A FEM-BEM coupling procedure through the Steklov-Poincarè operator

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ABSTRACT

Many advantages can be got in combining finite and boundary elements. It is the case, for example, of unbounded field problems where boundary elements can provide the appropriate conditions to represent the infinite domain while finite elements are suitable for more complex properties in the near domain.

However, in spite of it, other disadvantages can appear. It would be, for instance, the loss of symmetry in the finite elements stiffness matrix, when the combination is made.

On the other hand, in our days, with the strong irruption of the parallel proccessing the techniques of decomposition of domains are getting the interest of numerous scientists. With their application it is possible to separate the resolution of a problem into several subproblems. That would be beneficial in the combinations BEM-FEM as the loss of symmetry would be avoided and every technique would be applicated separately.

Evidently, for the correct application of these techniques it is necessary to establish the suitable transmission conditions in the interface between BEM domain and FEM domain.
In this paper, one parallel method is presented which is based in the interface operator of Steklov Poincaré.

INTRODUCTION

As it is well known, there are many types of complicated boundary and initial value engineering problems which can be solved applying the Finite Element Method (FEM). However, important difficulties can occur when attempting to solve an example which extends over an infinite domain.

On the other hand the Boundary Element Method (BEM) has been successful in treating problems extending over an infinite domain. However, in other type of situations, such as material non-linearities it is not so suitable.

With a convenient combination of both methods (BEM-FEM) numerous advantages can appear. For instance, in many unbounded problems the boundary elements can provide the suitable conditions to represent the infinite domain while the finite elements are more appropriate in the zones closer to load concentration where non linear behavior could be expected.

That is the typical situation of fluid-structure or soil-structure interaction. The structure must be analyzed carefully for design and then it is interesting to get a detailed information over all the domain. On the other hand, the rest is only interesting, in general, on its interface; its extension and homogeneity recommend the use of boundary elements.

Two techniques have been used mainly to do this combination [1]:

(i) Treatment of the discretized region with boundary elements as a finite superelement and combination with the FEM.

(ii) Treatment of the FEM region as an equivalent boundary element and combination with the BEM region.

The first technique is very interesting as it allows to solve the problem like one of the finite elements with the inconvenient of the loss of symmetry of the stiffness matrix.

Also important in recent years, due to the irruption of the parallel processing in the computation, is the development of numerical algorithms oriented to this type of resolution. The domain decomposition method for the numerical solution of differential boundary-value-problems is a demonstration of this evolution. This method is based on the partition of the computational domain Ω into subdomains of reduced size. Then the original differential problem is reformulated upon each subdomain, yielding a family of almost independent subproblems of lower computational complexity.

Obviously, from the physical point of view, consistency of the subdomain problem with the original one is ensured by enforcing suitable transmission of information between adjacent subregions using proper interface operators. The problem will be solved separately in several independent subproblems iteratively until the convergence.

Reorientating the domain decomposition method to the BEM-FEM coupling problem, numerous advantages can be obtained. It would allow, for instance, to calculate the FEM region and the BEM region separately omitting disadvantages such as the loss of symmetry of the stiffness matrix resulting of the coupling.

INTERFACE CONDITIONS

The BEM-FEM coupling will always be, at first, possible by the application of the proper interface conditions between BEM region and FEM region.

