WYPIWYG hyperelasticity

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Abstract

In this paper we describe a new promising procedure to model hyperelastic materials from given stress-strain data. The main advantage of the proposed method is that the user does not need to have a relevant knowledge of hyperelasticity, large strains or hyperelastic constitutive modelling. The engineer simply has to prescribe some stress strain experimental data (whether isotropic or anisotropic) in also user prescribed stress and strain measures and the model almost exactly replicates the experimental data. The procedure is based on the piece-wise splines model by Sussman and Bathe and may be easily generalized to transversely isotropic and orthotropic materials. The model is also amenable of efficient finite element implementation. In this paper we briefly describe the general procedure, addressing the advantages and limitations. We give predictions for arbitrary "experimental data" and also give predictions for actual experiments of the behaviour of living soft tissues. The model may be also implemented in a general purpose finite element program. Since the obtained strain energy functions are analytic piece-wise functions, the constitutive tangent may be readily derived in order to be used for implicit static problems, where the equilibrium iterations must be performed and the material tangent is needed in order to preserve the quadratic rate of convergence of Newton procedures.

1 Introduction

When materials undergo large strains, the behaviour is nonlinear. Once the small displacement hypothesis is not valid, different possible (Seth-Hill) strain measures may be considered. These measures are nonlinear in displacements (e.g. Bathe 1996 [1]). Furthermore, different stress measures may also be used. These stress measures may be defined from the strain measures by work conjugacy. When a material is tested at large strains, then the obtained stress-strain plot is generally nonlinear for whatever measures we employ. Different material effects may be hidden in such nonlinearity: elasticity, plasticity, viscoelasticity, creep, etc (e.g. Kojic & Bathe 2005 [2]). All these effects are nonlinear in nature when large displacements and strains are considered. However, there is a fundamental difference between truly elastic deformations and the rest of them, whether plastic or viscous. Elastic deformations must be recovered when the external actions decease and that recovery must take place without any energy dissipation, i.e. energy is also completely recovered. This observation (or fundamental constitutive hypothesis) implies that during elastic processes the stresses are a function of the total elastic strain (not of

their history) and derived form a stored energy function. The existence of that stored energy function guarantees that no energy is dissipated and the input work is fully recovered when the initial state is recovered. Material models that consider the existence of such stored energy function are known as hyperelastic and are the only ones that guarantee truly elastic behaviour (e.g. Ogden 1986 [3] and Simó & Hughes 1998 [4]).

If the assumption of the existence of a stored energy function solves the problem of mathematical and physical consistency (i.e. elastic materials behave truly elastically) without directly resorting to the Bernstein compatibility conditions, it introduces the new problem of obtaining that energy function. The stored energy may not be directly measured; its change may only be measured through the experimental stress-strain curves. However at the same time it is rather impossible to define an explicit expression for a stored energy function which yields a given material behaviour. Hence, a large variety of stored energy function forms or "shapes" have been proposed motivated on the behaviour of different materials. The Ogden model (e.g. Ogden 1997 [3], Ogden 1972 [5]), the Mooney-Rivlin model (Mooney 1940 [6], Rivlin 1948 [7]), the Yeoh model (Yeoh 1990 [8]), the Arruda-Voice model (Arruda & Boyce 1993 [9]) and the Blatz-Ko models (Blatz & Ko 1962 [10]) are just some well-known examples. These models frequently contain some material constants that must be obtained using an optimization procedure to yield a possible "best fit". A global minimum is of course seldom guaranteed and some special procedures are frequently needed, as for example the use of the Levenberg-Marquardt algorithm (Twizell & Ogden 1983 [11]). Even with these inconveniences, the situation may be considered acceptable for the isotropic case, but for the transversely isotropic case or for the orthotropic case, the situation is rather worse since few energy functions are available and the predicted behaviour by those functions may deviate considerably from that obtained from experiments. Some of these anisotropic models are those of Holzapfel (e.g. Holzapfel 2000 [12] and therein references), Itskov and Aksel (Itskov & Atksel 2004 [13]), Diani et al (Diani et al 2004 [14]) and Holzapfel et al (Holzapfel et al 2000 [15]).

