

Dominance intensity measure within fuzzy weight oriented MAUT: An application

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A B S T R A C T

We introduce a dominance intensity measuring method to derive a ranking of alternatives to deal with incomplete information in multi-criteria decision-making problems on the basis of multi-attribute utility theory (MAUT) and fuzzy sets theory. We consider the situation where there is imprecision concerning decision-makers' preferences, and imprecise weights are represented by trapezoidal fuzzy weights. The proposed method is based on the dominance values between pairs of alternatives. These values can be computed by linear programming, as an additive multi-attribute utility model is used to rate the alternatives. Dominance values are then transformed into dominance intensity measures, used to rank the alternatives under consideration. Distances between fuzzy numbers based on the general-ization of the left and right fuzzy numbers are utilized to account for fuzzy weights.

An example concerning the selection of intervention strategies to restore an aquatic ecosystem contaminated by radionuclides illustrates the approach. Monte Carlo simulation techniques have been used to show that the proposed method performs well for different imprecision levels in terms of a hit ratio and a rank-order correlation measure.

1. Introduction

Most complex decision-making problems involve imprecise information [58]. It is frequently impossible to predict with certainty the alternative performances, since as they often reflect social or environmental impacts or are taken from statistics or measurements, they may be intangible.

Neither is it easy to elicit the relative importance of criteria by means of precise weights. Decision makers (DMs) may find it difficult to compare criteria or not want to reveal their preferences in public. Furthermore, in a group decision-making context, imprecision concerning preferences may be the result of a negotiation process. This situation is usually referred to as decision-making with imprecise information, with incomplete information or with partial information [49,50].

A number of papers on multi-attribute utility theory (MAUT) have dealt with incomplete information. For instance, Sage and White [54] proposed the model of *imprecisely specified multi-attribute utility theory* (ISMAUT), where preference information about both weights and utilities is assumed not to be precise. Malakooti [38] suggested a new efficient algorithm for ranking

alternatives when there is incomplete information about the preferences and the value of the alternatives. Ahn [1] extended Malakooti's work.

More recently, Jiménez et al. [29] accounted for missing information about some alternative performances. They proposed using the attribute range rather than redistributing the respective weights throughout the objective hierarchy.

Another possibility for dealing with imprecision within MAUT described in the literature attempts to apply the concept of pairwise and absolute dominance to eliminate inferior alternatives, leading to the so-called surrogate weighting methods [60,4], and adapted classical decision rules [46,56], respectively.

Eum et al. provided linear programming characterizations of dominance and potential optimality for alternatives when information about performances and/or weights is incomplete [21]. Lee et al. extended the approach to hierarchical structures [37], and Park developed the concepts of weak potential optimality and strong potential optimality [45]. In [39], the more general case considering imprecision, described by means of fixed bounds, appears in alternative performances, as well as in weights and utilities.

More recently, different dominance measuring methods, which use information about each alternative's intensity of dominance, have been proposed [2,41,40].

On the other hand, *stochastic multi-criteria acceptability analysis* (SMAA) is based on exploring the weight space in order to describe which scores would make each alternative the preferred

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option. Inaccurate or uncertain criteria values are represented by probability distributions and partial preference information. The SMAA-2 method [34] extended the original SMAA by considering all ranks in the analysis, and SMAA-O [36] was designed for problems where information for some or all criteria are ordinal. Different ways of handling dependent uncertainties within SMAA-2 have been analyzed in [35].

Sarabando and Dias [57] gave a brief overview of approaches proposed within the MAUT and MAVT (Multi-Attribute Value Theory) framework to deal with incomplete information.

The *analytical hierarchy process* (AHP) is another popular method, apart from MAUT, when the information on criteria is mainly cardinal and where attributes are fully compensatory. Incomplete information within AHP has also been addressed in the literature, such as [53], where an interval of numerical values is associated with each judgment in the pairwise comparisons or [33,25], dealing with missing data in AHP.

Outranking methods [5,23,44] overcome the assumption that attributes are fully compensatory and there is a true ranking of alternatives just waiting to be discovered. The most widely used outranking methods are ELECTRE and PROMETHEE. ELECTRE III improved ELECTRE II to deal with inaccurate, imprecise, uncertain and ill-determined data, while PROMETHEE III is based on intervals.

Studies concerning imprecision were also conducted using the theory of fuzzy sets [8,30], counting on the advances of research in arithmetic and the logical operators of fuzzy numbers, such as [62], suggesting the comparison of fuzzy numbers using a fuzzy measure of distance [17], proposing the non-additive fuzzy integral when there is dependence among criteria; or [43], introducing ordered weighted aggregation operators.

A modified fuzzy version of TOPSIS was proposed in [11], whereas [47] introduced a new specification of a fuzzy model, the fuzzy utility model which is applied for road route choice.

Preliminary works on the extension of AHP [13] to account for fuzzy numbers were proposed in [64,6]. A fuzzy integrated hierarchical decision-making approach was developed in [13] to solve the distribution center location selection problem. A fuzzy extension of AHP, FEHP, is provided in [12] to deal with the selection of global suppliers, where triangular fuzzy numbers are used in the DMs' comparison judgements, whereas the final priority of the considered criteria is based on a fuzzy synthetic extent analysis [14].

Developments related to fuzzy outranking methods, such as the utilization of the PROMETHEE method with trapezoidal fuzzy numbers proposed in [24] are reviewed in [7]. More recently, an extension of the ELECTRE I method for group decision-making in a fuzzy environment was introduced in [26].

In this paper we introduce a dominance measuring method that adapts the proposal in [41,27] to account for fuzzy weights, exploiting research reported in [62] on distances between fuzzy numbers based on the generalization of the left and right fuzzy numbers [19,3].

In Section 2 we review dominance-measuring methods proposed to deal with incomplete information within MAUT, which can be viewed as the groundwork of the proposed method. In Section 3 we outline the dominance-measuring method accounting for trapezoidal fuzzy weights. The approach is illustrated in Section 4 using an example concerning the selection of intervention strategies to restore an aquatic ecosystem. In Section 5, the performance of the proposed method is analyzed using Monte Carlo simulation techniques. Finally, some conclusions are discussed in Section 6.

2. Dominance-measuring methods

As cited in the previous section, one option described in the literature for dealing with imprecision within MAUT is to eliminate inferior alternatives based on the concept of dominance.

