Supersonic regime of the Hall-magnetohydrodynamics resistive tearing instability

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An earlier analysis of the Hall-magnetohydrodynamics (MHD) tearing instability [E. Ahedo and J. J. Ramos, Plasma Phys. Controlled Fusion **51**, 055018 (2009)] is extended to cover the regime where the growth rate becomes comparable or exceeds the sound frequency. Like in the previous subsonic work, a resistive, two-fluid Hall-MHD model with massless electrons and zero-Larmor-radius ions is adopted and a linear stability analysis about a force-free equilibrium in slab geometry is carried out. A salient feature of this supersonic regime is that the mode eigenfunctions become intrinsically complex, but the growth rate remains purely real. Even more interestingly, the dispersion relation remains of the same form as in the subsonic regime for any value of the instability Mach number, provided only that the ion skin depth is sufficiently small for the mode ion inertial layer width to be smaller than the macroscopic lengths, a generous bound that scales like a positive power of the Lundquist number. © 2012 American Institute of Physics. [http://dx.doi.org/10.1063/1.4739787]

I. INTRODUCTION

Magnetic reconnection driven by the tearing instability accounts for many important phenomena in space and laboratory plasmas. Numerous studies have been devoted to this topic, and the pioneering single-fluid theory[1] has been generalized to many other, more detailed physical models. These range from earlier kinetic[2–5] and two-fluid[6–9] analyses, to recent works that use the gyrokinetic formalism.[10,11] Also recently, the development of large scale two-fluid simulation codes[12,13] has renewed the interest in accurate two-fluid analytic results that could be used for code verification. Motivated by this, Refs. 14 and 15 revisited the simplest extension of single-fluid resistive magnetohydrodynamics (MHD), i.e., the two-fluid Hall-MHD model, deriving new analytic dispersion relations that apply to numerically relevant intermediate parameter regimes between the asymptotic limits where the classic results[1,3,6] hold. These novel intermediate dispersion relations were used in a successful benchmark[16] of the NIMROD code.[12] However, neither Ref. 14 nor Ref. 15 considered the parameter regime where the Hall-MHD tearing growth rate is comparable to the sound frequency: Ref. 14 considered separately the subsonic and hypersonic (zero-β) regimes, whereas the otherwise more comprehensive Ref. 15 considered only the subsonic regime. Tearing mode growth rates comparable or higher than the sound frequency can occur only at extremely low values of the ratio β between the squared sound and Alfven velocities, but there is merit in the somehow academic study of this regime. On the one hand, results here provide grounds for additional verification tests of the numerical codes, with the prominent new feature that the mode eigenfunctions become intrinsically complex. On the other hand, because of its apparent simplicity, the zero-β limit of the Hall-MHD tearing instability has long been the subject of studies, with conflicting results reported in the literature and

no clear-cut resolution of the issue to date. In this regard, both the early Ref. 8 and the more recent Ref. 17 claim a transition from the single-fluid dispersion relation to the electron-MHD dispersion relation for $\beta = 0$ and sufficiently large ion skin depth d_i, but Ref. 14 points out that, before the zero-β equations could transition from the single-fluid regime to the electron-MHD regime, the mode ion inertial width would become comparable to the macroscopic length scale thus invalidating the boundary layer asymptotics the analytic results are based on.

In this work, we carry out a detailed normal mode analysis of the two-fluid tearing instability, applicable to its supersonic regime. Our emphasis is in allowing general values of the Mach number defined as $M = \gamma/(k c_s)$, where γ and k are the instability growth rate and periodicity wavenumber and c_s is the sound velocity, but we keep the rest of the model as simple as possible for the sake of clarity and in order to facilitate the comparison with Refs. 8, 14, and 17. Thus, we adopt the Hall-MHD model with zero-Larmor-radius ions and polytropic closures considered in Ref. 15, in its resistive and massless electron version. In the same spirit, we order the dimensionless instability index Δ'/k and the ratio between the "guide" and "transverse" components of the equilibrium magnetic field as comparable to unity. The Hall parameter $\alpha = k d_i$ is constrained only by the condition that the tearing mode eigenfunction width be much smaller than the macroscopic equilibrium profile width L, so that the standard multiple-scale asymptotic matching technique can be used. Under these assumptions, we will obtain an exact solution for the maximal ordering of the Mach number $M \sim 1$ that matches the known subsonic solution[15] in its $M \to 0$ limit and provides a well defined hypersonic or zero-β result in its $M \to \infty$ limit. A salient feature of this solution is that, for $M \sim 1$, the eigenfunctions are intrinsically complex but the growth rate remains purely real. More interestingly, even though the form of the eigenfunctions depends on M, the

instability growth rate remains of the same form as in the subsonic regime for any value of M, subject only to the general condition that the mode width be smaller than the macroscopic length scale, something which is guaranteed if kd_i is below some bound that is always greater than $S^{1/3}$, the rather large cubic root of the Lundquist number. So we find that, for Mach numbers comparable or greater than unity, the electron-MHD regime cannot be reached within the validity range of the multiple-scale asymptotic matching analysis and that, in particular, the strictly $\beta = 0$ limit is always in the single-fluid dispersion relation regime.