In summary, the desire of an engineer is to just prescribe some stress-strain data in given stress and strain measures (for example nominal stresses and logarithmic strains) and let the "program" do the rest of the work such that the predicted behaviour exactly matches the prescribed experimental data of course keeping the truly elastic (hyperelastic) behaviour; i.e. it is a What Your Prescribe Is What You Get (WYPIWYG) model. The problem now is to obtain a stored energy function that does the job.

A handy procedure of this kind for isotropic materials has been introduced by Sussman and Bathe (Sussman & Bathe 2009 [16]). In their procedure they use piece-wise splines to interpolate the experimental data and obtain a continuum smooth representation of the behaviour of the material. Those analytic functions are used as intermediate tools to obtain the derivative of a stored energy function through an inversion formula in an also piece-wise setting. Once the derivative of the stored energy function is known, the energy function may also be obtained (although it is never needed in practice).

Their procedure is exact in representing the "prescribed" material behaviour (i.e. in also replicating possible measuring errors, which should be previously eliminated by the user) and hence may be used in substitution of any known isotropic material behaviour (Ogden, Mooney-Rivlin, etc.). This procedure, for the isotropic case is already available in the general purpose Finite Element code ADINA (ADINA R&D [17]).

The objective of the present paper is to show that the idea from Sussman and Bathe (Susmann & Bathe 2009 [16]) of using piece-wise splines interpolation-based models can be extended also to transversely isotropic materials and to orthotropic materials given some assumptions and some simplifying approximations. It will be shown that, although there is no experimental evidence for those assumptions to be valid in the anisotropic cases, they basically

affect the multiaxial nonproportional behaviour, still allowing for a perfect match of the given experimental data in the preferred directions. Here we note that equivalent assumptions are implicitly given by models which use an explicit form of the stored energy function and that the accuracy of those assumptions can be checked only through extensive experimental testing, not available at this moment in the literature to the authors' knowledge.

The layout of the paper is as follows. First, in Section 2 we address the continuum interpolation of experimental data through piece-wise splines. In Sections 3 and 4 the general procedure to obtain the stored energy function is presented. Although the procedure is different for different sets of experimental data, the general layout is common to all procedures. In Section 5 we show that any arbitrary user-prescribed "experimental" data can be almost exactly captured by the model. In Section 6 we show a prediction for a real material and actual experiments.

2 Initial continuum interpolation of discrete experimental data

One of the basic ingredients of the model is the spline-based interpolation of the stress-strain experimental measures. This technique interpolates the measured data points (x_i, y_i) using polynomials of up to third degree (cubic splines) between any two points.

The coefficients of each spline are forced to accomplish specific conditions to guarantee some smoothness requirements, being the resulting piecewise function twice continuously differentiable in all the experimental range and exactly passing over the data points. Physically, this means that we wish the elasticity moduli and its derivative to be continuous, which are attractive computational and smoothness requirements for hyperelastic behaviour. Some different boundary conditions at the ends of the interpolation range can be applied, although their effect over the resulting function is only reduced to a small region near the boundaries if the number of points is reasonably large.

The basic spline equation between two consecutive experimental points x_i and x_{i+1} is

$$P_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \qquad i = 1, \dots, N \tag{1}$$

where the number of points is N+1. The exact interpolation of the data points (x_i, y_i) gives two equations for each polynomial

$$y_i = P_i(x_i) \qquad y_{i+1} = P_i(x_{i+1}) \qquad i = 1, \dots, N \tag{2}$$

Between two subdomains, two additional conditions are established to enforce continuity of the first and second derivatives of the function across segments

$$P'_{i-1}(x_i) = P'_i(x_i) \qquad P''_{i-1}(x_i) = P''_i(x_i) \qquad i = 2, \dots, N \tag{3}$$

Two more (boundary) conditions are needed to complete the system of equations and be able to determine the set of 4N coefficients. A usual approach is to impose

$$P''_1(x_1) = P''_N(x_{N+1}) = 0 \tag{4}$$

which defines the so-called "natural" splines. Obviously, other boundary conditions may be applied.

