Use of Feynman diagrams in large-scale neural networks with nonlinear behaviour

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ABSTRACT

A new proposal to the study of large-scale neural networks is reported. It is based on the use of similar graphs to the Feynman diagrams. A first general theory is presented and some interpretations are given. A propagator, based on the Green's function of the neuron, is the basis of the method. Application to a simple case is reported.

1.- INTRODUCTION

Several models have been employed in the last years in order to obtain a close approach to the behaviour of neural networks. Most of them have been based on previous models obtained for artificial neural networks or in electrical simulations of the neurons. Computer simulations have widely employed in everyone of these studies.

A new approach will be presented in this paper. Its basis is the use of diagrams similar to the ones employed by R. P. Feynman in Quantum Electrodynamics[1]. They have been successful used in Many-Body Systems[2] and some other areas of Physics. As it will be shown it is possible to establish a close parallelism between these systems and some types of neural networks. The theory that is going to be reported here is just a first trial. Our model is merely "functional" and not necessarily corresponding to specific anatomical structures (e.g., dendrites and axons). Much more work will be necessary to perform, until some real results could be obtained. As a matter of fact, some analytical expressions will be needed related with the interaction among neurons as well as the real Green's functions for the propagation of perturbations. Because these analytical expressions are not known to us, just general equations will be given as well as a possible interpretation. When these expressions be known, particular results will be able to be obtained.

1.1.- Models for excitable cells and networks.

1.1.a.- Models depending on time: Hodging and Huxley model.

Most models for excitable membrane retain the general Hodging and Huxley [3] format, and can be written in the form

$$C\frac{dV}{dt} + I_{ion}(V,W_1,\ldots,W_n) = I(t) \qquad \frac{dW_i}{dt} = \phi\,\frac{[W_{i,\infty}(V)-W_i]}{\tau_i(V)} \tag{1}$$

where V denotes membrane potential, namely a deviation from a reference or *rest* level, C is membrane capacity, and I_{ion} is the sum of V- and t- dependent currents through the various ionic channel types. I(t) is the applied current. The variables $W_i(t)$ are used to describe the fraction of channels of a given type that are in various conducting states, for instance open or closed, at time t. The first-order kinetics for W_i involve V -dependence in the equilibrium function $W_{i,\infty}$ and in the time constant τ_i. ϕ is a time scale factor that may depend on i. If the current, I_j, for channel type j can be suitably modeled as Ohmic, then it might be expressed as

$$I_j = \bar{G}_j\,\sigma_j(V,W_1,\ldots,W_n)\,(V-V_j) \tag{2}$$

where G_j is the total conductance with all j-type channels open, that is, the product of single channel conductance with the total number of j channels. σ_j is the fraction of j channels that are open and it may depend on several of the variables W_i. V_j is the reversal potential for this ionic species. In this model for squid giant axon, Hodgkin and Huxley have taken three variables W_i, denoted as **m**, **h** and **n**, to describe the fractions, $\mathbf{m}^3\mathbf{h}$ and $\mathbf{n}^4$, of open Na^+ channels and K^+ channels, respectively.

Several different models have been presented following the preceding one. Depending on the aspects to be considered, some models may contain many variables and represent numerous channel types, especially if they seek to account for rather detailed aspects of spike shape and dependence upon many different pharmacological agents. On the other hand, if qualitative or semiquantitative characteristics of spike generation and input-output relations are adequate, then a reduced two- or three-variable model may suffice. One of this models has been the one reported by Morris and Lecar [4] in the context of electrical activity of the barnacle muscle fiber. The model incorporates a voltage-gated Ca^{2+} channel and a voltage-gated, delayed rectifier K^+ channel. The calcium current here plays a role in spike generation analogous to that of the sodium current in the Hodgkin and Huxley model. Some details about this model can be seen at the original Morris and Lecar paper [4].

In order to study the oscillations emerging with non-zero frequency, the above set of differential equations have been linearized [5] and the partial derivatives evaluated at the singular points. Different behaviours are obtained and, in some cases, on oscillation is present for a constant external excitation.

1.1.b.- Models depending on space and time: the Cable Equation.