II. GENERAL MODEL EQUATIONS

Following the physical model and the notation of Ref. 15 as closely as possible, we consider the system

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}, \tag{1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2}$$

$$\mu_0 \boldsymbol{j} = \nabla \times \boldsymbol{B}, \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{v} = 0, \tag{4}$$

$$\rho \frac{D\boldsymbol{v}}{Dt} = \boldsymbol{j} \times \boldsymbol{B} - \nabla(p_i + p_e), \tag{5}$$

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} + \eta \boldsymbol{j} + \frac{1}{en}(\boldsymbol{j} \times \boldsymbol{B} - \nabla p_e), \tag{6}$$

$$p_s n^{-\Gamma_s} = \text{const} \quad (s = i, e), \tag{7}$$

with a static, force-free equilibrium of constant density and temperatures,

$$\rho_0, p_{s0} = \text{const}, \quad \boldsymbol{j}_0 \times \boldsymbol{B}_0 = 0, \quad \boldsymbol{v}_0 = 0, \quad \boldsymbol{E}_0 = \eta \boldsymbol{j}_0 \simeq 0. \tag{8}$$

The emphasis in this model is to retain the Hall-MHD effects in the generalized Ohm's law and the plasma compressibility effects. The most restrictive assumptions are the polytropic equation of state, which would apply to high collisionality regimes, and the neglect of finite ion Larmor radius effects, which would apply to low ion beta.

A one-dimensional equilibrium slab geometry is also assumed, with the inhomogeneity along the x direction and the magnetic field of the form $\boldsymbol{B}_0 = B_{0y}(x)\boldsymbol{e}_y + B_{0z}(x)\boldsymbol{e}_z$, so that the force-free condition requires that its magnitude B_0 be constant. Typical sheet pinch profiles are

$$B_{0y}(x) = \epsilon_B B_0 \tanh\frac{x}{L}, \qquad B_{0z}(x) = [B_0^2 - B_{0y}^2(x)]^{1/2}, \tag{9}$$

and we treat ϵ_B as a parameter of order unity so as to allow for arbitrary guide fields.

Linearizing the above system for normal mode perturbations independent of z, with periodic spatial variation along the y direction and growth rate γ,

$$f(x, y, t) - f_0(x) = f_1(x)\exp(\gamma t + iky), \tag{10}$$

we obtain

$$(c_s^2 k^2 + \gamma^2)\rho_1 = -\gamma^2 \rho_0 \xi' - \frac{kB_{0z}}{\mu_0}\left(kB_{1z} + \frac{B_{0y}'}{B_{0z}}iB_{1x}\right), \tag{11}$$

$$\gamma^2 \mu_0 (\rho_0 \nabla^2 \xi + \rho_1') = ik(B_{0y}\nabla^2 B_{1x} - B_{0y}''B_{1x}), \tag{12}$$

$$\eta \nabla^2 B_{1x} = \gamma \mu_0 (B_{1x} - ikB_{0y}\xi) - \frac{m_i}{e\rho_0}kB_{0y}\left(kB_{1z} + \frac{B_{0y}'}{B_{0z}}iB_{1x}\right), \tag{13}$$

$$\eta \gamma \rho_0 \nabla^2 B_{1z} = kB_{0y}^2\left(kB_{1z} + \frac{B_{0y}'}{B_{0z}}iB_{1x}\right) - \gamma^2 \mu_0(B_{0z}\rho_1 - \rho_0 B_{1z})$$
$$- \frac{m_i}{e}\gamma(B_{0y}\nabla^2 B_{1x} - B_{0y}''B_{1x}) - \gamma^2 \mu_0 \rho_0 \frac{B_{0y}'}{B_{0z}}B_{0y}\xi. \tag{14}$$

