ROC* method, which was identified as the best approach. Although *SMAA-2* slightly outperforms *DME2*, *DME2* could be used when incomplete information about weights is expressed not just as weight intervals but also as weights satisfying linear or non-linear constraints, weights represented by fuzzy numbers or weights fitting normal probability distributions.

The performance of *DME1* and *DME2* is compared in <13> with other existing approaches (*SW* methods, modified decision rules and the *AP1* and *AP2* methods) when ordinal information represents imprecision concerning weights. As regards average hit ratios, *DME2* and *ROC* outperform the other methods and, according to the *paired-samples t-test*, there is no significant difference between the two. However, *ROC* can be only applied when ordinal relations regarding attribute weights are provided.

Other dominance measuring method was proposed in <7> where imprecise weights are represented by trapezoidal fuzzy weights. Dominance values

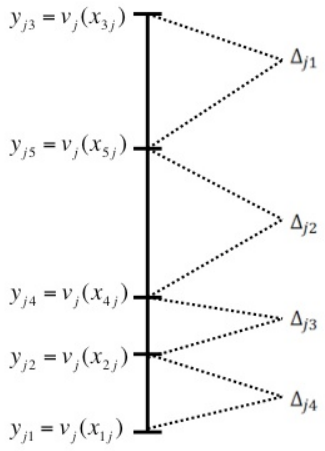

Figure 1: Ranking of alternatives and differences between consecutive alternatives for the attribute X_j.

are transformed into dominance intensity measures taking into account the distance between fuzzy numbers based on the generalization of the left and right fuzzy numbers defined by Tran and Duckstein <31>. An example concerning the selection of intervention strategies to restore an aquatic ecosystem contaminated by radionuclides illustrates the approach, and Monte Carlo simulation techniques are again used to analyze its performance for different imprecision levels.

As mentioned above, the ordinal information about the DM preferences considered in Sarabando and Dias <27> is the same as in this paper, i.e., a ranking of the alternatives in each attribute and also of the difference between the values of consecutive alternatives. Therefore, Sarabando and Dias's method can be used to analyze the performance of the proposed method.

We denote by V_j the set of constraints concerning component values in attribute X_j. For instance, A_3 could be the best of five alternatives for attribute X_j for the DM, followed by A_5, A_4, A_2 and A_1 ($v_j(x_{3j}) \geq v_j(x_{5j}) \geq v_j(x_{4j}) \geq v_j(x_{2j}) \geq v_j(x_{1j})$). Moreover, the ranking of differences between consecutive alternatives could be $\Delta_{j2} \geq \Delta_{j1} \geq \Delta_{j4} \geq \Delta_{j3}$, with $\Delta_{j2} = v_j(x_{5j}) - v_j(x_{4j})$, $\Delta_{j1} = v_j(x_{3j}) - v_j(x_{5j})$, $\Delta_{j4} = v_j(x_{2j}) - v_j(x_{1j})$ and $\Delta_{j3} = v_j(x_{4j}) - v_j(x_{2j})$, as illustrated in Fig. 1.

Sarabando and Dias <27> used the *ROC* method to derive a weight vector. Besides, they propose an adaptation of the *ROC* method, ΔROC, to compute a vector of values for each attribute that can approximately represent all the vectors' values compatible with the available ordinal information:

1. Determine a rank order centroid for $m-1$ variables:

$$\Delta_{jk} = \frac{1}{m-1} \sum_{l=rank(\Delta_{jk})}^{m-1} \frac{1}{l}, k = 1, ..., m-1,$$

where $rank(\Delta_{jk})$ is 1 when Δ_{jk} is the best, 2 when it is the second best, and so on.

2. The approximate values for the levels in attribute X_j, are:

$$y_{jt} = 0, \text{ if } rank(y_{jt}) = m,$$

$$y_{ji} = \sum_{k=rank(y_{ji})}^{m-1} \Delta_{jk}, \text{ if } i = 1, ..., m, \text{ and } rank(y_{ji}) \neq m,$$

where $rank(y_{ji})$ is 1 when y_{ji} is the best, 2 when it is the second best, and so on.

For the example in Figure 1 with 5 alternatives, we have $\Delta_{j2} = 25/48$, $\Delta_{j1} = 13/48$, $\Delta_{j4} = 7/48$ and $\Delta_{j3} = 1/16$. Then, $y_{j1} = 0$, $y_{j2} = \Delta_{j4} = 7/48$, $y_{j3} = \Delta_{j1} + \Delta_{j2} + \Delta_{j3} + \Delta_{j4} = 1$, $y_{j4} = \Delta_{j3} + \Delta_{j4} = 10/48$ and $y_{j5} = \Delta_{j2} + \Delta_{j3} + \Delta_{j4} = 35/48$.

Finally, the additive model, see Eq. (1), is used to evaluate and rank the alternatives under consideration.