The set of equations shown above are able to be the basis for a model mainly interested on temporal behaviour. But no results can be obtained for the spatial one. It is in this point where the Cable Theory can offer some new results. As it is well known, the cable equation is a partial differential equation that neurophysiologist usually express as

$$\lambda^2 \frac{\partial^2 V}{\partial x^2} - V = \tau \frac{\partial V}{\partial t} \tag{3}$$

Here V represents the voltage difference across the membrane (interior minus exterior) as the deviation from its resting value, i.e., $V = V_i - V_e - E_r$; x represents distance along the axis of the membrane cylinder, and λ is the length constant of the core conductor. t represents time and τ is the membrane time constant of the passive membrane.

The above equation has many solutions. The problem is to construct a solution that satisfies not only the partial derivative equation but also the boundary conditions and the initial conditions. The two basic methods to construct basic solutions from which the more complicated solutions are obtained are separation of variables and the fundamental solutions or Green's function. In the first approach the solution is

$$V(x,t) = [A\sin(\alpha x) + B\cos(\alpha x)]\, e^{-(1+\alpha^2)t} \tag{4}$$

whereas in the second one is

$$V(x,t) = C_o(\pi t)^{-1/2}\, e^{-(t+x^2/4t)} \tag{5}$$

where x can be extended from $-\infty$ to $+\infty$. The singularity, an instantaneous point charge, is located at $x = 0$ when $t = 0$. If the amount of this charge is Q_0 coulombs, then for a semi-infinite length, extending from $x = 0$ to $x = +\infty$, the value of C_0 is $Q_0/(\lambda c_m)$ where λc_m represents the membrane capacitance of a λ length of the cylinder (in farads), implying that C_0 has the dimension of volts.

1.1.c.- Green's function method.

The importance of using the Green's function method is the possibility to obtain from it, the general solution for any external disturbance to the system. This is so because if we have a general equation in the form

$$\Gamma \, |u\rangle = |\Phi\rangle \tag{6}$$

where Γ is a linear differential operator which does not depend explicitly on x or t, then the Green's function associated with this equation is the solution of

$$\Gamma \; G(x-x',t-t') = \delta(x-x')\delta(t-t') \tag{7}$$

and then

$$|u\rangle = G \, |\Phi\rangle \tag{8}$$

In Cable Theory, the Γ function should be given by

$$\Gamma = \lambda^2 \frac{\partial^2}{\partial x^2} - \tau \frac{\partial}{\partial t} - 1 \tag{9}$$

No external action is present in eq. (7).

2.- PARALLELISM WITH FEYNMAN APPROACH TO THE MANY BODY PROBLEM.

As it is known from Quantum Mechanics and Solid State Theory, the general equation for obtaining the state of any system is the Schrödinger equation which with a perturbing potential $V(\nabla)$, may be written

$$\left[+\frac{\nabla^2}{2m} + i\frac{\partial}{\partial t} - V(\nabla) \right] \psi(k,t) = 0 \tag{10}$$

This equation has the associated Green's function equation (in k-space)

$$\left[-\frac{k^2}{2m} + i\frac{\partial}{\partial t} - V(k) \right] G^+(k,t-t') = \delta(t-t') \tag{11}$$

where V(k) is the Fourier transform of $V(\nabla)$. The solution to this equation may be written as an integral equation

$$G^+(k,t-t') = G_0^+(k,t-t') + \int_{-\infty}^{+\infty} dt'' G_0^+(k,t-t'') V(k) G^+(k,t''-t') \tag{12}$$

with G^+_0 as Green's function for the unperturbed Schrödinger equation. It is possible to obtain the perturbation expansion for G^+ in terms of G_0^+ by iterating

$$G^+(k,t-t') = G_0^+ \int_{-\infty}^{+\infty} dt'' G_0^+(k,t-t') V(k) G_0^+(k,t''-t') + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dt'' dt''' G_0^+ V G_0^+ G_0^+ V G_0^+ + \ldots$$

If we now use for G_0

$$G_0^+(k,t-t') = -i\,\theta_{t-t'}\,e^{-i\epsilon_k(t-t')} \qquad \text{with} \qquad \theta_{t-t'} = \begin{cases} 1 & \text{for } t-t'>0 \\ 0 & \text{for } t-t'<0 \end{cases} \tag{14}$$

this solution corresponds to the propagator employed by Feynman as the cornerstone of his theory for the many-body problem (MBP).

The above equation, (10), is somehow similar to the cable equation (3) reported previously. Hence, if we are able to obtain the Green's function for a neural system, it could get a similar interpretation to the analogous solution in MBP. This will be the objective of the next paragraph.

3.- PROPAGATOR INTERPRETATION OF THE NEURAL MODELS.