Here, we have defined the squared sound velocity $c_s^2 = m_i^{-1}\sum_s \Gamma_s T_{s0}$ and the Lagrangian displacement variable $\xi = \gamma^{-1}v_{1x}$; the prime ($'$) denotes the derivative with respect to x and the Laplacian operator is $\nabla^2 = d^2/dx^2 - k^2$. This form of the linearized Eqs. (11)–(14) was given in Ref. 15, except that the last term of Eq. (14) (which is inconsequential in the subsonic analysis) was omitted there. Further algebraic reduction is carried out by eliminating the perturbed density ρ_1 and introducing the perturbed magnetic field variable,

$$Q = B_{1z} + \frac{B_{0y}'}{kB_{0z}}iB_{1x}, \tag{15}$$

to be used primarily instead of B_{1z}. Using also the equilibrium relation $B_{0y}^2(x) + B_{0z}^2(x) = B_0^2$, and the definitions of the Alfven velocity $c_A = B_0(\mu_0\rho_0)^{-1/2}$ and the ion skin depth $d_i = m_i(e^2\mu_0\rho_0)^{-1/2}$, we arrive at the *still exact* linearized system for the variables (B_{1x}, ξ, Q)

$$\frac{\eta}{\mu_0}\nabla^2 B_{1x} - \gamma B_{1x} = -B_{0y}\left(i\gamma k\xi + k^2 d_i c_A \frac{Q}{B_0}\right), \tag{16}$$

$$\frac{\gamma^2}{c_A^2}\left(\frac{c_s^2}{\gamma^2 + k^2 c_s^2}\xi'' - \xi\right) = \frac{\gamma^2}{\gamma^2 + k^2 c_s^2}\left(\frac{B_{0z}Q}{B_0^2}\right)' + \frac{i}{kB_0^2}(B_{0y}\nabla^2 B_{1x} - B_{0y}''B_{1x}), \tag{17}$$

$$\frac{\eta\gamma}{\mu_0}\nabla^2 Q - \left[\gamma^2 + \frac{k^2 c_A^2}{\gamma^2 + k^2 c_s^2}\left(\gamma^2 + k^2 c_s^2\frac{B_{0y}^2}{B_0^2}\right) - i\gamma kd_i c_A\frac{B_{0y}B_{0y}'}{B_0 B_{0z}}\right]Q$$
$$= \frac{\gamma^4}{\gamma^2 + k^2 c_s^2}B_{0z}\xi' - \frac{\gamma d_i c_A}{B_0}(B_{0y}\nabla^2 B_{1x} - B_{0y}''B_{1x})$$
$$+ \frac{i\eta\gamma}{\mu_0 k}\left[\left(\frac{B_{0y}'}{B_{0z}}\right)''B_{1x} + 2\left(\frac{B_{0y}'}{B_{0z}}\right)'B_{1x}'\right]. \tag{18}$$

Equation (18) follows from a combination of Eqs. (13)–(15) where the last term of Eq. (14) is canceled by the term proportional to ξ from the right-hand-side of Eq. (13).

This general form of the linearized resistive-Hall-MHD equations shows clearly its marginally stable ideal-MHD solution, applicable to the "outer" region away from $x = 0$:

$$\eta = 0, \quad \gamma = 0, \quad Q = 0, \quad B_{0y}\nabla^2 B_{1x} - B''_{0y}B_{1x} = 0, \quad (19)$$

whose discontinuity at $x = 0$ defines the tearing stability index, positive for an unstable mode,

$$\Delta' = \frac{B'_{1x}\big|_{x=0^-}^{x=0^+}}{B_{1x}(0)}. \quad (20)$$

For the equilibrium profiles of Eq. (9), we have $\Delta' = 2k[(kL)^{-2} - 1]$.

With little loss of generality, we can write an alternative form of the system (16)–(18) by just assuming $0 < \Delta'/k \sim 1$, thus $k \sim 1/L$, and anticipating that the growth rate will scale as a fractional (smaller than 1) power of the resistivity, which allows to neglect $\eta k^2/\mu_0$ compared to γ. Substituting also Eq. (16) for $\nabla^2 B_{1x}$ in Eqs. (17) and (18), the result is

$$B''_{1x} = -\frac{\mu_0 \gamma}{\eta}\left[B_{0y}\left(ik\xi + \frac{k^2 d_i c_A}{\gamma}\frac{Q}{B_0}\right) - B_{1x}\right], \quad (21)$$

$$\frac{\gamma^2}{c_A^2}\left(\frac{c_s^2}{\gamma^2 + k^2 c_s^2}\xi'' - \xi\right) = \frac{\gamma^2}{\gamma^2 + k^2 c_s^2}\left(\frac{B_{0z}Q}{B_0^2}\right)'$$
$$- \frac{i\mu_0\gamma B_{0y}}{\eta k B_0^2}\left[B_{0y}\left(ik\xi + \frac{k^2 d_i c_A}{\gamma}\frac{Q}{B_0}\right) - B_{1x}\right], \quad (22)$$