3. A new dominance intensity method based on triangular fuzzy numbers and a distance notion

We consider that a DM's preferences are represented by ordinal information, for both weights and component values. Consequently, the DM provides a ranking of attribute importance. Without loss of generality we assume that attribute weights are indexed in descending order $\mathbf{w} = (w_1, w_2, ..., w_n) \in W : w_1 \geq w_2 \geq ... \geq w_n \geq 0$, $\sum_{j=1}^{n} w_j = 1$.

In this specific case, the optimization problem for deriving pairwise dominance values is non-linear, see Eq. (2), since it incorporates the product of pairs of variables (attribute weights and component values) in the objective function. We can simplify the problem by applying the *ROC* method on the basis of the available ordinal information about weights. This has been demonstrated to derive a good representation of the set W, as cited in Section 2. The *ROC* method is generalized to cases that include weak orders or partial orders in <28>. We denote by $(w_1^c, ..., w_n^c)$ the weight vector resulting from the *ROC* method, which is the centroid of W.

The optimization problem is now linear since only the component values are under consideration. Thus, this problem could be solved using the simplex method, the dominance matrix D and the dominance measuring methods (*AP1*, *AP2*, *DME1* and *DME2*) applied to rank the considered alternatives.

In this paper, we propose computing the following rather than pairwise dominance values (D_{kl}):

$$v_{kl} = \sum_{j=1}^{n} w_j^c v_{kj}^c - \sum_{j=1}^{n} w_j^c v_{lj}^c, \tag{4}$$

where $(w_1^c, ..., w_n^c)$ is the centroid or center of gravity of the polytope representing the weight space and $(v_{k1}^c, v_{l1}^c), ..., (v_{kn}^c, v_{ln}^c)$ are the centroids or centers of gravity of the polytopes in the n attributes delimited by the constraints accounting for alternatives A_k and A_l. Note that the centroid is considered as the most representative point that verifies the constraints that delimit the polytope. Moreover, $D_{kl} \leq v_{kl} \leq -D_{lk}$.

The centroid of the polytope associated with constraints on component values in the attribute X_j for the alternatives A_k and A_l is:

$$\mathbf{v}_j^c = (v_{kj}^c, v_{lj}^c) = \frac{\int\limits_{[0,1]^2} V_j^{kl} dv}{\int\limits_{[0,1]^2} dv},$$

where V_j^{kl} is the set of constraints concerning component values in the attribute X_j for alternatives A_k and A_l. Note that $V_j^{kl} \subset V_j$, which includes the constraints concerning component values in the attribute X_j for all the alternatives.

Some techniques have been proposed to find the center of gravity of a polytope, see, e.g., Lahdelma et al. <8>; Lahdelma and Salminen <10>; Mármol et al. <12>. In Section 4 we propose a method to derive the endpoints of a polytope delimited by constraints representing the ordinal information on component utilities. The centroid can then be computed by averaging these endpoints.

As it would be very simplistic to represent a constraint set as just a point, we have built a normalized triangular fuzzy number as follows. We assign possibility 1 to the value v_{kl} and, as $D_{kl} \leq v_{kl} \leq -D_{lk}$, the possibility linearly decreases to D_{kl} and $-D_{lk}$. However, as v_{kl} is computed from centroids, a better option is to consider the following symmetric triangular fuzzy number (see Fig. 2):

$$\widetilde{I}_{kl} = (I_{kl}^L, v_{kl}, I_{kl}^U), \tag{5}$$

where $I_{kl}^L = v_{kl} - m_{kl}$ and $I_{kl}^U = v_{kl} + m_{kl}$, and

$$m_{kl} = min\{(-D_{lk} - v_{kl}), (v_{kl} - D_{kl})\},$$

with membership function (see Fig 2.)

$$\mu_{\widetilde{I}_{kl}}(x) = \begin{cases} \dfrac{x - I_{kl}^L}{v_{kl} - I_{kl}^L}, & \text{if } I_{kl}^L \leq x \leq v_{kl} \\ 1, & \text{if } x = v_{kl} \\ \dfrac{x - I_{kl}^U}{v_{kl} - I_{kl}^U}, & \text{if } v_{kl} \leq x \leq I_{kl}^U \\ 0, & \text{otherwise} \end{cases}.$$

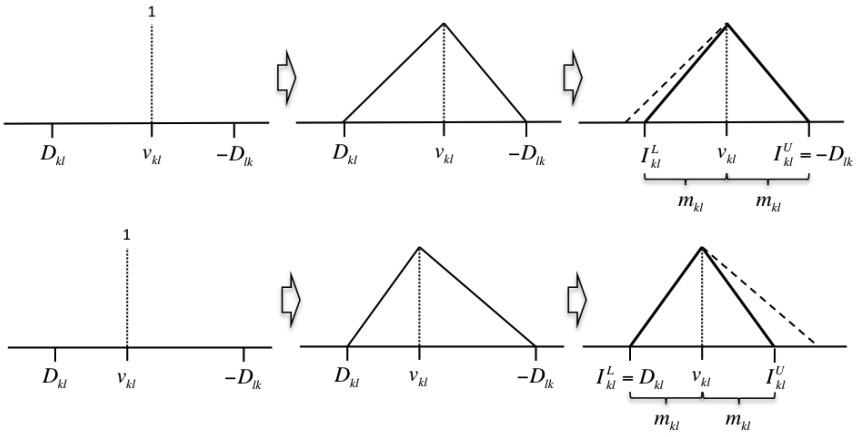

Figure 2: Building $\widetilde{I}_{kl}$.