3.1.- Single perturbation propagator.

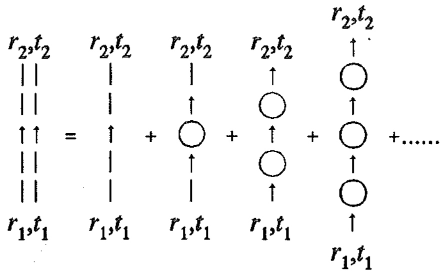
According to previous results, we could define the *classical propagator* for the neural system as

$G^+(r_2, t_2, r_1, t_1)$ = function defining a perturbation (for instance, a voltage) present in the one-neuron system at point r_1 at time t_1 and propagating to another point r_2 at later time t_2. It is related with the probability of the perturbation motion.

This propagator in the absence of external interactions will be called the *free propagator*. The quantity G^+ is a "retarded" propagator and, by definition, it is equal to zero for $t_2 \leq t_1$. If there are several possibilities for going from one place to another, the total propagator from point 1 to point 2 will be just the sum of each propagation process taken separately

P(2,1) = P(process I) + P(process II) + ...

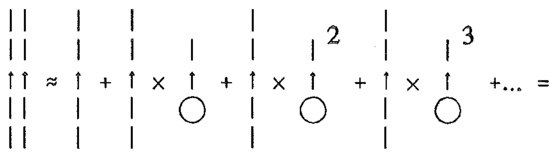
and if there are internal perturbations in the path from one point to another, given by V_{lm}, where l and m are the space-time coordinates where the process takes place, we can write the above equation in the form

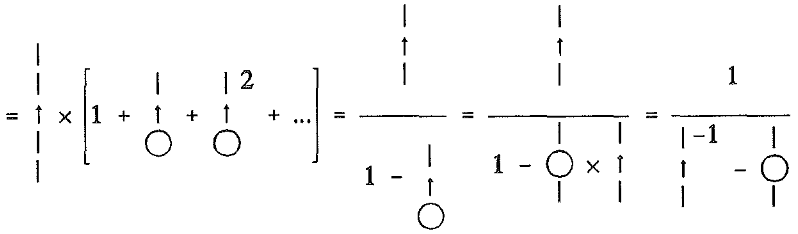


(15)

where each one of the terms in the sum gives every one of the possible ways for going from point 1 to point 2. Circles indicate possible perturbations, either internal or external to the system.

In the simplest case, where all the perturbations have the same expression, this sum gives



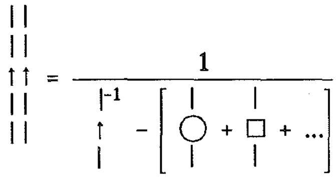


(16)

which may be then translated into

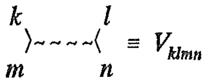
$$G^{+} \approx \frac{1}{(G_0)^{-1} - V_{lm}} \tag{17}$$

If other different types of perturbations are possible, the new equation should read



3.2.- Interacting neuron system.

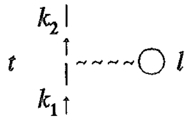
Let us have now a many neuron system consisting of N neurons interacting by means of two-neurons interactions $V(|r_j - r_i|)$, depending on the characteristics of the connections. Such interaction may be represented diagrammatically by a wiggly line



(19)

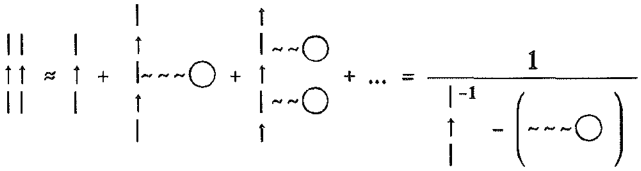
k, l, m and n represent neuron states. m and n have no relation with similar terminology in the Hodging and Huxley model.