Let us consider a decision-making problem with m alternatives, A_i , $i = 1, \dots, m$, and n attributes, X_j , $j = 1, \dots, n$, where incomplete information about input parameters was incorporated into the decision-making process. U_i ($\mathbf{u}_i = (u_{i1}, \dots, u_{in}) \in U_i$) define the feasible region for utilities associated with alternative A_i over each attribute. Different methods can be used to build utility functions depending on the level of knowledge and features of the attribute under consideration. When there is in-depth and precise knowledge about the attribute, the DM can directly construct a piecewise linear utility function by providing the best and the worst attribute values and some intermediate values with their respective imprecise utilities. Methods based on lotteries, such as *fractile method* and the *extreme gambles method* [22], are used when DMs have little knowledge about or are inexperienced in the domain. The GMAA decision support system, which includes the combination of two slightly modified standard procedures for utility assessment, is introduced in [28]. Incomplete information is entered as value intervals in response to the probability questions that the DM is asked, checking for consistency.

On the other hand, W defines the feasible region for weights, representing the relative importance of criteria as follows:

- ordinal relations, $\mathbf{w} \in W = \{\mathbf{w} = (w_1, \dots, w_n) : w_1 \geq w_2 \geq \dots \geq w_n\}$,
- value intervals, $\mathbf{w} \in W = \{\mathbf{w} = (w_1, \dots, w_n) : w_j \in [w_j^L, w_j^U], j = 1, \dots, n\}$,
- intervals for weight ratios (trade-offs), $\mathbf{w} \in W = \{\mathbf{w} = (w_1, \dots, w_n) : w_j/w_k \in [w_{jk}^L, w_{jk}^U], j = 1, \dots, n\}$,
- linear inequality constraints for weights, $\mathbf{w} \in W = \{\mathbf{w} = (w_1, \dots, w_n) : A\mathbf{w} \leq c\}$, or
- nonlinear inequality constraints for weights, $\mathbf{w} \in W = \{\mathbf{w} = (w_1, \dots, w_n) : g(\mathbf{w}) \leq 0\}$.

There are many weighting methods that use different questioning procedures to elicit weights, such as *SWING weighting* and *SMARTS* [20], *pricing out method* and *TRADEOFFS weighting* [31], *AHP* [52], or *preference programming* [55]. Most are adapted to account for imprecision.

We assume an additive model, which is considered a valid approximation in most real decision-making problems for the reasons described in [48,59], and is widely used within MAUT,

$$u(A_i) = \sum_{j=1}^n w_j u_{ij} = \mathbf{w}^T \mathbf{u}_i.$$

Given two alternatives A_k and A_l , the alternative A_k dominates A_l if $D_{kl} \geq 0$, D_{kl} being the optimum value of the optimization problem:

$$\begin{aligned} D_{kl} &= \min\{u(A_k) - u(A_l) = \mathbf{w}^T(\mathbf{u}_k - \mathbf{u}_l)\} \\ \text{s.t. } &\mathbf{u}_k \in U_k, \mathbf{u}_l \in U_l \\ &\mathbf{w} \in W. \end{aligned} \quad (1)$$

This concept of dominance is called *pairwise dominance*. Another type of dominance, known as *absolute dominance*, can be employed [56]. Absolute dominance considers the following optimization problems:

$$U_k = \max\{u(A_k) = \mathbf{w}^T \mathbf{u}_k \mid \mathbf{w} \in W, \mathbf{u}_k \in U_k\} \quad \text{and}$$

$$L_k = \min\{u(A_k) = \mathbf{w}^T \mathbf{u}_k \mid \mathbf{w} \in W, \mathbf{u}_k \in U_k\}.$$

Alternative A_k absolutely dominates A_l if $L_k \geq U_l$, i.e., the lower bound of A_k exceeds the upper bound of A_l . Note that if A_k absolutely dominates A_l , then A_k dominates A_l , but the reverse does not hold.

Note that this dominance approach often results in almost no prioritization of alternatives or too many non-dominated

alternatives [32]. However, pairwise and absolute dominance values can be used to further prioritize competitive alternatives, and hence recommend the best alternative and fully rank alternatives. An example of how these dominance values have been employed is the adaptation of four classical decision rules to encompass an imprecise decision context [46,56].

A recent approach is to use information about each alternative's intensity of dominance, known as *dominance-measuring methods*. The starting point for dominance-measuring methods is the assessment of the so-called *dominance matrix*:

$$D = \begin{pmatrix} - & D_{12} & \dots & D_{1m-1} & D_{1m} \\ D_{21} & - & \dots & D_{2m-1} & D_{2m} \\ D_{31} & D_{32} & - & D_{3m-1} & D_{3m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ D_{m1} & D_{m2} & \dots & D_{mm-1} & - \end{pmatrix},$$

where D_{kl} values are computed from expression (1).

The first two dominance-measuring methods were proposed in [2], where both *dominating* and *dominated measures* were computed from the paired dominance values, leading to a *net dominance*. This is used as a measure of the strength of preference in the sense that a greater net value is better.

Two new dominance-measuring methods were proposed in [41]. The first one was based on the idea first suggested by Ahn and Park. Dominating and dominated measures are computed, but they are then combined into a preference intensity rather than a net dominance measure. In the second method, alternatives were ranked on the basis of a *global preference intensity measure*. Paired dominance values were first transformed into preference intensities depending on the preference for alternatives A_k and A_l . Then a global preference intensity measure was derived for each alternative A_k . This global preference intensity measure was calculated as the sum of the preference intensities of alternative A_k compared with the other alternatives. This was used as the measure of strength of preference.

The above four methods could be applied for different representations of imprecision concerning weights. Monte Carlo simulation techniques were applied to analyze and compare their performance with other approaches, such as *surrogate weighting methods* [60] and [4], and *adapted classical decision rules* for ordinal relations regarding attribute weights, i.e., DMs ranked attributes in descending order of importance. The results showed that the preference intensity measure based method performs better than the adaptation of classical decision rules in terms of the identification of the best alternative and the overall ranking of alternatives, closely rivaling the *rank-order centroid weights method*, which was identified as the best approach.