Note that alternative A_k is better than A_l in the positive portion of the interval I_{kl}. However, alternative A_l is better than A_k in the negative portion.

Then, normalized triangular fuzzy numbers $\widetilde{I}_{kl}$ could be used in conjunction with a distance notion proposed in Tran and Duckstein <31>, to define a *dominance intensity* measure as follows: If we consider the location of the triangular fuzzy number $\widetilde{I}_{kl}$ regarding the crisp value 0, then we have two possibilities (see Fig. 3): if $v_{kl} < 0$, then the dominance intensity of alternative A_k over A_l can be computed as minus the distance of the fuzzy number $\widetilde{I}_{kl}$ to the crisp value 0. Otherwise ($v_{kl} \geq 0$), the dominance intensity is the distance of the fuzzy number $\widetilde{I}_{kl}$ to the crisp value 0.

Note that in both cases we are already taking into account the possibility of $\widetilde{I}_{kl}$ being located completely on the right and on the left of zero, respectively, see cases *b*) and *d*) in Fig. 3. In the case *d*) alternative A_k dominates A_l, whereas in the *b*) A_l dominates A_k. This constitutes a difference with respect to the *DME2* method, in which the dominance intensity of A_k over A_l is 1 ($DI_{kl} = 1$) when $D_{kl} \geq 0$ (alternative A_k dominates A_l), whereas $DI_{kl} = 0$ when $D_{lk} \geq 0$ (alternative A_l dominates A_k). Therefore, it does not consider the strength of dominance, i.e., there is no difference between $D_{kl} = 0.1$ or $D_{kl} = 1.5$, where $DI_{kl} = 1$ in both cases. However, we use the distance of $\widetilde{I}_{kl}$ to zero as the dominance intensity.

Finally, a dominance intensity measure for each alternative A_k, DIM_k, is derived as the sum of the dominance intensities of alternative A_k regarding the other alternatives. This measure is used as a measure of the strength of preference in the sense that greater dominance intensity is better.

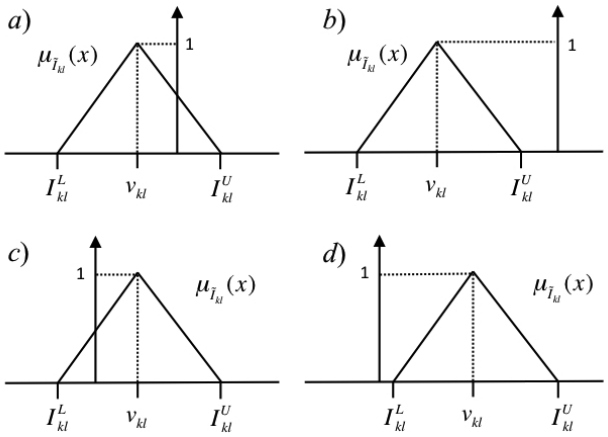

Figure 3: Locations of triangular fuzzy numbers.

Briefly, the method can be implemented as follows:

1. Compute values v_{kl} by averaging the endpoints of the polytope delimited by constraints representing the ordinal information on component utilities using the method described in Section 4.
2. Build the triangular fuzzy numbers $\widetilde{I}_{kl} = (I_{kl}^L, v_{kl}, I_{kl}^U)$. To do this, first compute pairwise dominance values, D_{kl}, solving the optimization problem in Eq. (2). Note that the method proposed in Section 4 can again be used, since one of the endpoints is the optimal solution, which can be identified by just evaluating the endpoints in the objective function.
3. Compute the *dominance intensities* as follows:
   - If $v_{kl} \geq 0$, then $DI_{kl} = d(\widetilde{I}_{kl}, 0, f)$, where d refers to Tran and Duckstein's distance <31>, and f is a weight function for differentiating a risk-averse, risk-neutral or risk-prone DM, as explained later.
   - Else ($v_{kl} < 0$), $DI_{kl} = -d(\widetilde{I}_{kl}, 0, f)$.
4. Compute a *dominance intensity measure* for each alternative A_k,

$$DIM_k = \sum_{l=1, l\neq k}^{m} DI_{kl}.$$

5. Rank alternatives according to DIM_k values, where the alternatives with the maximum and minimum DIM_k are the best and worst, respectively.