One possible process is the interaction of the neuron with a modification in its state, changing from k_1 to k_2. The process occurs at a time t and can written as



(20)

This type of diagram will be call *Bubble diagram.* In the same way as we did before, we can add diagrams with several "bubbles" in the same perturbation line. The result should be

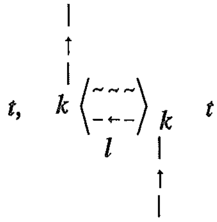


(21)

and hence

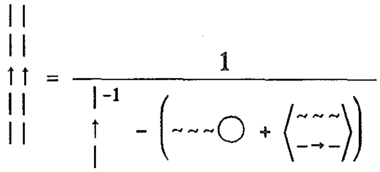
$$G^{+}(k,\omega) = \frac{1}{G_0^{-1} - (\,V_{klkl}\,)} \tag{22}$$

The second type of interaction corresponds to a perturbation going through a particular neuron and after interaction with a second neuron, this last one get the perturbation and the first one remains unperturbed. This phenomenon will be represented by



(23)

and it is called *Open oyster diagram.* In a similar way as it was performed for eq. (35), we can add for every possible situation and obtain

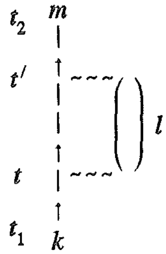


(24)

And in the same way as before, we get

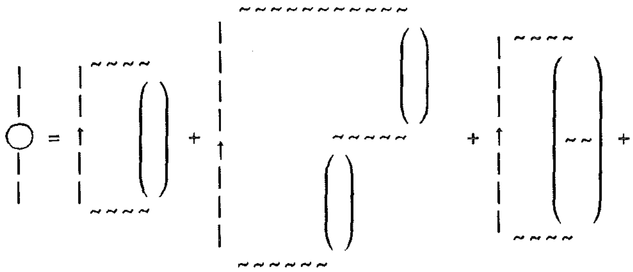
$$G^{+}(k,\omega) = \frac{1}{G_0^{-1} - (\,V_{klkl} + V_{lkkl})} \tag{25}$$

Moreover, another type of processes are still possible. They correspond with an interaction lasting a certain time and again with a change in the state of the neuron. This situation is written as



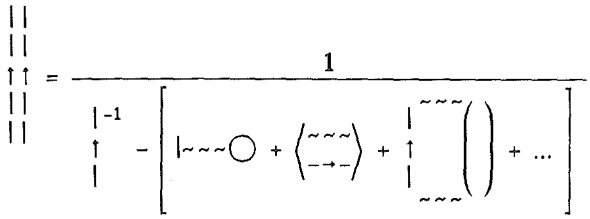
(26)

This diagram should be added to the above graphs in a similar way as we have done for previous types of interaction. The sum would give now



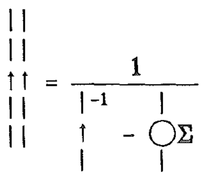
(27)

The situation in this case corresponds to a state that could be called "self energy" of the system and corresponds to the so called Dyson equation in MBP. We call it "self-energy" because no external perturbation is present. The corresponding total equation is given by



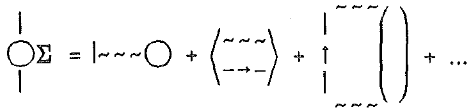
(28)

or



(29)

where



(30)

is the sum of all proper (irreducible) self-energy parts.

Translated into functions, eq. (30) becomes

$$G(k,\omega) = \frac{1}{G_0^{-1} - \Sigma(k,\omega)} \tag{31}$$

where

$$- i \ \Sigma(k,\omega) \equiv \tag{32}$$

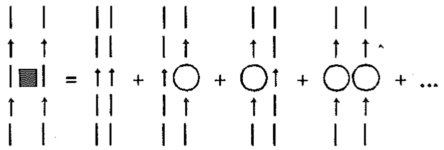
is the Fourier transform of the corresponding interactions.

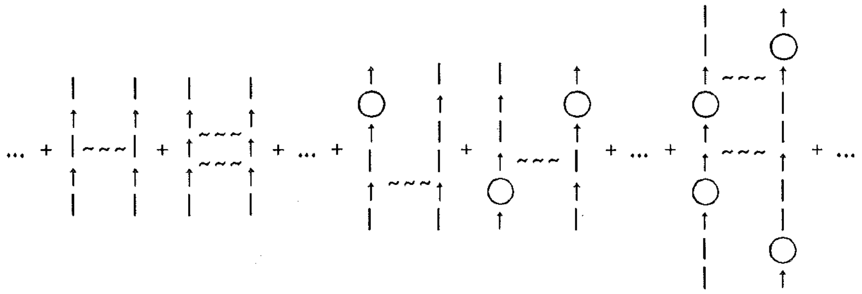
From these new diagrams is possible now to study the collective interactions in a many-neuron system. In order to do it, it is necessary first to considerer the two-excitations propagator.

3.3.- Two excitations propagator.