Normalizing each subdomain and taking the first derivative at the N+1 points as independent variables, it can be shown that the previous 4N equations reduces to the following tridiagonal system of N+1 equations

$$\begin{gathered} 2y'_1 + y'_2 = 3\,(y_2 - y_1), \\ y'_{i-1} + 4y'_i + y'_{i+1} = 3\,(y_{i+1} - y_{i-1}), \quad i = 2, \ldots, N \\ y'_N + 2y'_{N+1} = 3\,(y_{N+1} - y_N). \end{gathered} \tag{5}$$

which can be solved very efficiently even for large values of N. Finally, the spline coefficients can be easily computed from this solution and, hence, the interpolation is fully determined for the experimental domain. In case the range for strains for a specific problem is larger than that given by the experimental data, the extrapolation given by the end-point conditions are used. Of course, alternatively, the user may prescribe some "guessed" extrapolation data.

3 Special decomposition of the stored energy function.

As it is well known, the deformation gradient **F** may be decomposed into an stretch part and a rotation part. The stretch part is that of interest to compute strain and stress measures. The right polar decomposition provides the relation

$$\mathbf{F} = \mathbf{RU} \tag{6}$$

where **R** and **U** are the (orthogonal) rotation tensor and the (symmetric) right stretch tensor, respectively. It is well known that for an isotropic hyperelastic material the strain energy density W is an invariant of the right stretch tensor **U**. As a direct consequence, $W(\mathbf{U})$ may be regarded as a function of any invariants of that tensor, and particularly of the three principal stretches λ_i, that is

$$W(\mathbf{U}) = W(\lambda_1, \lambda_2, \lambda_3) \tag{7}$$

Moreover, if the material is incompressible, the widely accepted Valanis-Landel hypothesis (Valanis & Landel 1967 [18]) allows the decomposition of the strain energy function $W(\lambda_1, \lambda_2, \lambda_3)$ into a sum of three independent, but equal in form, functions $\omega(\lambda_i)$. Taking into account the relation between the principal stretches and the principal logarithmic (Hencky) strains, $E_i = ln\lambda_i$, the previous additive decomposition can be rewritten in terms of E_i, taking the equivalent form

$$W(\mathbf{E}) = W(E_1, E_2, E_3) = \omega(E_1) + \omega(E_2) + \omega(E_3) \tag{8}$$

where $\mathbf{E} = \ln\mathbf{U}$ represents the symmetric second-order Hencky strain tensor in the material configuration.

Focusing now on the description of a transversely isotropic material due to the existence of a preferred direction of anisotropy it is obvious that there exists a rotation tensor **Q** for which the isotropic invariance relation $W(\mathbf{E}) = W(\mathbf{QEQ}^T)$ is not fulfilled (an arbitrary rotation not parallel to the preferred direction, $\mathbf{a}_0$, would be an example). Therefore, for this type of materials the potential W is no longer an invariant of **E** and, in a general deformation state in which strain principal directions are not coincident with preferred material directions, it would not be correct to assume formulations only based on principal strains. Instead, for this particular case, W has to be considered as a function of the direction that characterizes the anisotropic behavior as well, that is, it must be $W = W(\mathbf{E},\mathbf{a}_0)$. Evidently, for an orthotropic

material, the other two preferred directions $\mathbf{b}_0$ and $\mathbf{c}_0$ have to be added as arguments of W, resulting in that case

$$W = W(\mathbf{E}, \mathbf{a_0}, \mathbf{b_0}, \mathbf{c_0}) \tag{9}$$

An easy way to consider all these dependences of W is to simply employ the 6 components of $\mathbf{E}$ in the material basis $\{\mathbf{a_0}, \mathbf{b_0}, \mathbf{c_0}\}$ as the independent variables. Hence

$$W = W(E_{11}, E_{22}, E_{33}, E_{12}, E_{23}, E_{13}) \tag{10}$$

Before providing an additive decomposition for this more complicated case, we try to expand the Valanis-Landel decomposition for incompressible isotropic materials in a general basis. First, note that $\omega(E)$ can be expanded as

$$\omega(E) = aE + bE^2 + H.O.T = \omega_I(E) + \omega_{II}(E) + H.O.T \tag{11}$$