The distance defined by Tran and Duckstein <31> for the *generalization of left and right fuzzy numbers (GLRFN)* <4> is used in $d(\widetilde{I}_{kl}, 0, f)$. A fuzzy set $\tilde{a} = (a_1, a_2, a_3, a_4)$ is called a *generalization of the left and right fuzzy numbers*

($GLRFN$) when its membership function is defined as

$$\mu_{\widetilde{a}}(x) = \begin{cases} L\left(\dfrac{a_2 - x}{a_2 - a_1}\right), & \text{if } a_1 \leq x \leq a_2 \\ 1, & \text{if } a_2 \leq x \leq a_3 \\ R\left(\dfrac{x - a_3}{a_4 - a_3}\right), & \text{if } a_3 \leq x \leq a_4 \\ 0, & \text{otherwise,} \end{cases}$$

where L and R are strictly decreasing functions defined in $[0, 1]$ and satisfying the conditions:

$$L(x) = R(x) = 1 \text{ if } x \leq 0 \text{ and } L(x) = R(x) = 0 \text{ if } x > 0.$$

Triangular fuzzy numbers are special cases of $GLRFN$ with $L(x) = R(x) = 1 - x$ and $a_2 = a_3$. A $GLRFN$ is denoted as $\tilde{a} = (a_1, a_2, a_3, a_4)_{L\tilde{a}-R\tilde{a}}$ and an α-cut of $\tilde{a}$ is a crisp set that contains all the elements that have a membership value greater than or equal to α:

$$\tilde{a}(\alpha) = (\tilde{a}_L(\alpha), \tilde{a}_R(\alpha)) = (a_2 - (a_2 - a_1)a_3 L_{\tilde{a}}^{-1}(\alpha), a_3 - (a_4 - a_3)a_3 R_{\tilde{a}}^{-1}(\alpha)).$$

Tran and Duckstein <31> define the distance between two $GLFRN$ fuzzy numbers $\tilde{a}$ and $\widetilde{b}$ as

$$d^2(\widetilde{a}, \widetilde{b}, f) = \frac{\int_0^1 \left\{ \begin{array}{l} \left[\frac{\widetilde{a}_L(\alpha)+\widetilde{a}_R(\alpha)}{2} - \frac{\widetilde{b}_L(\alpha)+\widetilde{b}_R(\alpha)}{2}\right]^2 + \\ +\frac{1}{3}\left[\left(\frac{\widetilde{a}_L(\alpha)+\widetilde{a}_R(\alpha)}{2}\right)^2 + \left(\frac{\widetilde{b}_L(\alpha)+\widetilde{b}_R(\alpha)}{2}\right)^2\right] \end{array} \right\} \times f(\alpha)(d\alpha)}{\int f(\alpha)(d\alpha)}.$$

The function $f(\alpha)$ is positive continuous in $[0, 1]$ and serves as a weight function. The distance is computed as the weighted sum of distances between two intervals across all α-cuts from 0 to 1. Moreover, it flexibilizes DM participation. For example, $f(\alpha) = \alpha$ looks to be reasonable when the DM is risk-neutral, whereas a risk-averse DM would put more weight on information at a higher α level by using functions such as $f(\alpha) = \alpha^2$ or a higher power of α. A constant ($f(\alpha) = 1$), or even a decreasing function f, could be used for a risk-prone DM.

For the particular case of the distance from a triangular fuzzy number $\tilde{a} = (a_1, a_2, a_3)$ to a constant (specifically 0), we have:

1. If $f(\alpha) = \alpha$, then

$$\begin{aligned} d^2(\widetilde{a}, 0, f) = \; & a_2^2 + \frac{1}{3}a_2(a_3 + a_1) + \frac{1}{18}[(a_3 - a_2)^2 + (a_2 - a_1)^2] - \\ & - \frac{1}{18}[(a_2 - a_1)(a_3 - a_2)]. \end{aligned}$$

2. If $f(\alpha) = 1$, then

$$\begin{aligned} d^2(\widetilde{a}, 0, f) = \; & a_2^2 + \frac{1}{2}a_2(a_3 + a_1) + \frac{1}{9}[(a_3 - a_2)^2 + (a_2 - a_1)^2] - \\ & - \frac{1}{9}[(a_2 - a_1)(a_3 - a_2)]. \end{aligned}$$

3. If $f(\alpha) = \alpha^2$, then

$$d^2(\widetilde{a}, 0, f) = a_2^2 + \frac{1}{4}a_2(a_3 + a_1) + \frac{1}{144}[(a_3 - a_2)^2 + (a_2 - a_1)^2] - \\ -\frac{1}{96}[(a_2 - a_1)(a_3 - a_2)].$$

4. Set of endpoints and centroid of a polytope delimited by constraints representing ordinal information

In this section we propose a method for deriving the set of endpoints of a polytope delimited by the following constraints:

1. A ranking of the variables under consideration $y_1, ..., y_m \in [0, 1]$, $y_1 \geq y_2 \geq ... \geq y_m$.
2. A ranking of the differences between consecutive variables in the above ranking $\Delta_j = y_j - y_{j+1}, j = 1, .., m - 1$.