A two-excitations propagator gives the probability that if one perturbation appears into the many-neuron system at point $\mathbf{r}_1$ at time t_1 and another at $\mathbf{r}_3$ at time t_3, then one of the perturbations will be observed at $\mathbf{r}_2$ at later time t_2 and the other at $(\mathbf{r}_4, t_4)$. This can be evaluated as the sum of the probabilities for all the different ways this could happen. We could write this situation, diagrammatically, as

$$P(r_4 t_4,\ldots,r_1 t_1) \equiv$$



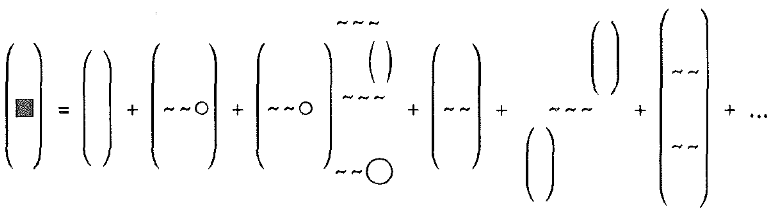


(33)

where the wiggle line stands for the possible direct interactions between neurons and the circles corresponds to the proper "self-energy" parts or to other types of perturbations, external or internal.

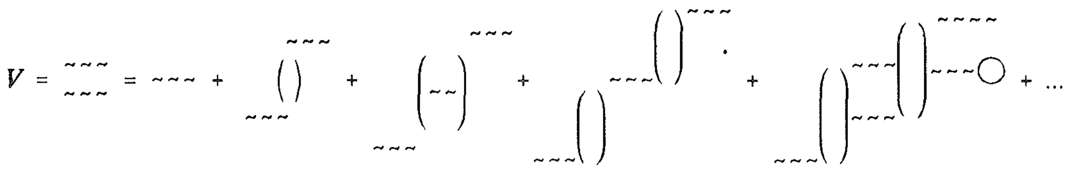
3.4.- Collective Perturbation Propagator.

A Collective Excitation is essentially a regular variation in the properties of the system when no external perturbation is present. It seems plausible that such waves might be described by a propagator which propagates a disturbance from one point to another, analogous to the way a single particle propagator propagates a single particle in many-body theory. It is easy to get such a propagator from above equations by letting $(\mathbf{r}_3, t_3) = (\mathbf{r}_4, t_4)$ and $(\mathbf{r}_1, t_1) = (\mathbf{r}_2, t_2)$. This yields the "*Collective Perturbation Propagator*", given diagrammatically by



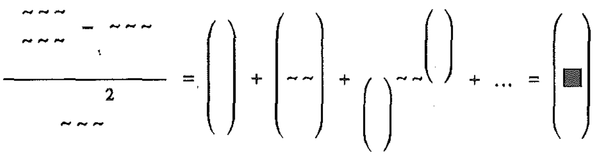
(34)

where several types of internal perturbations have been indicated, both single-neuron and between neurons. If we write now the interactions between neurons at the same-level, as



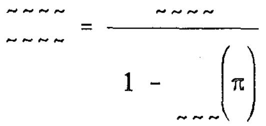
(35)

we obtain



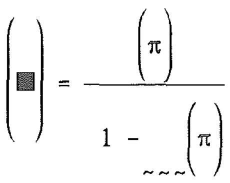
(36)

We can apply now a the Dyson-like equation, similar to the one previously obtained, and get



(37)

We find

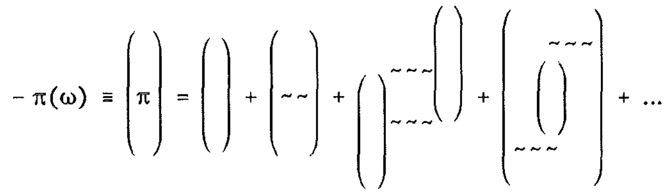


(38)

that can be expressed as

$$F = \frac{\pi(\omega)}{1 + V_k \pi(\omega)} \tag{39}$$

being



(40)

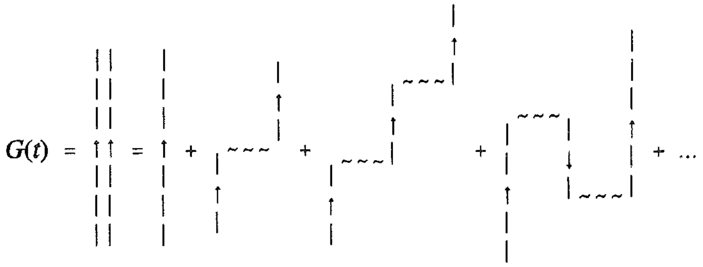
the sum over all irreducible interaction parts. Thus, the perturbation propagator has been expressed in terms of the sum over irreducible *interaction* parts, π.