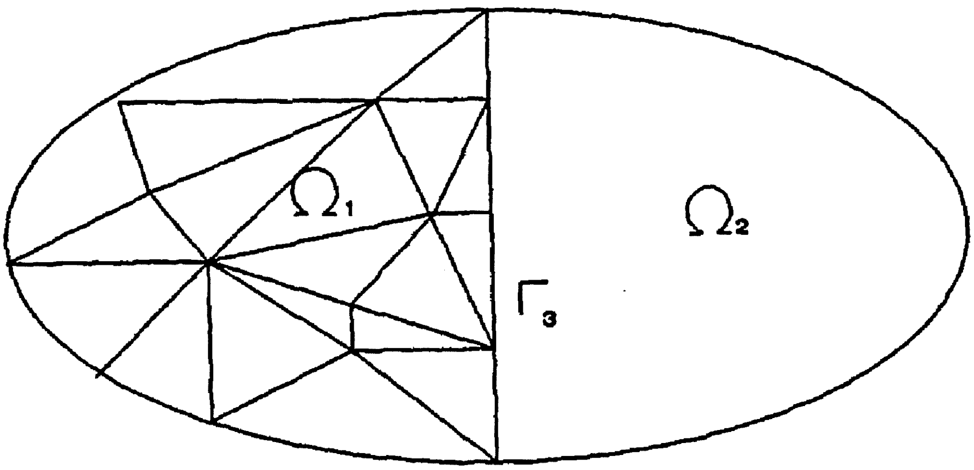


Figure 1. Division of Ω into two subdomains

We consider a domain Ω like the one represented in figure (1) divided in two subregions Ω_1 and Ω_2 separated by the interface Γ_3 . The first of them Ω_1 discretized according to the FEM and the second Ω_2 according to the BEM. Designating as u_3^i, p_3^i the potential and flux respectively over the interface Γ_3 for the region i (i=1,2 in a bidimensional problem) the interface restrictions will correspond to the potential continuity (i) and the flux equal and opposite across the interface:

$$
\begin{aligned}
&\text{(i) Compatibility:} \qquad && u_3^1 = u_3^2 \\
&\text{(ii) Equilibrium:} \qquad && p_3^1 = p_3^2
\end{aligned}
\qquad (1)
$$

The consideration of the compatibility condition (Equation 1(i)) to solve a parallel problem is immediate by assuming the same Dirichlet condition λ in the interface between both subdomains remaining the rest of boundary conditions invariable:

$$\begin{array}{lll} Lu_i=f & in & \Omega_i \\ u_i=\bar{u} & in & \partial\Omega\cap\partial\Omega_i \\ u_3^i=\lambda & in & \Gamma_3 \end{array} \quad (2)$$

where L is a partial differential operator and f is a given datum.

To introduce the equilibrium condition (Equation 1(ii)) in the interface we will use the Steklov-Poincarè operator (also called Schur complement after the discretization).

According to it, supposing a region such as Ω_1 discretized with the FEM, the consideration of variables separately in Ω_1 and in the interface Γ_3 drives to the resolution of an equations system:

$$\begin{pmatrix} K_{11F} & K_{13F} \\ K_{13F}^t & K_{33F} \end{pmatrix} \begin{pmatrix} u_1 \\ u_3^1 \end{pmatrix} = \begin{pmatrix} b_{1F} \\ b_{3F} \end{pmatrix} \quad (3)$$

or, posing the problem in the interface Γ_3:

$$[K_{33F}-K_{13F}^tK_{11F}^{-1}K_{13F}]\,u_3^1=b_{3F}-K_{13F}^tK_{11F}^{-1}b_{1F} \quad (4)$$

where

$$S_1=K_{33F}-K_{13F}^tK_{11F}^{-1}K_{13F} \quad (5)$$

is the Schur complement [7] referred to the subdomain Ω_1, which as it can be observed in equation (4) represents the nodal loads in the interface Γ_3 for a Laplace problem with homogenous Dirichlet conditions in the external boundary.

Considerating a Laplace problem, of equation (4) is obtained that:

$$b_{3F}=S_1u_3^1+K_{13F}^tK_{11F}^{-1}b_{1F} \quad (6)$$

represents the nodal loads for the interface Γ_3 belonging to the subregion Ω_1.

Posing the problem in the BEM subregion Ω_2 considered like an equivalent finite element [1], a similar expression to FEM is obtained:

$$\begin{pmatrix} K_{22c} & K_{23c} \\ K_{32c} & K_{33c} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3^2 \end{pmatrix} = \begin{pmatrix} b_{2c} \\ b_{3c} \end{pmatrix} \tag{7}$$

where $K_{23}^c <> K_{32}^c$.