where a and b are constants, $\omega_I(E)$ includes the linear terms and $\omega_{II}(E)$ the second-order ones. Then, Equation (10) results in

$$W(\mathbf{E}) = \omega(E_1) + \omega(E_2) + \omega(E_3) \approx a(E_1 + E_2 + E_3) + b(E_1^2 + E_2^2 + E_3^2) \tag{12}$$

$$W(\mathbf{E}) \approx a(tr\mathbf{E}) + b(tr\mathbf{E}^2) = b(tr\mathbf{E}^2) = b\mathbf{E}\colon\mathbf{E} \tag{13}$$

where the incompressibility condition $tr\mathbf{E} = \mathbf{E}\colon\mathbf{I} = 0$ has been used. If we now represent the tensor $\mathbf{E}$ in a general basis not coincident with the Lagrangian material axes, it yields

$$W(\mathbf{E}) \approx b\mathbf{E}\colon\mathbf{E} = b(E_{11}^2 + E_{22}^2 + E_{33}^2 + 2E_{12}^2 + 2E_{23}^2 + 2E_{13}^2) \tag{14}$$

i.e.

$$W(\mathbf{E}) \approx \omega_{II}(E_{11}) + \omega_{II}(E_{22}) + \omega_{II}(E_{33}) + 2\omega_{II}(E_{12}) + 2\omega_{II}(E_{23}) + 2\omega_{II}(E_{13}) \tag{15}$$

which, as it is clear, is a decomposition only valid to the second order. It can be easily shown that if higher order terms are considered, then the additive decomposition of W has terms with coupled strain components. These additional terms should be considered to ensure that $W(\mathbf{E})$ is invariant up to higher orders under generic rotations. In the principal axes of deformation, Ogden (Ogden 1974 [19]) has shown that the strain energy given in eqn. (8) is valid to the fifth order, providing some theoretical support to the Valanis-Landel hyphotesis, with is also verified through experiments (e.g. Treolar 1944 [20], Ogden 1997 [3])

Motivated by this additive uncoupled second-order decomposition of W for isotropic materials, we postulate a similar decomposition of the strain energy function for orthotropic materials as

$$W = \omega_{11}(E_{11}) + \omega_{22}(E_{22}) + \omega_{33}(E_{33}) + 2\omega_{12}(E_{12}) + 2\omega_{23}(E_{23}) + 2\omega_{13}(E_{13}) \tag{16}$$

where, as explained above, W has been expressed as a function of the six components of the symmetric tensor $\mathbf{E}$ in the reference frame defined by the material preferred directions $\{\mathbf{a_0}, \mathbf{b_0}, \mathbf{c_0}\}$. Note that, in this case, six different functions have been used, which is in accordance with the number of the unknown parameters needed to describe an orthotropic incompressible material at small strains with the volumetric and isochoric behaviors fully uncoupled. We want to note that such a function is used to describe the deviatoric stress-strain

behavior within the framework of small strains, so its applicability to multiaxial large deformations using linear relations to logarithmic strains is somewhat justified.

This last decomposition is the definitive expression that we will employ in the following sections, in which we give insight into the general procedure to obtain piecewise spline representations of the first derivative of the unknown functions ω_{ij}.

4 Procedure to obtain the stored energy function

In order to easily introduce the method, but without loss of the generality of the procedure, we particularize the strain energy function to the transversely isotropic case and briefly explain the methodology to obtain the first derivative of the unknown functions ω_{ii}. There exist different procedures to determine the shear terms ω_{ij} $(i \neq j)$ depending on the available set of measured data points. These procedures are addressed in detail elsewere (Latorre & Montans 2013 [21]). In a similar way as explained herein, all the components of *W* could be calculated in the more general orthotropic case (Latorre & Montans 2013 [22])