3.5.- Evaluation of propagator by summation of graphs: Application to a simple case.

Propagators may be evaluated either by solving the differential equation they obey or by expressing them as a perturbation series with the aid of graphs and summing over selected sets of graphs to infinite order. The latter technique has the advantage of being systematic and to a high degree automatic. In some cases the summation may be carried out over all graphs and is exact. The main rules we will apply are

a) Factor of $iG_0(w)$ for the forward propagating lines.
b) Factor of $iG_0^{\cdot}(w)$ for the backwards propagating lines.
c) Factor of $-iV_k$ for each interaction wiggle.

In order to obtain some particular results we are going to apply some of the above techniques to a very simple case. The situation to be studied is just the propagator for a collective perturbation with just horizontal interactions between neurons. The basic configuration can be represented as



(41)

If we just consider forward propagation we can obtain

$$iG(\omega) = iG_0\left[1 + iG_0(-iV_k) + (iG_0)^2(-iV_k)^2 + ...\right] = \frac{iG_0}{1 - G_0V_k} \tag{42}$$

The free propagator could be taken, in its easier form, as a *"displacement wave"* with the form

$$G_0(t) = -i\left[\theta_t e^{-i\omega_0 t} + \theta_{-t} e^{+i\omega_0 t}\right] \tag{43}$$

which has the Fourier transform

$$G_0(\omega) = \frac{1}{\omega - \omega_0} - \frac{1}{\omega + \omega_0} = \frac{2\omega_0}{\omega^2 - \omega_0^2} \tag{44}$$

G_0 includes propagation both forward and backwards in time. With this approach we obtain

$$G(\omega) = \frac{2\omega_0}{1 - 2\omega_0\left(\dfrac{V_k}{\omega^2 - \omega_0^2}\right)} \tag{45}$$

which is the perturbation law. Depending on the V_k value, a particular law would be obtained.

We are going to solve, at this point, a particular case. It is just a simple example of many other situations that can be worked out. We will consider just propagation forward in time. This situation is the corresponding to a pulse-like interaction, V_k, lasting a time T and with amplitude A. This interaction will have the form

$$V_k(t) = \begin{cases} A & \text{for } |t| < T/2 \\ 0 & \text{for } |t| > T/2 \end{cases} \tag{46}$$

Its Fourier transform is

$$V(\omega) = A\,T\,\frac{\sin\dfrac{\omega T}{2}}{\dfrac{\omega T}{2}} \tag{47}$$

With this result, the propagator G will be

$$G(\omega) = \frac{1}{(\omega - \omega_0) - A\,T\,\dfrac{\sin\omega T/2}{\omega T/2}} \tag{48}$$

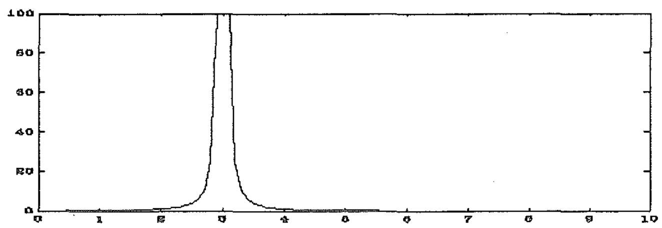


Figure 1.- Frequency response of the free propagator.

If we apply a pulse with $T = 0$, we obtain the natural response of the system. The result is given in Fig.1, where we have taken $\omega_0 = 3$. If we change the value of the pulse width, some major changes appear in the resulting propagator. As it can be seen in Figs. 2-3, if we move from $T = 0.1$ to $T = 10$, several peaks appear in $(G(\omega))^2$. A logarithmic scale has been taken in the vertical axis. As it can be seen, several new frequencies have been created in the system, being the number a function of the pulse amplitude. The interval between peaks depends strongly on the value given to T, being smaller for larger T values. As a matter of fact, the whole spectrum is densely covered by peaks, although with a very small amplitude. These results are in a good agreement with similar results obtained by other conventional methods[5]. Moreover, several other results can be obtained by varying the pulse amplitude and other parameters of our model. The result of this analysis will be published elsewhere.