First, we build the set of vertices for the polytope under consideration, denoted by V. The first vertex to be added to V is $(0, 0, 0, ..., 0)$, since this vector satisfies all constraints. To build a new vertex we assign a value 1 to the position corresponding to the best-ranked variable in the above vertex. Next, we assign a value 1 to the position corresponding to the second ranked variable in the previous vertex, leading to a new vertex, and so on, until we reach the worst-ranked variable, which yields the vertex, $(1, 1, ..., 1)$.

Note that when the variables are ranked in descending order ($y_1 \geq y_2 \geq ... \geq y_m$), then it is trivial to derive V, $V = \{(0, 0, 0, 0, ..., 0), (1, 0, 0, 0, ..., 0), (1, 1, 0, 0, ..., 0), (1, 1, 1, 0, ..., 0), ..., (1, 1, 1, 1..., 1)\}$.

Now, we consider the ranking of the differences between consecutive variables in the above variable ranking, $\Delta_j = y_j - y_{j+1}, j = 1, .., m-1$. We denote by EP and M the sets to which we add endpoints of the polytope and the differences between consecutive variables, respectively. Both sets are initially empty, and EP will contain all the endpoints of the polytope when the procedure ends.

Then, we progressively add to M the difference between consecutive variables according to the available ranking, i.e., first we add the best-ranked difference, then the second-ranked and so on. Each time a new difference is added to M, the vertices in V associated with the best variable of each element in M are averaged, and the resulting vector is added to EP. The procedure ends when all differences have been added to M, i.e., $M = \{\Delta_1, \Delta_2, ..., \Delta_{m-1}\}$. Finally, we add the endpoint (0,0,0,...,0) to EP.

The algorithm for deriving the endpoints is as follows:

- Step 1. Build the set of vertices V considering the ranking of the variables y_1,..., y_m. Set $V = \emptyset$.
  - Add (0,0,...,0) to V.
  - For $i = 1, ..., m$:
    - Identify the i-th best-ranked variable in the ranking, y_{jbest}.

    - Assign value 1 to the element in the previous vertex added to V corresponding to y_{jbest}.
    - Add the new vertex to V.

  Note that there will be $m+1$ elements in V at the end of Step 1.

- Step 2. We consider the ranking of the differences between consecutive variables $\Delta_i = y_i - y_{i+1}, j = 1, ..., m-1$. Set $M = \emptyset$ and $EP = \emptyset$.

  For $i = 1, ..., m-1$:

  - Add the i-th best-ranked difference to M.
  - Identify the vertices associated with the best variable of each element in M.
  - Compute and add the average of the considered vertices to EP.

- Step 3. Add (0,0,...,0) and (1,1,...,1) to EP.

Finally, the EP set contains all the endpoints of the polytope, whose average yields to the centroid of the polytope.

Next, we illustrate the method with the example shown in Fig. 1, i.e., we consider a problem with five variables and the following rankings: $y_3 \geq y_5 \geq y_4 \geq y_2 \geq y_1$ and $\Delta_2 \geq \Delta_1 \geq \Delta_4 \geq \Delta_3$, with $\Delta_1 = y_3 - y_5$, $\Delta_2 = y_5 - y_4$, $\Delta_3 = y_4 - y_2$ and $\Delta_4 = y_2 - y_1$.

Then, the algorithm would work as follows:

- Step 1: From $y_3 \geq y_5 \geq y_4 \geq y_2 \geq y_1$, we have associated the vertices as follows: V=$\{(0,0,0,0,0)$, $y_3 : (0,0,1,0,0)$, $y_5 : (0,0,1,0,1)$, $y_4 : (0,0,1,1,1)$, $y_2 : (0,1,1,1,1)$, $y_1 : (1,1,1,1,1)\}$.

- Step 2: $M = \emptyset$ and $EP = \emptyset$.

  i=1:

  - $\Delta_2 = y_5 - y_4$ is the best-ranked difference, so $M = \{\Delta_2\}$.
  - y_5 is the best variable corresponding to Δ_2, then the vertex in V corresponding to y_5, (0,0,1,0,1), is added to EP.

  i=2:

  - $\Delta_1 = y_3 - y_5$ is the second-ranked difference, so $M = \{\Delta_2, \Delta_1\}$.
  - y_5 and y_3 are the best variables corresponding to the differences in M, Δ_2 and Δ_1, respectively. We compute the average of the associated vertices, $y_5 : (0,0,1,0,1)$ and $y_3 : (0,0,1,0,0)$, which we add to EP. $EP = \{(0,0,1,0,1), (0,0,1,0,1/2)\}$.

  i=3:

  - $\Delta_4 = y_2 - y_1$ is the third-ranked difference, so $M = \{\Delta_2, \Delta_1, \Delta_4\}$.