Mateos et al. consider different cases with incomplete information about weights [40], specifically, weight intervals, weights fitting independent normal probability distributions or weights represented by triangular or trapezoidal fuzzy numbers. Monte Carlo simulation is again used to compare dominance-measuring methods with *SMAA* and *SMAA-2 methods* where intervals account for imprecision in weights.

The dominance concept has also been used within fuzzy multi-criteria analysis. Although the most widely used definition is based on the maximum degree of membership of the two considered fuzzy numbers (see, e.g. [16,15]), other definitions have been proposed, such as the comparison of overlapped and non-overlapped areas among the two fuzzy numbers [61], or their arithmetic difference [67]. In [66], Yeh and Deng combine arithmetic difference with the notion of fuzzy reference set to address the fuzzy ranking problem.

The dominance-measuring method that we introduce in this paper adapts the proposal in [41,27] to account for fuzzy weights,

exploiting research reported in [62] on distances between fuzzy numbers based on the *generalization of the left and right fuzzy numbers* [19,3].

3. A dominance-measuring method accounting for trapezoidal fuzzy weights

First, let us review several concepts on fuzzy sets that will be used in the proposed method. A fuzzy set $\tilde{a} = (a_1, a_2, a_3, a_4)$ is called a *generalization of the left and right fuzzy numbers (GLRFN)* when its membership function is defined as

$$\mu_{\tilde{a}}(x) = \begin{cases} L\left(\frac{a_2-x}{a_2-a_1}\right) & \text{if } a_1 \leq x \leq a_2 \\ 1 & \text{if } a_2 \leq x \leq a_3 \\ R\left(\frac{x-a_3}{a_4-a_3}\right) & \text{if } a_3 \leq x \leq a_4 \\ 0 & \text{otherwise,} \end{cases}$$

where L and R are strictly decreasing functions defined in $[0,1]$ and satisfying the conditions:

$$L(x) = R(x) = 1 \quad \text{if } x \leq 0,$$

$$L(x) = R(x) = 0 \quad \text{if } x \geq 0.$$

For $a_2 = a_3$, we have the classical Dubois-Prade definition of right and left triangular fuzzy numbers [19]. Trapezoidal fuzzy numbers are a special case of *GLRFN* with $L(x) = R(x) = 1 - x$. A *GLRFN* is denoted as $\tilde{a} = (a_1, a_2, a_3, a_4)_{L\tilde{a}-R\tilde{a}}$ and a α -cut of \tilde{a} is defined as

$$\tilde{a}(\alpha) = (\tilde{a}_L(\alpha), \tilde{a}_R(\alpha)) = (a_2 - (a_2 - a_1)a_3L_{\tilde{a}}^{-1}(\alpha), a_3 - (a_4 - a_3)a_3R_{\tilde{a}}^{-1}(\alpha)).$$

Tran and Duckstein [62] define the distance between two *GLRFN* fuzzy numbers \tilde{a} and \tilde{b} as

$$D^2(\tilde{a}, \tilde{b}, f) = \int_0^1 \left\{ \left[\frac{\tilde{a}_L(\alpha) + \tilde{a}_R(\alpha)}{2} - \frac{\tilde{b}_L(\alpha) + \tilde{b}_R(\alpha)}{2} \right]^2 + \frac{1}{3} \left[\left(\frac{\tilde{a}_L(\alpha) + \tilde{a}_R(\alpha)}{2} \right)^2 + \left(\frac{\tilde{b}_L(\alpha) + \tilde{b}_R(\alpha)}{2} \right)^2 \right] \right\} \times f(\alpha) d\alpha / \int_0^1 f(\alpha) d\alpha.$$

The function $f(\alpha)$, which serves as a weight function, is positive continuous in $[0,1]$, the distance being computed as a weighted sum of distances between two intervals across all the α -cuts from 0 to 1. DM participation is flexible thanks to the presence of function f . For example, when the DM is risk-neutral, $f(\alpha) = \alpha$ seems to be reasonable. A risk-averse DM might want to attach more weight to information at a higher α level by using other functions, such as $f(\alpha) = \alpha^2$ or a higher power of α . A constant ($f(\alpha) = 1$), or even a decreasing function f , can be utilized for a risk-prone DM.

For the particular case of the distance of a trapezoidal fuzzy number $\tilde{a} = (a_1, a_2, a_3, a_4)$ to a constant (specifically 0), we have

1. If $f(\alpha) = \alpha$, then

$$D^2(\tilde{a}, 0, f) = \left(\frac{a_2 + a_3}{2} \right)^2 + \frac{1}{3} \left(\frac{a_2 + a_3}{2} \right) [(a_4 - a_3) - (a_2 - a_1)] + \frac{2}{3} \left(\frac{a_3 - a_2}{2} \right)^2 + \frac{1}{9} \left(\frac{a_3 - a_2}{2} \right) [(a_4 - a_3) + (a_2 - a_1)]$$

$$+ \frac{1}{18}[(a_4 - a_3)^2 + (a_2 - a_1)^2] - \frac{1}{18}[(a_2 - a_1)(a_4 - a_3)].$$

2. If $f(\alpha) = 1$, then

$$D^2(\tilde{a}, 0, f) = \left(\frac{a_2 + a_3}{2}\right)^2 + \frac{1}{2}\left(\frac{a_2 + a_3}{2}\right)[(a_4 - a_3) - (a_2 - a_1)] + \frac{1}{3}\left(\frac{a_3 - a_2}{2}\right) + \frac{1}{6}\left(\frac{a_3 - a_2}{2}\right)[(a_4 - a_3) + (a_2 - a_1)] + \frac{1}{9}[(a_4 - a_3)^2 + (a_2 - a_1)^2] - \frac{1}{9}[(a_2 - a_1)(a_4 - a_3)].$$

3. If $f(\alpha) = \alpha^2$, then

$$D^2(\tilde{a}, 0, f) = \left(\frac{a_2 + a_3}{2}\right)^2 + \frac{1}{4}\left(\frac{a_2 + a_3}{2}\right)[(a_4 - a_3) - (a_2 - a_1)] + \left(\frac{a_3 - a_2}{2}\right)^2 + \frac{1}{12}\left(\frac{a_3 - a_2}{2}\right)[(a_4 - a_3) + (a_2 - a_1)] - \frac{1}{144}[(a_4 - a_3)^2 + (a_2 - a_1)^2] + \frac{1}{96}[(a_2 - a_1)(a_4 - a_3)].$$

Let us now introduce the new dominance-measuring method. We now consider that imprecise weights are represented by trapezoidal fuzzy numbers \tilde{w}_j , $j = 1, \dots, n$. Different direct and indirect methods for constructing fuzzy sets have been proposed for one and multiple experts in the literature [18]. In [9,10], the advantages, disadvantages and limitations of four elicitation methods are discussed, whereas [63] reviews three basic views of the representation of membership functions, together with fundamental measurements of linguistic terms of linguistic variables.