Writing the problem in the interface Γ_3 will remain:

$$[K_{33c} - K_{32c} K_{22c}^{-1} K_{23c}] = b_{3c} - K_{32c} K_{22c}^{-1} b_{2c} \tag{8}$$

where

$$S_2 = K_{33c} - K_{32c} K_{22c}^{-1} K_{23c} \tag{9}$$

is the Schur complement for the region Ω_2 and like in the FEM

$$b_{3c} = S_2 u_3^2 + K_{32c} K_{22c}^{-1} b_{2c} \tag{10}$$

represents the nodal loads for the interface Γ_3 belonging to the subregion Ω_2 in a Laplace problem.

Considering equations (6) and (10), the equilibrium condition (equation 1(ii)) can be expressed as the resolution of the problem

$$S_1 u_3^1 - B_1 + S_2 u_3^2 - B_2 = 0 \tag{11}$$

where

$$\begin{aligned} B_1 &= -K_{13F}^t K_{11F}^{-1} b_{1F} \\ B_2 &= -K_{32c} K_{22c}^{-1} b_{2c} \end{aligned} \tag{12}$$

which is equivalent to solve separately two Neumann problems in the interface Γ_3.

As in the limit $u_3^1 = u_3^1$ is verified, equation (11) could be solved by an iterative refinement procedure [4] where the solution is obtained according to the expression:

$$u_3^{n+1} = u_3^n + \rho d^n \tag{13}$$

being ρ a relaxation parameter and d^n the correction to be added to u_3^n which is determined by

$$d^n = A^{-1} r^n \tag{14}$$

where

$$A=\frac{(S_1^{-1}+S_2^{-1})}{2} \tag{15}$$

and the residual

$$r^n=\frac{(B_1-S_1u_3^1)+(B_2-S_2u_3^2)}{2} \tag{16}$$

In this way, if equation (13) is expressed for a Laplace problem with homogenous Dirichlet conditions in the external boundary, will be obtained:

$$u_3^{n+1}=u_3^n-\frac{\rho}{4}(u_3^1+S_2^{-1}S_1u_3^1+u_3^2+S_1^{-1}S_2u_3^2) \tag{17}$$

where

$$\begin{aligned} S_2^{-1}S_1u_3^1&=u_3^2\\ S_1^{-1}S_2u_3^2&=u_3^1 \end{aligned} \tag{18}$$

As the same condition $p_{i3} = p_1^3 + p_2^3$ has been considered over the interface between both subdomains, equation (17) will be:

$$u_3^{n+1}=u_3^n-\frac{\rho}{2}(u_3^1+u_3^2) \tag{19}$$

The resolution procedure of equation (13) is similar to the preconditioned gradient algorithm with preconditioner [5]

$$M=(S_1^{-1}+S_2^{-1})/4 \tag{20}$$

used when both subregions are discretized with the FEM.

DOMAIN DECOMPOSITION ALGORITHM

According to the treatment given to the transmission conditions in the interface, both compatibility and equilibrium, a parallel iterative procedure is obtained for the resolution of problems with BEM-FEM coupling. It would be as follows:

- Given an initial value $u_3=u_3^0$ in the interface Γ_3

- Calculation of the unknown solution $u_i^{n+½}$ separately in the BEM domain and in the FEM domain corresponding with the following Dirichlet problem (enforcement of the compatibility conditions):

$$
\begin{aligned}
&Lu_i^{n+1/2}=f \quad in \quad \Omega_i \\
&u_i^{n+1/2}=u_3^n \quad in \quad \Gamma_3 \\
&u_i^{n+1/2}=\bar{u} \quad in \quad \partial\Omega\cap\partial\Omega_i
\end{aligned}
\tag{21}
$$

- Calculation of the unknown value u_i^n for each subdomain by the resolution of the Neumann problem (enforcement of the equilibrium conditions):

$$
\begin{aligned}
&Lu_i^n=0 \quad in \quad \Omega_i \\
&u_i^n=0 \quad in \quad \partial\Omega\cap\partial\Omega_i \\
&\frac{\partial u_i^n}{\partial n}=\frac{1}{2}\left(\frac{\partial u_1^{n+1/2}}{\partial n}+\frac{\partial u_2^{n+1/2}}{\partial n}\right) \quad in \quad \Gamma_3
\end{aligned}
\tag{22}
$$

- Reinitialization of the value u_3 in the interface by the iterative refinement algorithm of equation (19) using the values calculated in equation (22):

$$
u_3^{n+1}=u_3^n-\rho\,(u_1^n+u_2^n) \tag{23}
$$

and iteration until the convergence.