– y_5, y_3 and y_2 are the best variables corresponding to the differences in M, respectively. We compute the average of vertices $y_5 : (0, 0, 1, 0, 1)$, $y_3 : (0, 0, 1, 0, 0)$ and $y_2 : (0, 1, 1, 1, 1)$, which we add to EP.
$EP = \{(0, 0, 1, 0, 1), (0, 0, 1, 0, 1/2), (0, 1/3, 1, 1/3, 2/3)\}$.

i=4:

– $\Delta_3 = y_4 - y_2$ is the worst-ranked difference, so $M = \{\Delta_2, \Delta_1, \Delta_4, \Delta_3\}$.

– y_5, y_3, y_2 and y_4 are now the best variables corresponding to the differences in M, respectively. We compute the average of vertices $y_5 : (0, 0, 1, 0, 1)$, $y_3 : (0, 0, 1, 0, 0)$, $y_2 : (0, 1, 1, 1, 1)$ and $y_4 : (0, 0, 1, 1, 1)$, which we add to EP.
$EP = \{(0, 0, 1, 0, 1), (0, 0, 1, 0, 1/2), (0, 1/3, 1, 1/3, 2/3), (0, 1/4, 1, 1/2, 3/4)\}$.

- Step 3: (0,0,0,0,0) and (1,1,1,1,1) are added to EP.

Finally, EP=$\{$(0,0,1,0,1),(0,0,1,0,1/2),(0,1/3,1,1/3,2/3),(0,1/4,1,1/2,3/4),(0,0,0,0,0),(1,1,1,1,1)$\}$, and the centroid is derived by averaging the endpoints in EP, yielding (1/6,19/72,5/6,11/36,47/72).

Note importantly that the method proposed in this section can also be used if we have the ranking of alternatives for each attribute under consideration, but the information about the differences between the values of consecutive alternatives is not available. This situation is less stressful on DMs and makes the method suitable for more real decision-making problems, in which the expert is often reluctant or may find it difficult to provide much information about his/her preferences. Notice also that the differences between the values of consecutive alternatives may be hard to quantify.

However, the method cannot be used if the available ordinal information is partial rather than complete, i.e., some alternatives are not included in the rankings available for some attributes.

5. Performance analysis based on Monte Carlo simulation techniques

In this section we analyze and compare the performance of the proposed method, DIM, with other dominance measuring methods (*AP1*, *DME1* and *DME2*) and with the method proposed by Sarabando and Dias <27>, which represents the imprecision concerning the DM's preferences in the same way as in this paper.

We set out to carry out a simulation study for different scenarios accounting for different numbers of alternatives and attributes. In accordance with previous simulations performed in the literature, we identify six different levels for the alternatives ($m = 3, 5, 7, 10, 15, 20$) and five different levels for the attributes ($n = 3, 5, 7, 10, 15$), yielding 30 design scenarios.

The process would be as follows for each scenario:

1. Generate component values randomly from a uniform distribution in (0,1), yielding an $m \times n$ matrix. This matrix has to be normalized making the smallest and largest values from each column zero and one, respectively. Note that dominated alternatives have to be removed in the simulation since they are not useful for analyzing the performance of the considered methods. From each row of the above matrix we derive the ranking of alternatives in each attribute and the ranking of the differences between consecutive alternatives.
2. Generate attribute weights randomly. First, we select $n-1$ independent random numbers from a uniform distribution on $(0,1)$, and rank these numbers. Suppose the ranked numbers are $1 \geq r_{n-1} \geq ... \geq r_2 \geq r_1 > 0$. The differences between consecutive ranked numbers are then used as the target weights $w_n^T = 1 - r_{n-1}, w_{n-1}^T = r_{n-1} - r_{n-2}, ..., w_1^T = r_1$. The resulting weights will sum 1 and be uniformly distributed in the weight space <5>. They are used to derive the ranking of attribute weights. Note that these weights will be the TRUE weights. The TRUE ranking of alternatives is computed using the component value matrix from the previous step and the TRUE weights.
3. Compute a ranking of alternatives for each method according to their algorithms using just the ordinal information obtained from the component value matrix and weights.
4. Compare the rankings provided by each method with the TRUE ranking. We use two measures of efficacy, the hit ratio and the rank-order correlation <2; 3>. The hit ratio is the proportion of all cases in which the method selects the same best alternative as in the TRUE ranking. Rank-order correlation represents how similar the overall rank structures of alternatives are in the TRUE ranking and in the ranking derived from the method. It is calculated using Kendall's τ (Winkler and Hays <34>):

$$\tau = 1 - \frac{2 \times (\text{number of pairwise preference violations})}{\text{total number of pair preferences}} = \frac{S}{m(m-1)/2},$$

where S is the difference between the number of concordant (ordered equally) and discordant (ordered differently) pairs and m is the total number of alternatives.
If there are tied (same value) observations then the denominator $m(m-1)/2$ has to be replaced by

$$\sqrt{[m(m-1)/2 - \sum_{i=1}^{t} t_i(t_i-1)/2][m(m-1)/2 - \sum_{i=1}^{t} u_i(u_i-1)/2]},$$

where t is the number of tied observation sets, t_i is the number of tied observations in the TRUE ranking, and u_i is the number of tied observations in the ranking derived from the method.