Then, the optimization problem (1) can be now represented by

$$\begin{aligned} \tilde{D}_{kl} &= \min \left\{ \sum_{j=1}^n \tilde{w}_j(\mathbf{u}_k - \mathbf{u}_l) = \sum_{j=1}^n (w_{j1}, w_{j2}, w_{j3}, w_{j4})(\mathbf{u}_k - \mathbf{u}_l) \right. \\ &= (D_{kl1}, D_{kl2}, D_{kl3}, D_{kl4}) \} \\ \text{s.t. } \mathbf{u}_k &\in U_k, \mathbf{u}_l \in U_l \\ \tilde{\mathbf{w}} &\in W, \end{aligned} \quad (2)$$

where $\tilde{\mathbf{w}} \in W = \{\tilde{\mathbf{w}} = (\tilde{w}_1, \dots, \tilde{w}_n) : \tilde{w}_j, j = 1, \dots, n\}$, being \tilde{w}_j trapezoidal fuzzy numbers.

Then, the first step in the proposed method is to compute the above trapezoidal fuzzy numbers, see Fig. 1. Consequently, each

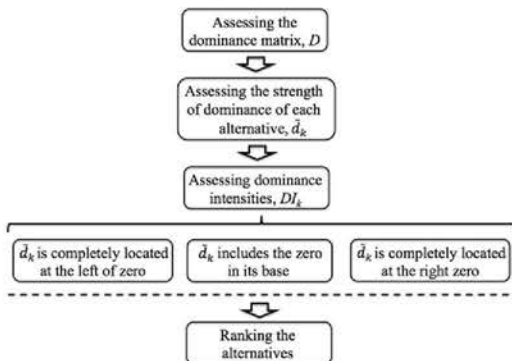


Fig. 1. Flowchart of the proposed method.

element in the dominance matrix is a trapezoidal fuzzy number:

$$D = \begin{pmatrix} - & \tilde{D}_{12} & \dots & \tilde{D}_{1m-1} & \tilde{D}_{1m} \\ \tilde{D}_{21} & - & \dots & \tilde{D}_{2m-1} & \tilde{D}_{2m} \\ \tilde{D}_{31} & \tilde{D}_{32} & - & \tilde{D}_{3m-1} & \tilde{D}_{3m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{D}_{m1} & \tilde{D}_{m2} & \dots & \tilde{D}_{mm-1} & - \end{pmatrix}.$$

Next, we compute the strength of dominance of alternative A_k by adding the trapezoidal fuzzy numbers in the k th row of D

$$\tilde{d}_k = (d_{k1}, d_{k2}, d_{k3}, d_{k4}) = \sum_{l=1}^m \tilde{D}_{kl} = \left(\sum_{l=1}^m D_{kl1}, \sum_{l=1}^m D_{kl2}, \sum_{l=1}^m D_{kl3}, \sum_{l=1}^m D_{kl4} \right).$$

Finally, a *dominance intensity*, DI_k , for each alternative A_k is computed as the proportion of the positive part of the fuzzy number \tilde{d}_k by the distance of the fuzzy number to zero. Specifically, the dominance intensity for alternative A_k is computed according to the location of \tilde{d}_k as follows:

1. If \tilde{d}_k is completely located at the left of zero, then DI_k is minus the distance of \tilde{d}_k to zero, because there is no positive part in \tilde{d}_k .
2. If \tilde{d}_k is completely located at the right of zero, then DI_k is the distance of \tilde{d}_k to zero, because there is no negative part existing in \tilde{d}_k .
3. If \tilde{d}_k includes the zero in its base, then the fuzzy number will have a part on the right of zero that we denote \tilde{d}_k^R and another part on the left of zero that we denote \tilde{d}_k^L . DI_k is the proportion that represents \tilde{d}_k^R with respect to \tilde{d}_k by the distance of \tilde{d}_k to zero less the proportion that represents \tilde{d}_k^L with respect to \tilde{d}_k by the distance of \tilde{d}_k to zero.

Next, we analyze each one of these cases in more detail:

- If $d_{k4} < 0$, see Fig. 2(a), then the dominance intensity of alternative A_k is defined as $DI_k = -D(\tilde{d}_k, 0, f)$.
- If $d_{k1} > 0$, see Fig. 2(b), then the dominance intensity of alternative A_k is defined as $DI_k = D(\tilde{d}_k, 0, f)$.
- If $d_{k1} < 0$ and $d_{k2} > 0$, see Fig. 2(c), the corresponding trapezoidal fuzzy number is divided by the vertical axis (at zero) into two parts. The left part \tilde{d}_k^L represents the proportion:

$$\frac{-d_{k1} \frac{-d_{k1}}{d_{k2} - d_{k1}}}{\frac{-d_{k1} \frac{-d_{k1}}{d_{k2} - d_{k1}}}{2} + \frac{(d_{k1})^2}{(d_{k4} + d_{k3} - d_{k2} - d_{k1})(d_{k2} - d_{k1})}} = \frac{(d_{k1})^2}{(d_{k4} + d_{k3} - d_{k2} - d_{k1})(d_{k2} - d_{k1})},$$

whereas the right part \tilde{d}_k^R represents the proportion

$$\frac{d_{k4} + d_{k3} - d_{k2} - d_{k1} - \frac{(-d_{k1})(-d_{k1})}{2(d_{k2} - d_{k1})}}{\frac{d_{k4} + d_{k3} - d_{k2} - d_{k1}}{2}} = \frac{d_{k2}(-d_{k2} + d_{k3} + d_{k4}) - d_{k1}(d_{k3} + d_{k4})}{(d_{k2} - d_{k1})(d_{k4} + d_{k3} - d_{k2} - d_{k1})}.$$