For a transversely isotropic material, taking $\mathbf{e}_3$ as the preferred direction of the material, *W* takes the specific form

$$W = \omega_{11}(E_{11}) + \omega_{11}(E_{22}) + \omega_{33}(E_{33}) + 2\omega_{13}(E_{23}) + 2\omega_{13}(E_{13}) \tag{17}$$

with only three different functions to determine. The strain energy *W* is expressed in a reference frame for which $E_{12} = 0$, so the term ω_{12} is not considered in the decomposition. We study the case in which the available experimental measures are the tension-compression stress distribution $\tilde{\sigma}_1(\tilde{E}_1)$ and the transverse strain distribution $\tilde{E}_2(\tilde{E}_1)$, both obtained from a uniaxial test performed in the (transversely) isotropic axis $\mathbf{e}_1$. Hence, in view of the decomposition of *W*, the following relations hold

$$\begin{aligned} \tilde{\sigma}_1(\tilde{E}_1) &= \omega'_{11}(\tilde{E}_1) + p, \\ 0 &= \omega'_{11}(\tilde{E}_2(\tilde{E}_1)) + p, \\ 0 &= \omega'_{33}(\tilde{E}_3(\tilde{E}_1)) + p. \end{aligned} \tag{18}$$

with p representing a pressure-like quantity (hydrostatic pressure) required to maintain incompressibility. Note that the principal strains are subjected to the incompressibility condition of the material, i.e. $E_1 + E_2(E_1) + E_3(E_1) = 0$. If the pressure p is eliminated from the above equations, they reduce to

$$\tilde{\sigma}_1(\tilde{E}_1) = \omega'_{11}(\tilde{E}_1) - \omega'_{11}(\tilde{E}_2(\tilde{E}_1)) \tag{19}$$

$$\omega'_{11}(\tilde{E}_2(\tilde{E}_1)) = \omega'_{33}(\tilde{E}_3(\tilde{E}_1)) \tag{20}$$

From the first equation we can obtain a piecewise representation of ω'_{11}, as we explain below. Then, the second equation will provide the function ω'_{33}. This last equation can be regarded as a compatibility condition between terms ω_{11} and ω_{33}.

First, the data points $\tilde{\sigma}_1(\tilde{E}_1)$ are fit using a piecewise cubic spline, as shown in Section 2. We call $\sigma_1(E_1)$ that piecewise spline function. Secondly, each transversal strain measure $\tilde{E}_2$ is regarded as a scalar, $\tilde{a}$ say, multiplied by each longitudinal strain measure $\tilde{E}_1$. Note that, since the parameter $\tilde{a}$ may take a different value for each measured data point, the relation $\tilde{E}_2 = \tilde{a}\tilde{E}_1$ must not be regarded as a linear approximation of the distribution $E_2(E_1)$. Hence

$$\tilde{\sigma}_1(\tilde{E}_1) = \omega'_{11}(\tilde{E}_1) - \omega'_{11}(\tilde{a}\tilde{E}_1) \tag{21}$$

which is an expression that can be approximately inverted to obtain each value $\omega'_{11}(\tilde{E}_1)$ through:

$$\omega'_{11}(\tilde{E}_1) \approx \sum_{k=0}^{\infty} \sigma_1(\tilde{a}^k \tilde{E}_1) \tag{22}$$

The spline function $\sigma_1(E_1)$ makes possible the calculation of the terms in the summation. We call this solution the *inversion formula*. To prove it, assume now that $\tilde{E}_2 = \tilde{a}\tilde{E}_1$ is a linear relationship (that is, a is constant) and simply substitute the solution provided in Eqn. (22) into eqn. (21) to obtain

$$\sum_{k=0}^{\infty} \sigma_1\left(a^k \tilde{E}_1\right) - \sum_{k=0}^{\infty} \sigma_1\left(a^{k+1} \tilde{E}_1\right) = \sigma_1\left(a^0 \tilde{E}_1\right) = \sigma_1\left(\tilde{E}_1\right) = \tilde{\sigma}_1\left(\tilde{E}_1\right) \tag{23}$$

provided that $a = \tilde{E}_2/\tilde{E}_1 < 1$, which is a very usual condition. Also note that the equality $\sigma_1(\tilde{E}_1) = \tilde{\sigma}_1(\tilde{E}_1)$ has been carried out due to the fact that the spline representation $\sigma_1(E_1)$ passes exactly through the measured data. However, in real situations $\tilde{a}$ may not be constant. Then the inverse formula given above is not exact but the series are still convergent if $\tilde{a} < 1$. In these cases, it can be shown that the error of the inversion formula is small and can be neglected for practical purposes.