NUMERICAL EXPERIMENTS

Let us use the algorithm described in the last section to solve a Poisson boundary-value problem:

$$
\begin{aligned}
&-\Delta u=f \quad in \quad \Omega \\
&u=g \quad in \quad \partial\Omega
\end{aligned}
\tag{24}
$$

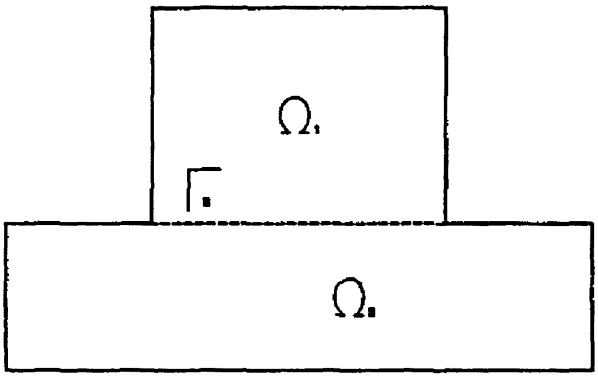
We will denote by D.O.F. the total number of degrees of freedom (i.e. of numerical unknowns) of the numerical scheme which is being used and by NIT the minimum number of iterations needed to reduce the initial error by a factor ϵ:

$$
\| e^{NIT} \| < \epsilon \, \| e^0 \| \tag{25}
$$

where e^k denotes error at step k and $\|.\|$ denotes the maximum norm at the gridpoints lying on the subdomain interface.

Moreover, we will use the average reduction factor per iteration (E.R.F.) [8] defined by:

$$
E.R.F.=\left(\|e^n\| \,/\, \|e^0\| \right)^{1/n} \tag{26}
$$



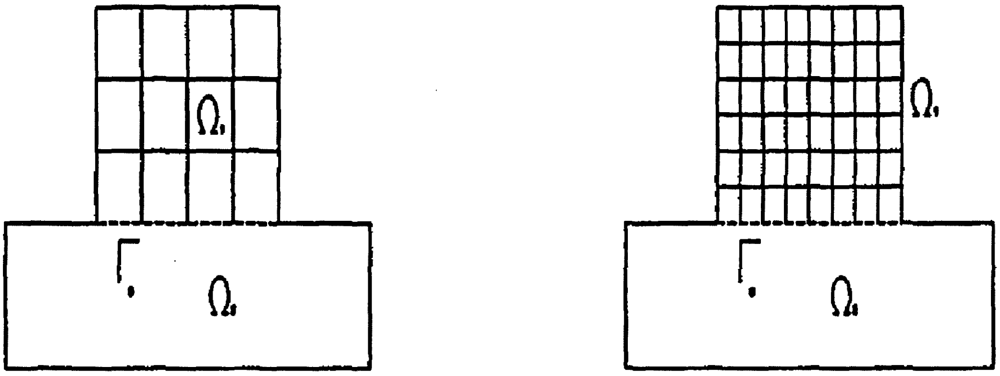


Figure 2(a) and (b). Meshes used in example 1.

In the first example (Fig.2), f and g in equation (24) will be taken to get an exact solution corresponding to the function:

$$u=4xy \tag{27}$$

The subdivision corresponding to Fig. 2a has been taken in such way that:

$$D.O.F. \begin{cases} \Omega_1 = 20 \\ \Omega_2 = 16 \\ \Gamma_3 = 3 \\ Total = 31 \end{cases} \tag{28}$$

and the one corresponding to Fig. 2b:

$$
D.O.F. \begin{cases} \Omega_1 = 63 \\ \Omega_2 = 40 \\ \Gamma_3 = 7 \\ Total = 94 \end{cases} \tag{29}
$$

We use linear elements over the region BEM and cuadrilateral finite elements of 4 nodes over the region FEM, both on uniform meshes.