The dominance intensity of alternative A_k is defined as

$$DI_k = \frac{d_{k2}(-d_{k2} + d_{k3} + d_{k4}) - d_{k1}(d_{k3} + d_{k4})}{(d_{k2} - d_{k1})(d_{k4} + d_{k3} - d_{k2} - d_{k1})} D(\tilde{d}_k, 0, f) - \frac{(d_{k1})^2}{(d_{k4} + d_{k3} - d_{k2} - d_{k1})(d_{k2} - d_{k1})} D(\tilde{d}_k, 0, f).$$

- If $d_{k3} < 0$ and $d_{k4} > 0$, Fig. 2(d), the corresponding trapezoidal fuzzy number is again divided by the vertical axis into two

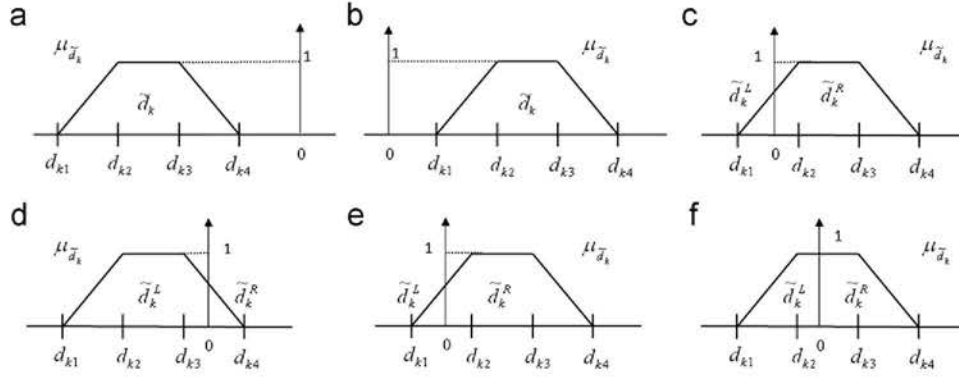


Fig. 2. Possible locations of \tilde{d}_k .

parts, \tilde{d}_k^L and \tilde{d}_k^R , represented by the proportions

$$\frac{d_{k4} + d_{k3} - d_{k2} - d_{k1}}{2} \frac{d_{k4}}{d_{k4} - d_{k3}} - \frac{d_{k4}}{2} \frac{d_{k4}}{d_{k4} - d_{k3}}$$

$$= \frac{d_{k4}(d_{k4} - d_{k2} - d_{k1}) - d_{k3}(d_{k3} - d_{k2} - d_{k1})}{(d_{k4} + d_{k3} - d_{k2} - d_{k1})(d_{k4} - d_{k3})}$$

and

$$\frac{\frac{d_{k4}}{2} \frac{d_{k4}}{d_{k4} - d_{k3}}}{d_{k4} + d_{k3} - d_{k2} - d_{k1}} = \frac{(d_{k4})^2}{(d_{k4} - d_{k3})(d_{k4} + d_{k3} - d_{k2} - d_{k1})},$$

respectively, and dominance intensity of alternative A_k is

$$DI_k = \frac{(d_{k4})^2}{(d_{k4} - d_{k3})(d_{k4} + d_{k3} - d_{k2} - d_{k1})} D(\tilde{d}_k, 0, f) - \frac{d_{k4}(d_{k4} - d_{k2} - d_{k1}) - d_{k3}(d_{k3} - d_{k2} - d_{k1})}{(d_{k4} + d_{k3} - d_{k2} - d_{k1})(d_{k4} - d_{k3})} D(\tilde{d}_k, 0, f).$$

- If $d_{k2} < 0$ and $d_{k3} > 0$, see Fig. 2(e), \tilde{d}_k^L and \tilde{d}_k^R are

$$\frac{\frac{d_{k4} - d_{k3}}{2} + d_{k3}}{d_{k4} + d_{k3} - d_{k2} - d_{k1}} = \frac{d_{k4} + d_{k3}}{d_{k4} + d_{k3} - d_{k2} - d_{k1}},$$

and

$$\frac{\frac{d_{k2} - d_{k1} - d_{k2}}{2}}{d_{k4} + d_{k3} - d_{k2} - d_{k1}} = \frac{-d_{k2} - d_{k1}}{d_{k4} + d_{k3} - d_{k2} - d_{k1}},$$

respectively, and the dominance intensity of alternative A_k is

$$DI_k = \frac{-d_{k2} - d_{k1}}{d_{k4} + d_{k3} - d_{k2} - d_{k1}} D(\tilde{d}_k, 0, f) - \frac{d_{k4} + d_{k3}}{d_{k4} + d_{k3} - d_{k2} - d_{k1}} D(\tilde{d}_k, 0, f).$$

Once the dominance intensity has been computed for each alternative A_k , DI_k , the alternatives are ranked accordingly, where the best (rank 1) is the alternative with greatest DI_k and the worst is the alternative with the least DI_k .

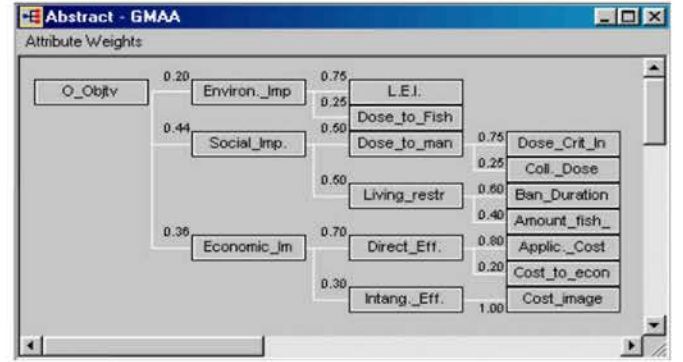


Fig. 3. Objective hierarchy for lake Svyatoye.

4. Application to the restoration of aquatic ecosystems

In this section we illustrate the application of the proposed method to the evaluation of intervention strategies for restoring a radionuclide contaminated aquatic ecosystem, lake Svyatoye, which was heavily contaminated after the Chernobyl accident [51]. This is a complex decision-making problem simultaneously accounting for environmental, social and economic impact.

Fig. 3 shows the objective hierarchy representing all the relevant aspects to be considered in the evaluation of the intervention strategies.