Once all the values $\omega'_{11}(\tilde{E}_1)$ are calculated using the inversion formula the corresponding piecewise spline function, $\omega'_{11}(E_1)$, is built. This last representation, together with the compatibility relation given in Eqn. (20), let us obtain the remaining function $\omega'_{33}(E_3)$ evaluated at the known points $\tilde{E}_3(\tilde{E}_1) = -\tilde{E}_1 - \tilde{E}_2(\tilde{E}_1)$. With the values $\omega'_{33}(\tilde{E}_3)$, the piecewise spline representation $\omega'_{33}(E_3)$ is finally built.

For convenience, a step-by-step outline of the previous process is given:

1. Measured data points: $\tilde{\sigma}_1(\tilde{E}_1)$ and $\tilde{E}_2(\tilde{E}_1)$ from a tension-compresion uniaxial test.
2. Build the piecewise spline function $\sigma_1(E_1)$ from $\tilde{\sigma}_1(\tilde{E}_1)$.
3. Obtain the values $\tilde{a} = \tilde{E}_2/\tilde{E}_1$ and calculate $\omega'_{11}(\tilde{E}_1)$ with the inversion formula.
4. From all the values $\omega'_{11}(\tilde{E}_1)$, construct the spline representation $\omega'_{11}(E_1)$.
5. Calculate $\omega'_{33}(\tilde{E}_3)$ using Eqn. (20) and $\tilde{E}_3(\tilde{E}_1) = -\tilde{E}_1 - \tilde{E}_2(\tilde{E}_1)$.
6. Form the piecewise spline function $\omega'_{33}(E_3)$ from all the values obtained in step 5.

5 Prediction of the behaviour for different analytical models

For the isotropic case, the strain energy function reduces to

$$W = \omega(E_1) + \omega(E_2) + \omega(E_3) \tag{24}$$

with only one function to be determined from experimental data. Since there are no preferred directions, the strain energy function is expressed in the principal Lagrangian axes and the shear terms are not explicitly considered. In order to use our model to obtain ω, the procedure detailed in the previous section can be followed with the transverse strain distribution initially prescribed as $\tilde{E}_2(\tilde{E}_1) = -(1/2)\tilde{E}_1$, as effectively occurs for an incompressible isotropic material. Hence, the isotropic model of Sussman & Bathe is recovered.

In Figure 1 the stress-strain distribution obtained from a uniaxial test modeled with the Ogden hyperelastic model (see for example Ogden 1997 [3])

$$W = \sum_{a=1}^{3} \omega(E_a); \quad \omega(E_a) = \sum_{p=1}^{3} \frac{\mu_p}{\alpha_p}(e^{E_a \alpha_p} - 1) \tag{25}$$

with material constants

$$\alpha_1 = 1.3 \quad \alpha_2 = 5 \quad \alpha_3 = -2 \tag{26}$$

$$\mu_1 = 630\ kN/m^2 \quad \mu_2 = 1.2\ kN/m^2 \quad \mu_3 = -10\ kN/m^2 \tag{27}$$

is represented (solid circles). With those "measured" data points and the prescribed strains $\tilde{E}_2(\tilde{E}_1) = -(1/2)\tilde{E}_1$, the function ω has been calculated using the inversion formula. Using this function, the hyperelastic stresses predicted by our model are calculated through

$$\sigma_1(E_1) = \omega'_{11}(E_1) - \omega'_{11}(E_2(E_1)) \tag{28}$$

and are also depicted (triangles) in Figure 1. Note that the agreement with the initial spline which interpolates the "experimental" points is exact, as should be expected due to the linearity of the transverse strain behavior.

Next, we show the capabilities of the orthotropic model to reproduce experimental results when the initial data points are obtained from three different uniaxial tests performed in the preferred directions of the material. With these data at hand, the three longitudinal strain energy terms ω_{ii} can be obtain with the spline methodology and the inversion formula. The remaining shear components of *W*, i.e. ω_{ij} $(i \neq j)$, could be obtained from the corresponding shear tests.