Choosing a random initial value u_3^0 in the interface Γ_3 and taking $\epsilon = 10^{-4}$, the results for the two different geometrical situations depicted in Fig. 2a and 2b are reported in Table 1.

D.O.F.	31	94
NIT	6	6
E.R.F.	0.098	0.1

Table 1. Results for example 1

It is observed that the number of iterations necessary to reduce the initial error by a factor ϵ is independent of the number of degrees of freedom, being the E.R.F. very similar.

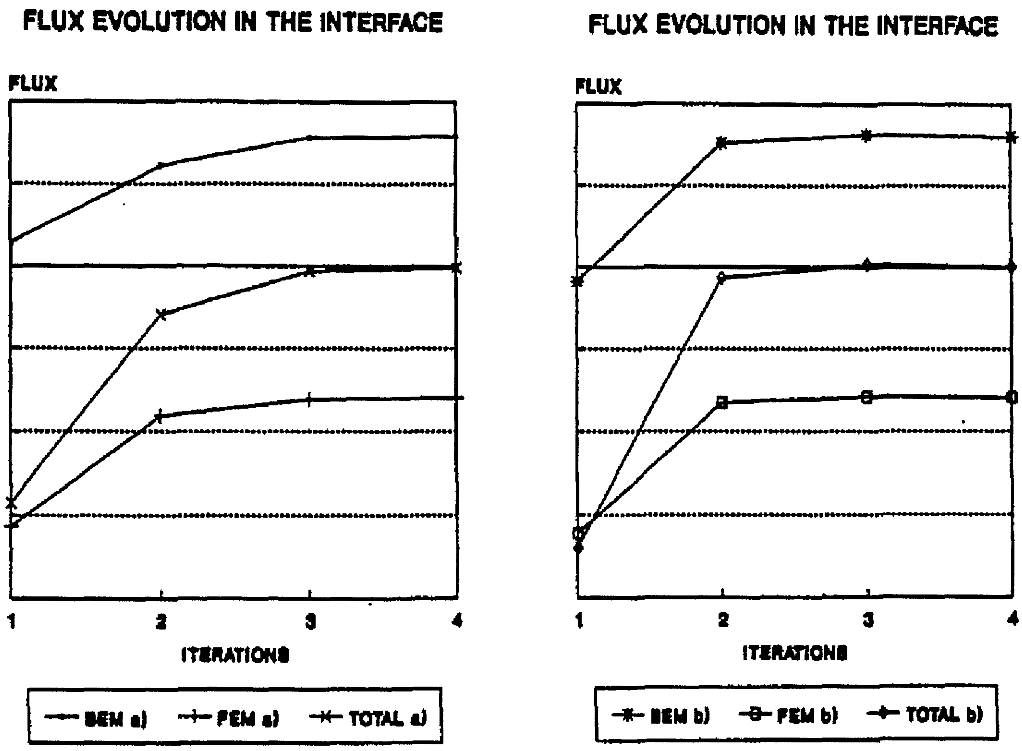


Figure 3(a) and (b). Evolution of the flux in the interface for example 1.

An evolution of the flux of a point on the interface can be observed in Fig.3a and 3b (corresponding with the mesh of Fig.2a and 2b respectively). There is a representation of the flux in a point belonging to BEM region and FEM region and the total value. The evolution is practically the same in both discretizations and the total flux reaches the zero value very quickly.

Now, we will consider a second example corresponding to a harmonic function:

$$u=x^3-3xy^2 \tag{30}$$

applied to the resolution of equation (24) in the domain represented in Fig.4 for which two different discretizations are solved.

Like in example 1, we will have for the subdivision of Fig. 4a:

$$D.O.F.\begin{cases}\Omega_1=21\\ \Omega_2=12\\ \Gamma_3=3\\ Total=28\end{cases} \tag{31}$$

and for the subdivision of Fig. 4b:

$$D.O.F.\begin{cases}\Omega_1= 65\\ \Omega_2= 24\\ \Gamma_3= 7\\ Total= 80\end{cases} \tag{32}$$

We use linear elements in the region BEM and cuadrilateral finite elements of 8 nodes in the region FEM, both on uniform meshes.