Environmental Impact (Environ_imp) is one of the main objectives of the decision analysis. It was divided into *Lake Ecosystem Index* (LEI), a simple and rational approach for measuring the ecological status of a lake and *Radiation Dose to Biota* (Dose_to_Fish). *Social Impact* (Social_imp.) was handled by two sub-objectives: *Minimizing Impact on Health* (Dose_to_Man) and *Living Restrictions* (Living_restr).

Regarding dose to man, we focused on the *Dose to Critical Individuals* (Dose Crit In), who should never receive radiation levels above thresholds for early health effects, and *Collective Dose* (Coll_Dose), which was linearly related to the increase in the risk of developing serious latent effects, mainly cancers. As regards living restrictions, other impacts were taken into consideration. These include countermeasures affecting the direct consumption of fish for food or its processing in the food industry, drinking water and water used by the food industry, the use of water for crops irrigation and the recreational uses of water bodies. For all these objectives, the attributes were the *Amount of Fish Affected* by restrictions (Amount_fish), as well as the duration of such restrictions (Ban_Duration).

Finally, *Economic Impact* (Economic_lm) was divided into *Direct Effects* (Direct_Eff.), more amenable to quantification, and

Intangible Effects (Intang_Eff.), like loss-of-image and adverse market reactions for the concerned area, which could also be subjectively rated by the user. The direct effects include the costs generated by the different bans on or restrictions to normal living conditions, which can be sub-divided into *Costs to the Economy* (Cost_to_econ) and the more subjective costs of lost recreation, and *Application Costs* (Applic_Cost), i.e., costs of chemical and physical remedial countermeasures.

Alternative performances were described under uncertainty by uniformly distributed intervals. On the other hand, imprecise component utilities were assessed, leading to classes of utility functions. Fig. 4 shows the corresponding weights, i.e., the indicators of the influence of the individual decision-making criteria. For more information about the problem, component utility and weight elicitation and the resulting ranking of alternatives, see [51].

The intervention strategies considered for evaluation are shown in Table 1, including the combination of chemical countermeasures with fishing bans so as to reduce the radiological and environmental impact.

The original imprecise weights elicited by DMs are shown in Fig. 4. The possibility of using trapezoidal fuzzy numbers rather than intervals was not considered at that time. However, DMs agreed at later meetings that trapezoidal fuzzy numbers would better reflect their preferences. As a starting point, weight intervals were transformed into trapezoidal fuzzy numbers, $\tilde{w}_j = (w_{j1}, w_{j2}, w_{j3}, w_{j4})$, as follows. Given an imprecise weight $[w_j^l, w_j^u]$, the base of the trapezoid is an interval such as $w_{j1} = w_j^l$ and $w_{j4} = w_j^u$, whereas the length of the lower base is a third part of the higher base, i.e., $w_{j2} = w_j^l + (w_j^u - w_j^l)/3$ and $w_{j3} = w_j^u - (w_j^u - w_j^l)/3$.

As performance intervals and classes of component utility functions are considered, the resolution of optimization problem (2) is equivalent to solving [42]:

$$\bar{D}_{kl} = u(A_k) - u(A_l) = \tilde{\mathbf{w}}^T \mathbf{z}_{kl}^*$$

where $\tilde{\mathbf{w}} \in W$, $\mathbf{z}_{kl}^* = (z_{kl1}^*, \dots, z_{kln}^*)$ and z_{klj}^* is the optimum value of the following optimization problem:

$$\begin{aligned} \min \quad & z_{klj} = u_j(x_{kj}) - u_j(x_{lj}) \\ \text{s.t.} \quad & x_{kj}^l \leq x_{kj} \leq x_{kj}^u, \quad j = 1, \dots, n \\ & x_{lj}^l \leq x_{lj} \leq x_{lj}^u, \quad j = 1, \dots, n \\ & u_j^l(x_{kj}) \leq u_j(x_{kj}) \leq u_j^u(x_{kj}), \quad j = 1, \dots, n \\ & u_j^l(x_{lj}) \leq u_j(x_{lj}) \leq u_j^u(x_{lj}), \quad j = 1, \dots, n. \end{aligned} \quad (3)$$

The optimal solution of problem (3) can be determined very simply for certain types of utility functions [42]. Specifically, if the utility function is monotonically increasing or decreasing, then $z_{klj}^* = u_j^l(x_{kj}^l) - u_j^u(x_{lj}^u)$ or $z_{klj}^* = u_j^u(x_{kj}^u) - u_j^l(x_{lj}^l)$, respectively.

The respective optimization problems were solved to compute the dominance matrix, and the strengths of dominance for each alternative are

$$\tilde{d}_1 = (-0.1438, 0.1334, 0.4118, 0.7174),$$

$$\tilde{d}_2 = (0.7752, 1.0954, 1.4157, 1.7712),$$

$$\tilde{d}_3 = (0.2493, 0.4764, 0.7035, 0.9730),$$

$$\tilde{d}_4 = (-0.1579, 0.3522, 0.8623, 1.3958),$$

$$\tilde{d}_5 = (0.2725, 0.5441, 0.8158, 1.1096),$$

$$\tilde{d}_6 = (-0.7043, -0.3217, 0.0609, 0.4724),$$

$$\tilde{d}_7 = (0.4777, 0.7281, 0.9782, 1.2489).$$

Finally, dominance intensities, Dl_k , are computed for each alternative A_k for the case of a risk-prone, neutral and risk-averse DM to obtain the corresponding ranking of alternatives, see Table 2. Table 2 also shows the ranking for the original problem in [51].

The three alternative rankings match the original ranking except for alternatives A_5 (*Lake liming*) and A_7 (*Wetland liming*), which switch positions in the ranking (second and third). This switch can be justified taking into account that the average overall utilities for both intervention strategies on which the ranking is based, are almost the same. Moreover, the upper overall utility for A_7 is greater than for A_5 [51].