We ran 20,000 trials for each of the 30 design scenarios, and replications were parallelized to save computational resources, mainly time.

Table 1 and Fig. 4 show the average hit ratio for each of the 30 design elements, i.e., the average values of 20,000 trials, considering a risk-neutral DM. We have marked the maximum hit ratio for each method across all 30 design scenarios in bold. The labels along the abscissa of the chart in Fig. 4 consist of two values corresponding to the number of alternatives and attributes, respectively. There are four columns for each label, representing the hit ratio or rank-order correlation levels for the considered methods.

Fig. 4 shows that the hit ratio decreases as the number of alternatives that there are for any given number of attributes grows, which is obvious. Additionally, the number of attributes also affects the hit ratio; it is greater the more attributes there are for any given number of alternatives.

DME1 and *DME2* methods clearly outperform the results provided by *AP1* in all scenarios. *DME1* and *DME2* are much better than *AP1* when there are a lot of alternatives. These results are consistent with the findings reported in <16>, in which imprecision is represented by value intervals in alternative performances, component values and weights, and <13>, in which ordinal information is considered for weights. The difference in the mean hit ratios between *DME2* and *AP1* is 1.236% for three and 6.262% for twenty alternatives.

We also find that the proposed method, *DIM*, outputs better results than the *DME1* and *DME2* methods in all scenarios. DIM performs much better than *DME1* and *DME2* at larger numbers of alternatives. The mean hit ratio is 81.38 for DIM and 77.24 and 78.39 for *DME1* and *DME2*, respectively.

DIM outputs very similar results to the *SD* method, and the difference between the average hit ratio is only 0.02. *DIM* outperforms the *SD* method in 10 scenarios, but the difference is lower than 0.54 in all cases. However, there are also five cases in which *SD* outperforms the *DIM* method, but now the difference is less than 0.02.

Furthermore, according to the *paired-samples* t-test (which computes the difference between the mean values of the two methods and tests whether the average differs from zero), there is no significant difference between the hit ratio means of the DIM and SD methods depending of the value of the significance level (significance level, two-tailed: 0.02546).

Table 2 and Fig. 4 show the rank-order correlations for each of the 30 design elements for a risk-neutral DM. Fig. 4 shows that the rank-order correlations increases proportionally to the number of attributes. Besides, the rank-order correlations for the *DME2*, DIM and SD methods also increases proportionally to the number of attributes.

DIM again outperforms *DME1* and *DME2* in all the scenarios, which also outperfoms *AP1*. The results output by the SD and *DIM* methods are again very similar, the difference between the average rank-order correlations being only 0.05. There are scenarios in which the *DIM* method is better than SD method, mainly when there are not many attributes, whereas SD slightly outperforms the *DIM* method for 10 or more alternatives. However, the difference is always lower than 0.02.

Figure 4: Hit ratio and rank-order correlation levels.

According to the *paired-samples* t-test, there is no significant difference between the hit ratio means (significance level, two-tailed: 0.064838).

The results for risk-prone and risk-averse DMs are similar. Table 3 shows the average hit ratios and rank-order correlations for both situations. Maximum values are marked in bold and correspond to the *DIM* method in all cases. The *DIM* method again outputs better results than the *DME1* and *DME2* methods, which are better than *AP1*. The SD and *DIM* methods are again very similar, and the difference between the hit ratio and rank-order correlation means of the DIM and SD methods are not significant (significance levels, two-tailed: 0.027587 and 0.057604, respectively, for a risk-prone DM; and 0.023705 and 0.077825, for a risk-adverse DM).

Table 1. Hit ratios. Risk-neutral DM

Alternatives	Criteria	Methods				
		$AP1$	$DME1$	$DME2$	DIM	SD
3	3	82.13	84.91	83.73	84.68	84.14
	5	84.42	85.21	84.92	85.41	85.36
	7	85.32	86.39	86.12	86.62	86.64
	10	85.37	86.79	86.82	87.16	87.15
	15	86.12	88.09	87.95	88.30	88.29
5	3	76.56	78.91	79.72	81.74	81.68
	5	77.57	80.24	80.22	81.92	81.92
	7	79.22	82.27	82.13	83.46	83.46
	10	79.72	82.43	82.77	83.90	83.90
	15	80.25	83.43	83.49	84.76	84.76
7	3	70.83	74.79	76.65	79.42	79.41
	5	74.19	76.64	77.83	80.19	80.19
	7	75.50	78.56	79.28	81.21	81.21
	10	76.59	80.02	80.41	82.40	82.40
	15	77.14	81.29	81.38	83.70	83.70
10	3	66.44	71.35	74.65	78.58	78.57
	5	70.40	73.52	75.39	78.26	78.26
	7	72.05	75.38	76.80	79.64	79.64
	10	73.72	77.43	78.37	81.22	81.20
	15	73.58	78.72	79.08	82.51	82.51
15	3	61.62	67.52	71.95	77.68	77.67
	5	66.34	69.64	71.93	76.39	76.40
	7	68.97	72.08	73.80	78.38	78.38
	10	70.20	73.80	75.46	80.12	80.12
	15	70.86	76.00	76.96	82.26	82.27
20	3	66.44	71.00	74.70	78.85	78.84
	5	63.63	67.17	70.18	74.84	74.86
	7	65.70	69.25	71.66	77.42	77.42
	10	67.78	71.46	73.01	79.06	79.07
	15	69.03	72.98	74.34	81.31	81.29
Mean		73.92	77.24	78.39	81.38	81.36