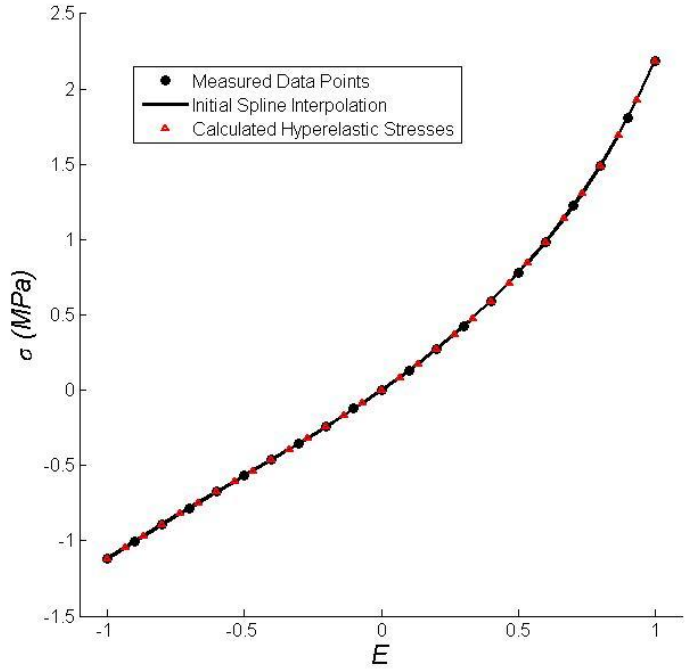

Figure 1: Initial piecewise spline interpolations of assumed data σ(E) from a uniaxial test performed on an isotropic (Ogden) material. Calculated stresses using the transversely isotropic splines-based model with initial distribution $\tilde{E}_2(\tilde{E}_1) = -(1/2)\tilde{E}_1$.

In Figure 2, three assumed stress-strain distributions $\tilde{\sigma}_i(\tilde{E}_i)$ obtained from the corresponding uniaxial tests are represented (solid marks). Also, the predicted results of the full orthotropic splines-based model are shown (hollow triangles). We note that the specific values of stresses and strains are irrelevant for the purpose of this section, since we just want to show the predictive capabilities of the procedure for arbitrary experimental data.

As can be seen in the figure, the results obtained with the model reproduce the "experimental" points very accurately. However, unlike for the isotropic case, the prediction of the data is not truly exact since the three compatibility conditions analogous to Eqn. (20) that appear in this case (one for each uniaxial test) are only approximately fulfilled in a least-squares sense. For more details of the underlying fitting process the reader is referred to (Latorre & Montáns 2013 [21]).

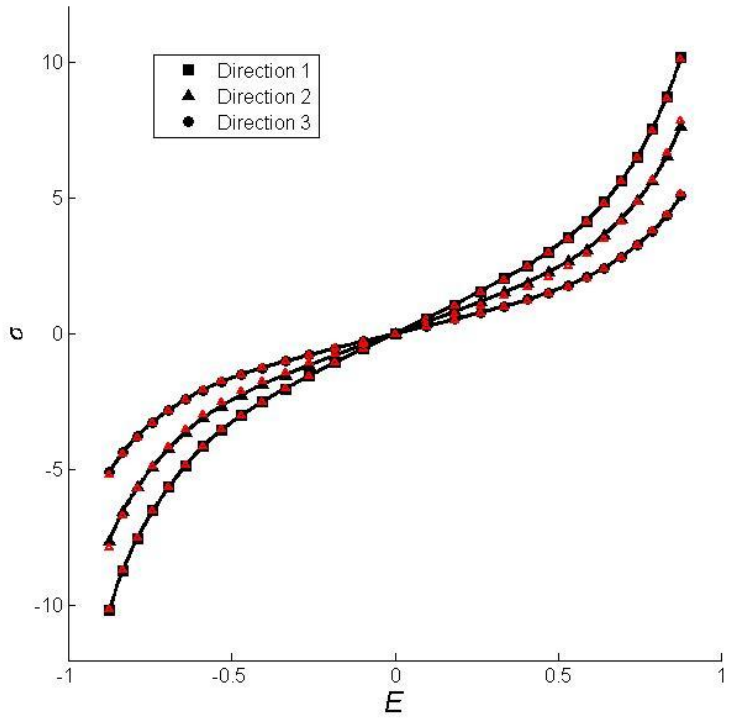

Figure 2: Initial spline interpolations of assumed data $\tilde{\sigma}_i(\tilde{E}_i)$ from uniaxial tests performed on an incompressible orthotropic material. Calculated stresses using the splines-based model.