To illustrate the assessment of dominance intensities listed in Table 2, we consider the case of the risk-prone DM and A_1 (*No action*) alternative. As $d_{11} = -0.1438 < 0$ and $d_{12} = 0.1334 > 0$, then

$$\tilde{d}_1^l = \frac{(d_{11})^2}{(d_{14} + d_{13} - d_{12} - d_{11})(d_{k2} - d_{k1})} = 0.06534,$$

$$\tilde{d}_1^R = \frac{d_{12}(-d_{12} + d_{13} + d_{14}) - d_{11}(d_{13} + d_{14})}{(d_{12} - d_{11})(d_{14} + d_{13} - d_{12} - d_{11})} = 0.9346,$$

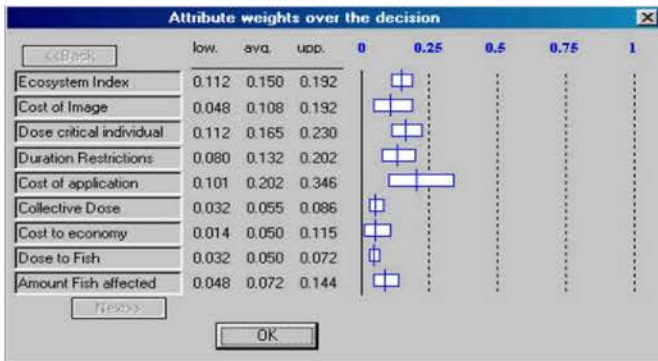


Fig. 4. Imprecise attribute weights.

Table 1
Intervention strategies.

Intervention strategies	Description
A_1 : No action	Natural evolution of the situation without intervention
A_2 : Potash	15 tonnes of potassium chloride added to the lake in April 1987
A_3 : Fertilizer	800 kg of fertilizer added to the lake between April and July 1987, 1988, 1989, 1990, i.e., 200 kg/month
A_4 : Food ban	Automatic fish consumption ban when ^{137}Cs content in fish is greater than 1000 Bq/kg
A_5 : Lake liming	15 tonnes of lime added to the lake in April 1987
A_6 : Sediment removal	250,000 m ² of sediments removed from the lake down to a depth of 10 cm from May to June 1990
A_7 : Wetland liming	30 ton of lime added to the catchment in May 1987

Table 2
Dominance intensities and ranking of alternatives.

Intervention strategies	Original ranking	Risk prone		Neutral		Risk averse	
		DI_k	Ranking	DI_k	Ranking	DI_k	Ranking
A_1	6	0.2853	6	0.2798	6	0.2850	6
A_2	1	1.2798	1	1.2754	1	1.2763	1
A_3	5	0.6174	5	0.6122	5	0.6129	5
A_4	4	0.6546	4	0.6472	4	0.6571	4
A_5	2	0.7055	3	0.7019	3	0.7041	3
A_6	7	0.0837	7	0.0811	7	0.0864	7
A_7	3	0.8719	2	0.8689	2	0.8702	2

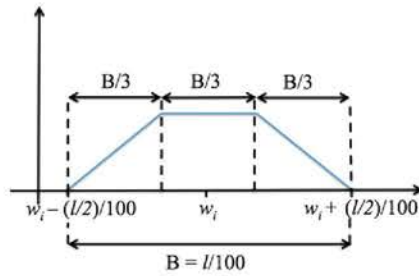


Fig. 5. Trapezoidal fuzzy number construction.

and

$$DI_1 = \tilde{d}_1^R \times D(\tilde{d}_1, 0, f) - \tilde{d}_1^L \times D(\tilde{d}_1, 0, f) = 0.9346 \times 0.1077 - 0.06534 \times 0.1077 = 0.2853.$$

5. Performance analysis based on monte carlo simulation techniques

The performance of the proposed dominance measuring method accounting for fuzzy weights has been analyzed on the basis of Monte Carlo simulation techniques using two measures of efficacy: hit ratio and rank-order correlation [2,39]. The *hit ratio* is the proportion of all cases in which the method selects the same best alternative as in the TRUE ranking. *Rank-order correlation* represents how similar the overall structures ranking alternatives are in the TRUE ranking and in the ranking derived from the proposed method. It is calculated using Kendall's τ [65]: $\tau = 1 - 2 \times (\text{number of pairwise preference violations}) / (\text{total number of pair preferences})$.

Four different levels of alternatives ($m = 3, 5, 7, 10$) and five different levels of attributes ($n = 3, 5, 7, 10, 15$) were considered in order to validate the results. Also, 20 000 trials were performed for each of the 20 design elements. For simulation purpose, different routines were implemented in Python v3.1.3, which provides the *Mersenne Twister method* pseudo-random number generator. It is one of the most extensively tested random number generators in existence and produces 53-bit precision floats and has a period of $2^{19937} - 1$.

First, component utilities for each alternative in each attribute are randomly generated from a uniform distribution in (0,1), leading to an $m \times n$ matrix. The columns in this matrix are normalized to make the smallest value 0 and the largest 1, and dominated alternatives are removed.

Next, attribute weights representing the relative attribute importance are generated. These weights are the *TRUE weights* and the derived ranking of alternatives will be denoted as the *TRUE ranking*. Those weights are then transformed into trapezoidal fuzzy numbers

Table 3
Measures of efficacy using 5% imprecision level.

Alternatives	Attributes	Hit ratio			Kendall's τ		
		Risk prone	Neutral	Risk averse	Risk prone	Neutral	Risk averse
3	3	99.67	99.67	99.68	99.60	99.61	99.61
3	5	99.19	99.22	99.25	99.02	99.06	99.07
3	7	98.39	98.40	98.43	98.05	98.07	98.10
3	10	96.34	96.35	96.37	95.92	95.96	95.97
3	15	90.69	90.70	90.76	89.65	89.70	89.73
5	3	99.19	99.19	99.21	99.20	99.21	99.21
5	5	98.80	98.84	98.85	98.48	98.51	98.51
5	7	98.33	98.39	98.38	97.22	97.25	97.25
5	10	97.44	97.53	97.59	94.03	94.07	94.07
5	15	95.68	95.84	95.91	83.83	83.90	83.92
7	3	98.98	98.98	98.99	99.14	99.15	99.15
7	5	98.39	98.43	98.42	98.40	98.41	98.41
7	7	97.74	97.82	97.83	97.10	97.13	97.13
7	10	96.72	96.83	96.87	93.84	93.90	93.91
7	15	95.30	95.68	95.75	83.36	83.44	83.46
10	3	98.47	98.47	98.47	99.11	99.12	99.12
10	5	97.82	97.85	97.87	98.27	98.29	98.30
10	7	96.99	97.07	97.12	96.98	97.01	97.01
10	10	95.95	96.15	96.16	93.58	93.64	93.65
10	15	94.52	94.84	94.95	82.57	82.63	82.64