Table 2. Rank-order correlation (Kendall's τ). Risk-neutral DM

Alternatives	Criteria	Methods				
		$AP1$	$DME1$	$DME2$	DIM	SD
3	3	76.12	78.97	77.12	78.93	77.55
	5	79.42	80.74	80.45	80.91	80.89
	7	79.99	81.23	80.96	81.83	81.82
	10	80.32	82.12	81.81	82.48	82.48
	15	81.06	83.16	83.01	83.63	83.63
5	3	74.53	76.98	76.83	78.40	78.34
	5	78.24	80.22	80.33	81.44	81.43
	7	79.36	81.67	81.63	82.72	82.72
	10	79.92	82.68	82.64	83.90	83.90
	15	79.91	83.23	83.28	84.41	84.41
7	3	73.45	76.47	76.95	78.45	78.43
	5	78.24	80.62	81.01	82.41	82.42
	7	79.28	81.82	82.34	83.46	83.46
	10	79.50	82.84	83.07	84.52	84.52
	15	79.75	83.63	83.80	85.31	85.31
10	3	72.05	75.98	76.81	78.41	78.42
	5	77.79	80.40	81.23	82.77	82.77
	7	78.85	81.75	82.44	84.07	84.07
	10	79.40	82.92	83.47	85.34	85.34
	15	79.35	83.60	84.01	86.23	86.23
15	3	69.43	75.25	76.03	77.84	77.87
	5	77.15	79.71	80.98	83.00	83.01
	7	78.29	81.34	82.40	84.73	84.74
	10	78.76	82.38	83.23	85.89	85.89
	15	78.91	83.03	83.67	86.91	86.91
20	3	72.00	75.93	76.72	78.40	78.41
	5	76.65	79.16	80.64	83.03	83.04
	7	77.57	80.57	81.88	84.76	84.78
	10	78.14	81.79	82.78	86.10	86.11
	15	78.29	82.53	83.29	87.20	87.20
Mean		77.72	80.76	81.16	82.92	82.87

Table 3. Results for a risk-prone and a risk-adverse DM

	Measure	$AP1$	$DME1$	$DME2$	DIM	SD
Risk-prone	Hit ratio	73.78	77.10	78.30	81.27	81.25
	Kendall's τ	77.69	80.743	81.16	82.89	82.85
Risk-adverse	Hit ratio	73.90	77.24	78.38	81.32	81.30
	Kendall's τ	77.70	80.74	81.18	82.92	82.86

6. Conclusions

We have proposed a new dominance measuring method to deal with ordinal information about the decision-maker's preferences, in both weights and component values. The decision maker provides a ranking of attribute importance. Besides, the method takes into account a ranking of the alternatives in each attribute and also a ranking of the difference of values between consecutive alternatives.

The proposed method uses the centroid of the polytope delimited by ordinal information and builds triangular fuzzy numbers, whose distances to the crisp value 0 are the basis for the definition of a dominance intensity measure.

The results of Monte Carlo simulation techniques applied demonstrate that the proposed method is clearly better at selecting the best alternative and ranking alternatives than other dominance measuring methods proposed in the literature. Its performance is very similar to the method proposed by Sarabando and Dias, which was developed to deal with decision-making problems with ordinal information about the decision-maker's preferences too. The *paired-samples t*-test shows that there is no significant difference between the two for a neutral, risk-prone and risk-averse decision-maker.

Sarabando and Dias's method is less computationally demanding, but its application is restricted to the discussed imprecise decision-making situation. On the other hand, the method proposed in this paper can also be used if we have the ranking of alternatives for each attribute under consideration, but the information about the differences between the values of consecutive alternatives is not available. The algorithm proposed to derive endpoints in the centroid computation still works. This situation is less stressful on DMs and makes the method suitable for much more real decision-making problems.

As a future research line we propose the use of simulation techniques to approximate the centroid and conduct the respective analysis of its performance when different types of partial ordinal information are available. Moreover, we also intend to use other types of fuzzy sets and different notions of associated distances to derive the final ranking of alternatives.

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