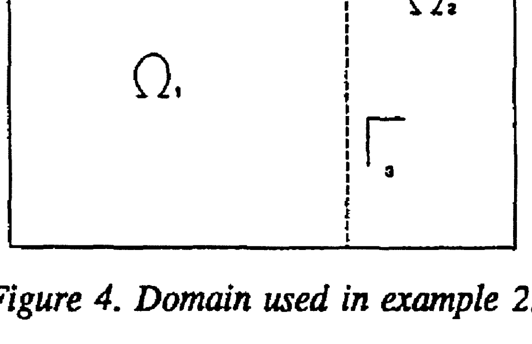


Figure 4. Domain used in example 2.

Figure 4(a) and (b). Meshes used in example 2.

Taking $\epsilon = 10^{-2}$ and a random initial value over the interface, the results are reported in Table 2.

D.O.F.	28	80
NIT	6	6
E.R.F.	0.386	0.423

Table 2. Results for example 2.

Like in the first example, the same conclusions are obtained. It is observed that the number of iterations to reduce the initial error by a factor is independent of the degrees of freedom. As it is logical when we consider a harmonic function the E.R.F. is superior to the one in the first example.

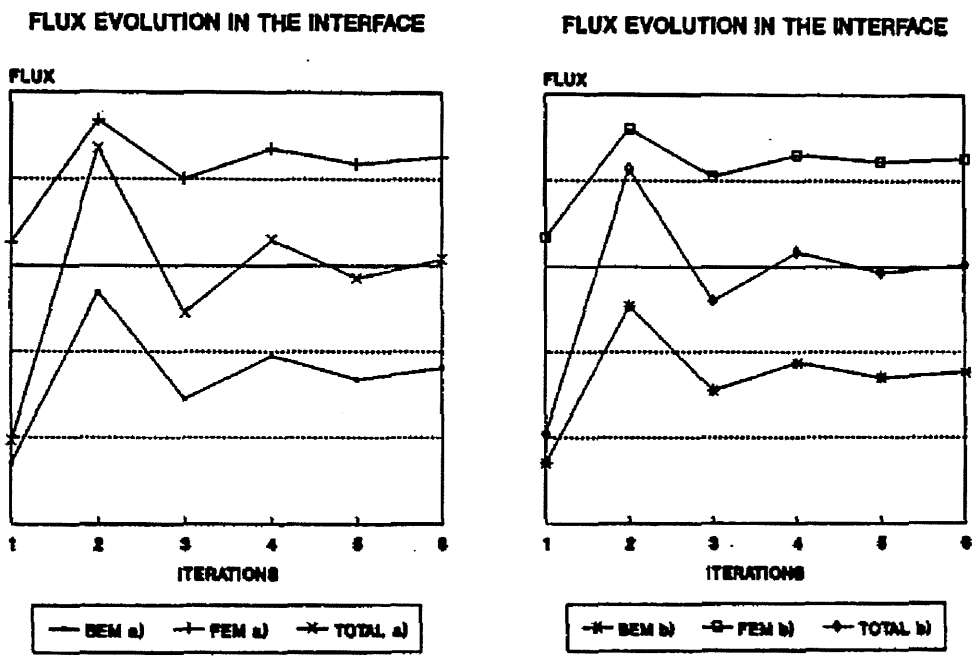


Figure 5. Evolution of the flux in the interface for example 2.

A representation of the flux in a point on the interface can be observed in Fig. 5 where the same conclusions as in Fig. 3 can be obtained.

CONCLUSIONS

With the numerical tests it is possible to affirm that the algorithm is valid to solve elliptic problems with a quick convergence in different subdomains separately. It makes possible the parallel implementation of problems BEM-FEM coupling avoiding some disadvantages implicit in the coupling.

Moreover, the numerical tests assure that the convergence is independent of the mesh step and the initial value u_3^0 in the interface.

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