$$\frac{\eta\gamma}{\mu_0}Q'' - \left[\gamma^2 + k^2 c_A^2\frac{\gamma^2 B_0^2 + k^2 c_s^2 B_{0y}^2}{(\gamma^2 + k^2 c_s^2)B_0^2} - ik d_i \gamma c_A\frac{B_{0y}B'_{0y}}{B_0 B_{0z}}\right]Q$$
$$= \frac{\mu_0\gamma^2 d_i c_A B_{0y}}{\eta B_0}\left[B_{0y}\left(ik\xi + \frac{k^2 d_i c_A}{\gamma}\frac{Q}{B_0}\right) - B_{1x}\right]$$
$$+ \frac{\gamma^4}{\gamma^2 + k^2 c_s^2}B_{0z}\xi' + \frac{i\eta\gamma}{\mu_0 k}\left[\left(\frac{B'_{0y}}{B_{0z}}\right)''B_{1x} + 2\left(\frac{B'_{0y}}{B_{0z}}\right)'B'_{1x}\right]. \quad (23)$$

III. ASYMPTOTIC THEORY OF THE MODE SINGULAR LAYER

The standard singular perturbation theory of the linear tearing mode retains the non-ideal and inertial terms of the system (21)-(23) only within a microscopic region near $x = 0$, where $\mathbf{k} \cdot \mathbf{B}_0 \simeq 0$. The solution obtained in this singular boundary layer smooths the discontinuity of the ideal-MHD "outer" solution, matching asymptotically its $x < 0$ and $x > 0$ branches. The non-ideal boundary layer must have a width much smaller than the equilibrium current sheet width L, but may include several distinct asymptotic sublayers depending on the plasma parameters. Then, within it, the equilibrium magnetic field components can be approximated by their lowest-order Taylor expansions about $x = 0$

$$B_{0y} \simeq B_0 x/L_B \quad \text{and} \quad B_{0z} \simeq B_0, \quad (24)$$

where, for the profiles of Eq. (9), $L_B = L/\epsilon_B$. Also, for the considered $\Delta'/k \sim 1$, we can use the so-called "constant-ψ approximation" whereby B_{1x} can be taken as constant within

the boundary layer whenever its locally large second derivative is not involved. This way, Eqs. (22) and (23) become a system for (ξ, Q) alone, decoupled from Eq. (21) which enters only in its integrated form across the layer to match the "outer" discontinuity parameter Δ'. Accordingly, the "inner" system for the singular layer becomes

$$\Delta' = \frac{\mu_0\gamma}{\eta}\int_{0^-}^{0^+} dx\left[1 - \frac{B_0 x}{B_{1x}(0)L_B}\left(ik\xi + \frac{k^2 d_i c_A}{\gamma}\frac{Q}{B_0}\right)\right], \quad (25)$$

$$\frac{\gamma^2}{c_A^2}\left(\frac{c_s^2}{\gamma^2 + k^2 c_s^2}\xi'' - \xi\right) = \frac{\gamma^2}{\gamma^2 + k^2 c_s^2}\frac{Q'}{B_0}$$
$$- \frac{i\mu_0\gamma x}{\eta k L_B}\left[\frac{x}{L_B}\left(ik\xi + \frac{k^2 d_i c_A}{\gamma}\frac{Q}{B_0}\right) - \frac{B_{1x}(0)}{B_0}\right], \quad (26)$$

$$\frac{\eta\gamma}{\mu_0}Q'' - \left[\gamma^2 + k^2 c_A^2\frac{\gamma^2 L_B^2 + k^2 c_s^2 x^2}{(\gamma^2 + k^2 c_s^2)L_B^2} - ik d_i \gamma c_A\frac{x}{L_B^2}\right]Q$$
$$= \frac{\mu_0\gamma^2 d_i c_A x}{\eta L_B}\left[\frac{B_0 x}{L_B}\left(ik\xi + \frac{k^2 d_i c_A}{\gamma}\frac{Q}{B_0}\right) - B_{1x}(0)\right]$$
$$+ \frac{\gamma^4}{\gamma^2 + k^2 c_s^2}B_0\xi' + \frac{i\eta\gamma}{\mu_0 k}\left(\frac{B'_{0y}}{B_{0z}}\right)''(0)B_{1x}(0). \quad (27)$$