Fig. 6. Hit ratio for 5%, 10%, 15% and 20% imprecision levels.

taking into account different imprecision levels (5%,10%,15% and 20%), as shown in Fig. 5. Given a precise weight w_i and a imprecision level $\%$, the higher base of the trapezoid is $[w_i - (l/2)/100,$

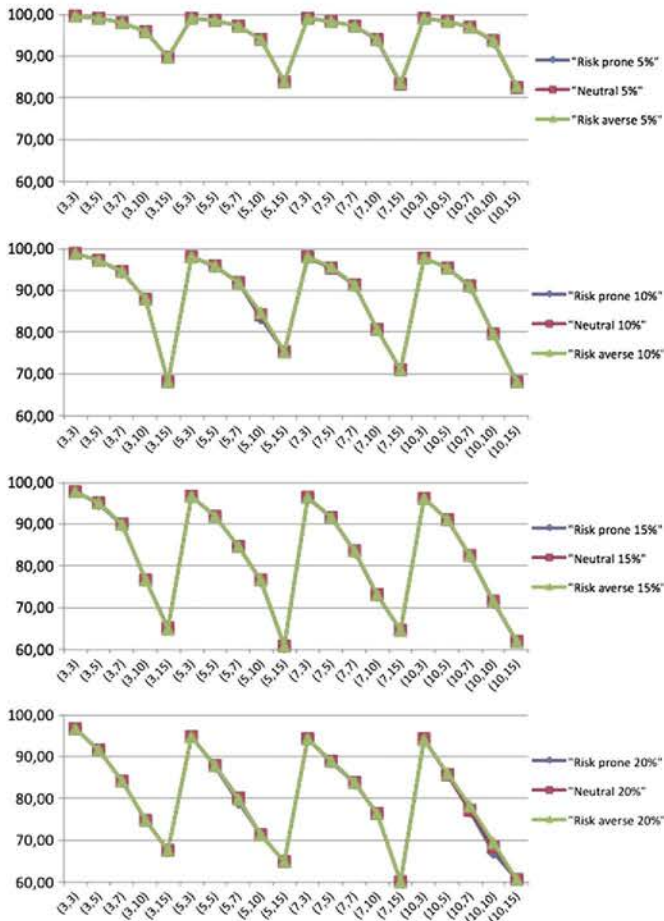


Fig. 7. Kendall's τ for 5%, 10%, 15% and 20% imprecision levels.

Table 4
Average hit ratio and Kendall's τ values.

Imprecision level (%)	Hit ratio (%)	Kendall's τ (%)
5	97.32	94.91
10	93.11	87.98
15	87.85	82.35
20	84.37	80.17

$w_i + (l/2)/100$], whereas the length of the lower base is a third of the length of the higher base, both centered at w_i .

Finally, the ranking of alternatives is computed and compared with the *TRUE ranking*. Table 3 exhibits the average hit ratio and Kendall's τ for each of the 20 design elements with a 5% imprecision level for the case of a risk-prone, neutral and risk-averse DM.

We can conclude that the hit ratio and the correlation coefficient are very similar for all three cases. The hit ratio is greater than 90.69% for all the design elements, whereas Kendall's τ is greater than 82.57%; and both measures are decreasing regarding the number of attributes and the number of alternatives.

Figs. 6 and 7 illustrate the hit ratio and Kendall's τ , respectively, for 5%, 10%, 15% and 20% imprecision levels for a risk-prone, neutral and risk-averse DM. The hit ratio and the Kendall's τ are again very similar for all three cases (there is a large overlap in all cases). Both measures are decreasing regarding the number of attributes, and this decrease is more pronounced the greater the imprecision is.

The hit ratio variability (the corresponding range for a given number of alternatives) is decreasing regarding the number of

alternatives for 5%, 10% and 15% imprecision levels. On the other hand, Kendall's τ tends to decrease with the number of alternatives. Note that when the number of alternatives is low, then each wrongly ranked alternative leads to a steep drop in Kendall's τ . On the other hand, when the number of alternatives is high, then several alternatives would have to be wrongly ranked to have a big impact on Kendall's τ .

Finally, Table 4 shows the average hit ratio and Kendall's τ values for the imprecision levels and a neutral DM. Both measures decrease against the level of imprecision, from 97.32% to 84.37% for the hit ratio and from 94.91% to 80.17% for Kendall's τ . We consider that values for a 20% imprecision level are rather good taking into account that this is quite high imprecision. Moreover, these outcomes predictably outperforms the results output in [40], since imprecision concerning weights is represented by weight intervals rather than fuzzy numbers in [40], i.e., less meaningful information about weights is available. On the other hand, if triangular rather than trapezoidal fuzzy weights are considered regarding weights [27] then hit ratio and Kendall's τ values improve with greater imprecision.

6. Conclusions

Dominance-measuring methods are becoming widely used in a decision-making context with incomplete information within MAUT and have been proved to outperform other approaches, like most surrogate weighting methods or the modification of classical decision rules for application in an imprecise decision context.

In this paper, a new dominance-measuring method has been proposed, in which weights representing the relative importance of decision-making criteria are described by means of trapezoidal fuzzy numbers. The method is based on pairwise dominances and on the distances between fuzzy numbers, using the generalization of the left and right fuzzy numbers. Moreover, the cases of risk-prone, neutral and risk-averse DMs have been considered and analyzed. Monte Carlo simulation techniques have been applied to analyze the performance of the proposed method on the basis of two measures of efficacy for different imprecision levels. Hit ratio and Kendall's τ values are very similar for the different types of DMs throughout the considered imprecision levels. The method performs well, the hit ratio and Kendall's τ values being higher than 84% and 80%, respectively, in the worst case (20% imprecision). As expected, the results are better than for similar studies accounting for weight intervals, whereas they are worse than when triangular rather than trapezoidal fuzzy weights are used, since triangular fuzzy weights provide more meaningful information about the weights.

In future research we proposed to look at other ways of representing imprecision in the problem. For instance, fuzzy numbers could be also used to represent alternative performances. Moreover, the DM could just provide an alternatives ranking for each criterion under consideration rather than performances and their utilities (assessment might be involved in real applications).

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