4.- CONCLUSIONS

From the above results some conclusions can be drawn. The first one is the possibility to apply Feynman-like

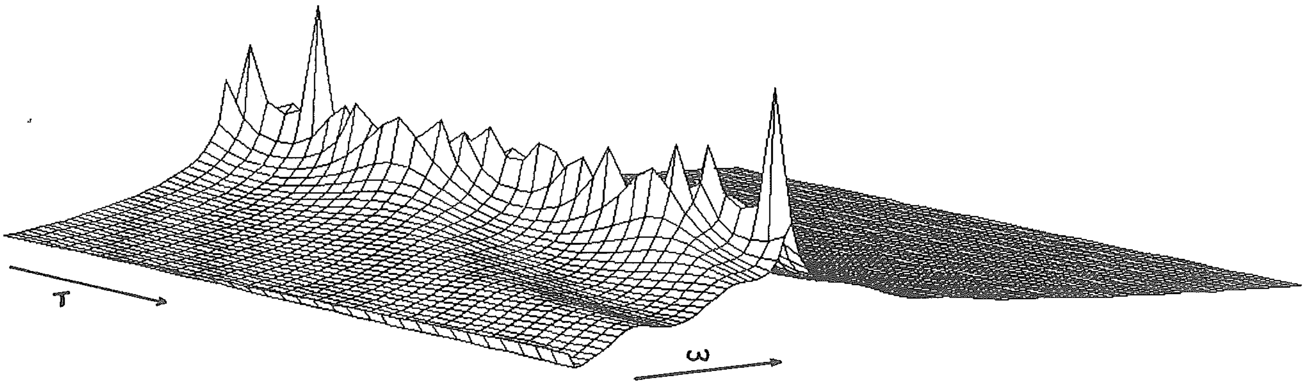


Figure 2.- Prpagator behaviour as a function of pulse width (from .1 to 10) and frequency (from 0 to 10). Pulse amplitude: 1.

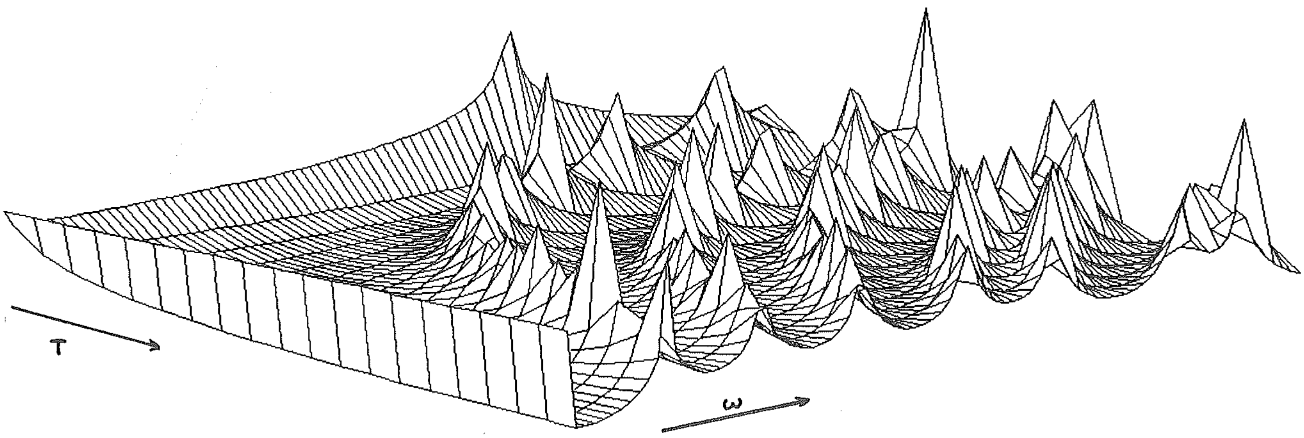


Figure 3.- Propagator behaviour as function of pulse width (from .1 to 6) and frequency (from 0 to 8). Pulse amplitude: 100.

diagrams to neural systems. Although we have just given one example, we believe that their employ can give help to the understanding of the neural system complex behaviour. Some more work will be necessary in order to determine the analytical expression for the real interactions between neurons as well as the Green's function for the different types of neurons. But we think that the present approach can open a new door to the study of these phenomena.

5.- ACKNOWLEDGEMENTS

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