Next, we express the above system in dimensionless form. Our basic dimensionless parameters are as follows: the inverse Lundquist number $S^{-1} = \epsilon_\eta = \eta k/(\mu_0 c_A) \ll 1$ which is the fundamental expansion parameter in the theory; the Hall parameter $\alpha = k d_i$; the ratio between the squared sound and Alfven velocities $\beta = c_s^2/c_A^2$; and the Alfven-normalized growth rate $\epsilon_\gamma = \gamma/(k c_A)$. Then, our equations will determine the dispersion relation in the dimensionless form $\epsilon_\gamma(\epsilon_\eta, \alpha, \beta, kL_B, \Delta'/k)$. A characteristic length in the microscopic layer analysis is

$$d_0 = (\epsilon_\gamma \epsilon_\eta)^{1/4}(L_B/k)^{1/2} \ll L_B, \quad (28)$$

which is the layer width in the cases where the "inner" tearing mode eigenfunction varies on a single length scale. In the cases where the eigenfunction varies on two distinct microscopic scales, with an "innermost" diffusive layer of width d_1 and an "intermediate" ion inertial layer of width d_2, it turns out that $d_1 \ll d_2 \ll L_B$ and $d_0 = (d_1 d_2)^{1/2}$.[15] We use d_0 to define the dimensionless scaled variables

$$\bar{x} = \frac{x}{d_0}, \quad \bar{\xi} = \frac{d_0 B_0}{L_B B_{1x}(0)}ik\xi, \quad \bar{Q} = \frac{d_0\alpha}{L_B\epsilon_\gamma}\frac{Q}{B_{1x}(0)}. \quad (29)$$

In terms of these, the dimensionless form of Eq. (25) is

$$k^{-1}\Delta' = \epsilon_\gamma^{5/4}\epsilon_\eta^{-3/4}(kL_B)^{1/2}\int_{-\infty}^{+\infty} d\bar{x}\left[1 - \bar{x}(\bar{\xi} + \bar{Q})\right]. \quad (30)$$

A set of parameters alternative to $(\epsilon_\eta, \alpha, \beta)$ is constituted by: the relative thickness $\delta = d_0/L_B = (\epsilon_\gamma\epsilon_\eta)^{1/4}(kL_B)^{-1/2} \ll 1$; the Mach number $M = \gamma/(kc_s) = \epsilon_\gamma\beta^{-1/2}$; and the scaled Hall parameter $\bar{\sigma} = \alpha\delta\epsilon_\gamma^{1/2}\epsilon_\eta^{-1/2}$. In terms of these, Eqs. (26) and (27) take the dimensionless form

$$\frac{1}{M^2+1}\frac{d^2\bar{\xi}}{d\bar{x}^2} - \delta^2 k^2 L_B^2 \bar{\xi} = \frac{M^2}{M^2+1}\frac{i}{\bar{\sigma}}\frac{d\bar{Q}}{d\bar{x}} + \bar{x}^2(\bar{\xi}+\bar{Q}) - \bar{x},$$
(31)

$$\delta^2 \frac{d^2\bar{Q}}{d\bar{x}^2} - \left(\epsilon_\gamma^2 + \frac{M^2+\delta^2\bar{x}^2}{M^2+1} - i\delta^2\bar{\sigma}\bar{x}\right)\bar{Q}$$
$$= -\frac{M^2}{M^2+1}i\bar{\sigma}\frac{d\bar{\xi}}{d\bar{x}} + \bar{\sigma}^2[\bar{x}^2(\bar{\xi}+\bar{Q}) - \bar{x}]$$
$$+ i\frac{\delta^6\bar{\sigma}}{\epsilon_\gamma^2}L_B^3\left(\frac{B'_{0y}}{B_{0z}}\right)''(0).$$
(32)

The analysis of Eqs. (30)–(32) to be pursued assumes $kL_B \sim \Delta'/k \sim 1$ (consistent with a general guide field $\epsilon_B \sim 1$) and will cover the whole range of M and the range of $\bar{\sigma}$ such that the singular layer is effectively thin, i.e., $\delta \lesssim d_2/L_B \ll 1$.

A. Subsonic regimes

In order to facilitate the comparison with the finite-Mach-number results to be shown next, it is worth summarizing the subsonic results of Ref. 15. That work singled out the orderings $\alpha \sim \epsilon_\eta^{1/5}$ and $\beta \sim \epsilon_\eta^{2/5}$, yielding $\epsilon_\gamma \sim \epsilon_\eta^{3/5}$, $\bar{x} \sim \bar{\xi} \sim \bar{Q} \sim 1$, and $\delta \sim M \sim \bar{\sigma} \sim \epsilon_\eta^{2/5}$, as defining the characteristic regime where the leading-order subsonic equations take their most general form. Applying these orderings to Eqs. (31) and (32), we get

$$\frac{d^2\bar{\xi}}{d\bar{x}^2} = \bar{x}^2(\bar{\xi}+\bar{Q}) - \bar{x},$$
(33)