6 Example: prediction of the behaviour of human soft tissue

Martins et al. present in their work (Martins et al 2011 [23]) experimental data from uniaxial tensile tests for transversely isotropic human living tissues and characterize the damage process

in the tissue samples. We are only interested in modelling the hyperelastic non-damaged behaviour, so we try to reproduce only some partial results they provide.

As in the examples from the previous section, Figure 3 shows the experimental data and the predicted results provided by the spline-based model. As can be observed, the hyperelastic stresses calculated reproduce very accurately the experimental measures in both directions.

The experimental data points are given in terms of stretches and Cauchy stresses, hence the proper conversion has been previously performed ($E_i = ln\lambda_i$). Furthermore, since only uniaxial tension measures are provided, both stress distributions $\sigma_i(E_i)$ have been regarded odd functions of the strains in order to be able to apply the inversion formula. If compression measures had been available, they should have been employed instead. However we note that the predictability capability of the experimental data would not have been much different in the range of interest.

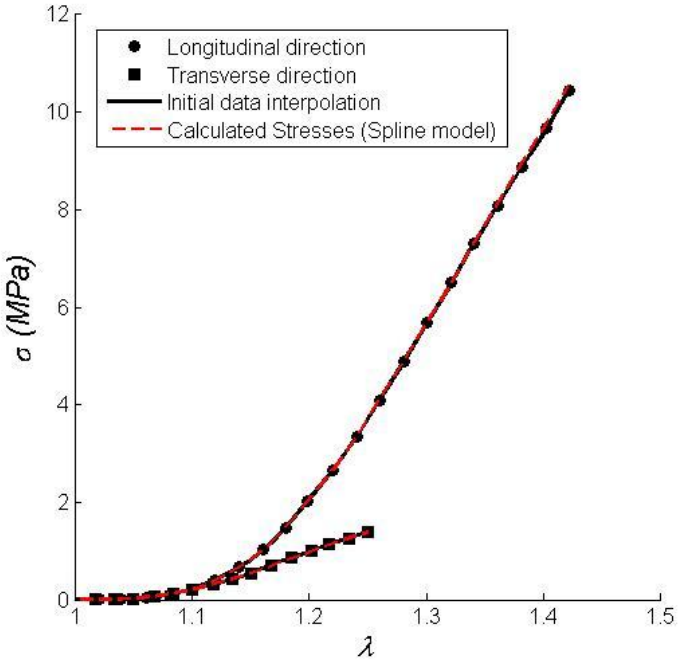

Figure 3: Measured Cauchy stress points $\tilde{\sigma}_3(\tilde{\lambda}_3)$ and $\tilde{\sigma}_1(\tilde{\lambda}_1)$ from uniaxial tests on human living tissues in the anisotropic and transverse directions (Martins et al 2011 [23]). Calculated stresses using the transversely isotropic spline-based model. The damage process is not considered.

6 Conclusions

In this paper we have presented a handy procedure to model large strain elasticity. The procedure is based on a piece-wise spline interpolation from which a stored energy function is derived. Since the model is based on a stored energy function, the behaviour is truly elastic in the sense that the strains and the introduced energy are fully recovered when the external actions cease, so no energy dissipation takes place.

The procedure is based on the idea from Sussman and Bathe for isotropic materials, but it is generalized to transversely isotropic materials and to orthotropic materials through an energy decomposition similar to that of Valanis and Landel that in the general case is an approximation. It is shown that arbitrary stress-strain plots (either from experiments or motivated whatsoever) are almost exactly replicated. Hence possible material instabilities are also captured by the model, but of course a check may be performed if this type of behaviour is not a desired feature. We have shown these properties through different examples.

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