$$\frac{d^2\bar{Q}}{d\bar{x}^2} - (\tau+\bar{x}^2)\bar{Q} = \sigma^2[\bar{x}^2(\bar{\xi}+\bar{Q}) - \bar{x}],$$
(34)

which is the system that was investigated in Ref. 15 and where the relevant subsonic parameters are $\sigma = \bar{\sigma}/\delta$ and $\tau = M^2(1+\beta)/\delta^2$. The different subsonic regimes of the Hall-MHD tearing instability are obtained from special asymptotic subsets of this system and are represented schematically in Fig. 1. There, the different parametric regions (PRs) corresponding to the different asymptotic forms of the subsonic dispersion relation are shown in a plane spanned by the basic primary parameters $\alpha = kd_i$ and $\beta = c_s^2/c_A^2$. The parametric region labeled PR0 corresponds to the general form of the subsonic regimes. Regions PR1, PR3, and PR5 are the domains of validity of the classic tearing mode dispersion relations in the single-fluid,[1] electron-MHD,[6] and ion-sound-gyroradius-width (also called semicollisional)[3] regimes, respectively. Regions PR2 and PR6 are the intermediate domains where the dispersion relations newly derived in Ref. 15 apply. Region PR4 is the validity domain of the intermediate dispersion relation derived in Ref. 14 for large guide fields and shown to apply also to arbitrary guide fields in Ref. 15. Two general conditions establish overall applicability bounds for these results, excluding the gray region in the diagram of Fig. 1. The first one is the subsonic condition $M \ll 1$. The second one is the condition that, in the electron-MHD regime where the
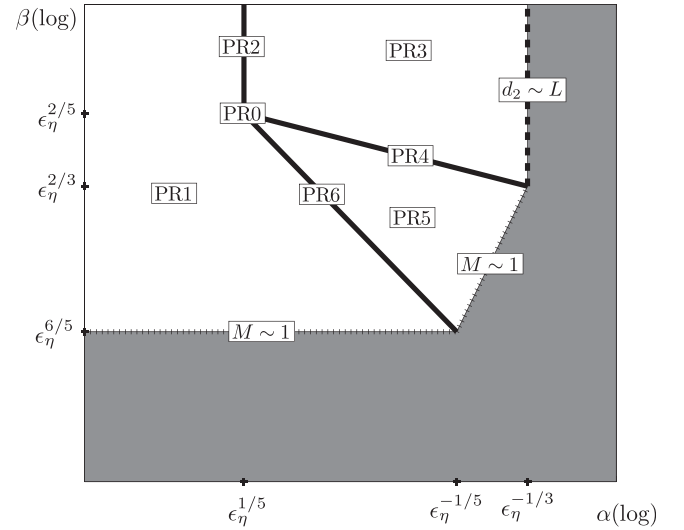

FIG. 1. Sketch of the locations of the seven asymptotic parametric regions (from PR0 to PR6) for the subsonic range of the tearing instability, as discussed in Ref. 15. That analysis excluded the grey region, which is limited by either $M \sim 1$ (dotted line), corresponding to the transonic range, or $d_2 \sim L$ (dashed line), marking the bound for a thin intermediate inertial sublayer.

mode singular layer splits into two different scale sublayers, the broader ion inertial sublayer should still be much narrower than the macroscopic equilibrium length scale, $d_2 \ll L$. The latter condition was not discussed in Ref. 15 but limits the applicability of the conventional electron-MHD dispersion relation to $\alpha = kd_i \ll \epsilon_\eta^{-1/3} \equiv S^{1/3}$. Except for this $d_2 \ll L$ condition, the domains defined in our subsonic diagram agree with the subsonic part of the diagram put forward in Ref. 17, which did not consider the $d_2 \ll L$ condition either and showed only the range of the intermediate regions PR2, PR4, and PR6 without obtaining the actual dispersion relations for them.

Of particular relevance to the finite-Mach-number analysis to follow is the low-β, subsonic region PR6. This region (whose dispersion relation covers asymptotically those of PR1 and PR5 too) is characterized by $\alpha^2\beta \sim \epsilon_\eta^{4/5}$ and $\epsilon_\eta^{6/5} \ll \beta \ll \epsilon_\eta^{2/5}$, yielding $\epsilon_\gamma \sim \epsilon_\eta^{3/5}$, $\bar{x} \sim \bar{\xi} \sim \bar{Q} \sim 1$, and $\delta \sim \epsilon_\eta^{2/5} \ll M \sim \bar{\sigma} \ll 1$, hence $\sigma^2 \sim \tau \gg 1$. Under these orderings, the singular layer system reduces to

$$\bar{Q} = -\frac{\sigma^2}{\tau}\frac{d^2\bar{\xi}}{d\bar{x}^2}, \qquad \left(1+\frac{\sigma^2}{\tau}\bar{x}^2\right)\frac{d^2\bar{\xi}}{d\bar{x}^2} - \bar{x}^2\bar{\xi} + \bar{x} = 0, \quad (35)$$

where $\sigma^2/\tau \to \bar{\sigma}^2/M^2$ since $\beta \ll \epsilon_\eta^{2/5}$ here.

B. Sonic-supersonic regimes

For the Mach numbers comparable or greater than unity that the present work is mainly concerned about, the characteristic regime where the leading-order equations take their most general form corresponds to the extension to $M \sim 1$ of the orderings in the above PR6: $\bar{x} \sim \bar{\xi} \sim \bar{Q} \sim 1$ and $\delta \ll M \sim \bar{\sigma} \sim 1$. Applying these orderings to Eqs. (31) and (32), they become

$$\frac{1}{M^2+1}\frac{d^2\bar{\xi}}{d\bar{x}^2} = \bar{x}^2(\bar{\xi}+\bar{Q}) - \bar{x} + \frac{M^2}{M^2+1}\frac{i}{\bar{\sigma}}\frac{d\bar{Q}}{d\bar{x}},$$
(36)

$$-\frac{M^2}{M^2+1}\bar{Q} = \bar{\sigma}^2[\bar{x}^2(\bar{\bar{\xi}}+\bar{Q})-\bar{x}] - \frac{M^2}{M^2+1}i\bar{\sigma}\frac{d\bar{\bar{\xi}}}{d\bar{x}}. \quad (37)$$

Clearly, the asymptotic approximation leading to this system (36) and (37) remains valid both for $M \gg 1$ (i.e., in the hypersonic or $\beta \to 0$ limit) and for $\bar{\sigma} \ll 1$. It is also valid for large values of $\bar{\sigma}$, provided the magnitudes of the normalized eigenfunctions $(\bar{\bar{\xi}}, \bar{Q})$ and their derivatives remain such that the neglected terms proportional to δ^2 in Eqs. (31) and (32) remain subdominant. The explicit solution to be obtained next will confirm that this is indeed the case, if we restrict ourselves to the validity domain of our multiple-scale singular perturbation analysis requiring the width d_2 of the ion inertial layer (when it becomes broader than the diffusive layer) to remain much smaller than the macroscopic equilibrium width L.

It should be pointed out that, whereas the subsonic system (33) and (34) for the normalized variables $(\bar{\bar{\xi}}, \bar{Q})$ is purely real, the finite-Mach-number system (36) and (37) is intrinsically complex and gives rise to complex eigenfunctions. However, these have definite parity properties such that the real parts of $\bar{\bar{\xi}}$ and $\bar{Q}$ are odd functions of $\bar{x}$ and their imaginary parts are even. Therefore, when taken to Eq. (30), the contribution of the imaginary parts of $\bar{\bar{\xi}}$ and $\bar{Q}$ to the right-hand-side integral vanishes and the resulting growth rate is always real (it was erroneously stated in Ref. 15 that the complex eigenfunctions would yield complex growth rates in the finite-Mach-number regime).

Equations (36) and (37) with eigenfunctions $\bar{\bar{\xi}} = \bar{\bar{\xi}}_R + i\bar{\bar{\xi}}_I$ and $\bar{Q} = \bar{Q}_R + i\bar{Q}_I$ admit the *exact* integral

$$\bar{Q}_I = -\bar{\bar{\xi}}_I = \bar{\sigma}\frac{d\bar{\bar{\xi}}_R}{d\bar{x}}, \quad (38)$$

$$\bar{Q}_R = -\frac{\bar{\sigma}^2}{M^2}\frac{d^2\bar{\bar{\xi}}_R}{d\bar{x}^2}, \quad \left(1+\frac{\bar{\sigma}^2}{M^2}\bar{x}^2\right)\frac{d^2\bar{\bar{\xi}}_R}{d\bar{x}^2} - \bar{x}^2\bar{\bar{\xi}}_R + \bar{x} = 0. \quad (39)$$

So, besides the parity argument, the imaginary parts of $\bar{\bar{\xi}}$ and $\bar{Q}$ cancel completely in Eq. (30) and do not contribute to the growth rate dispersion relation. Moreover, the real part system (39) is identical to the one that applies in the 6th subsonic asymptotic domain, Eq. (35). Therefore, the dispersion relation for Mach numbers comparable or greater than unity is identical to the subsonic one in PR6 (which also covers PR1 and PR5), as derived in Ref. 15

$$\epsilon_\gamma = \epsilon_\eta^{3/5}\left(\frac{\Delta'^2}{C^2k^3L_B}\right)^{2/5}f_6^{-4/5}\left(\frac{\alpha^2\beta}{\epsilon_\eta^{1/2}\epsilon_\gamma^{1/2}kL_B}\right). \quad (40)$$

Here, $C = 2\pi\Gamma(3/4)/\Gamma(1/4)$ and the numerically calculated function f_6 (Ref. 15) is plotted in Fig. 2. This function is well approximated by the analytic fit

$$f_6(u) \simeq \frac{1-u/4+\pi u^2/20}{1+Cu^{5/2}/20}, \quad (41)$$

which is also shown in Fig. 2. Profiles of the real components ξ_R and Q_R are plotted in Fig. 4 of Ref. 15 for

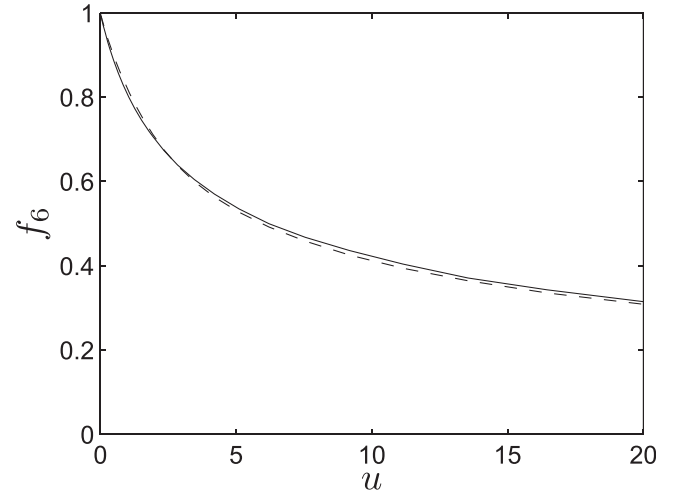FIG. 2. Function $f_6(u)$ (solid line). The dashed line is the approximate fit of f_6 in Eq. (41).

$\bar{\sigma}/M \to \sigma/\tau^{1/2} = 0.2$, 1, and 5. For $\bar{\sigma}/M = $ const, the imaginary components ξ_I and Q_I, Eq. (38), tend to zero for $\bar{\sigma} \ll 1$ in the subsonic regime[15] and satisfy $\xi_R \ll \bar{Q}_I = -\bar{\bar{\xi}}_I \ll Q_R$ for $\bar{\sigma} \gg 1$ in the hypersonic regime.

In conclusion, as we pass through finite values of the Mach number and enter the supersonic regime, the tearing mode eigenfunctions develop imaginary parts, but the growth rate remains purely real and the form of its dispersion relation remains unchanged. The form of this general dispersion relation (40) is the same as in the low-β subsonic regime and depends on the parameters α and β through the combination $\alpha\beta^{1/2} = kd_s$, where $d_s = d_i\beta^{1/2}$ is the ion sound gyroradius. For $\alpha\beta^{1/2} = kd_s \ll \epsilon_\eta^{2/5}$, the argument of f_6 is small and the dispersion relation (40) reduces to the single-fluid one[1] of PR1, independent of kd_s

$$\epsilon_\gamma = \epsilon_\eta^{3/5}\left(\frac{\Delta'^2}{C^2k^3L_B}\right)^{2/5}. \quad (42)$$

For $\alpha\beta^{1/2} = kd_s \gg \epsilon_\eta^{2/5}$, the argument of f_6 is large and the dispersion relation (40) reduces to the so-called semicollisional one[3] of PR5 that scales like $(kd_s)^{2/3}$

$$\epsilon_\gamma = \epsilon_\eta^{1/3}\alpha^{2/3}\beta^{1/3}\left(\frac{\Delta'}{\pi k^2L_B}\right)^{2/3}. \quad (43)$$

In this $kd_s \gg \epsilon_\eta^{2/5}$ region, the tearing eigenfunctions vary on two separate spatial scales forming two sublayers: a broader, non-diffusive one of width $d_2 \sim d_s \gg d_0$, and the innermost diffusive layer of width $d_1 \sim d_0^2/d_s$. The validity of the present multiple-scale singular perturbation analysis requires the broader sublayer width to be much smaller than the macroscopic lengths, that is, $kd_s \ll 1$. Staying within this limit, we verify that the corresponding eigenfunctions and growth rate are such that the δ^2 terms that were neglected to derive the considered supersonic systems (36) and (37) from the general Eqs. (31) and (32) are indeed subdominant. So, our general result (40) is valid throughout the supersonic regime, limited only by the $kd_s \ll 1$ or $\alpha \ll \beta^{-1/2}$ condition for
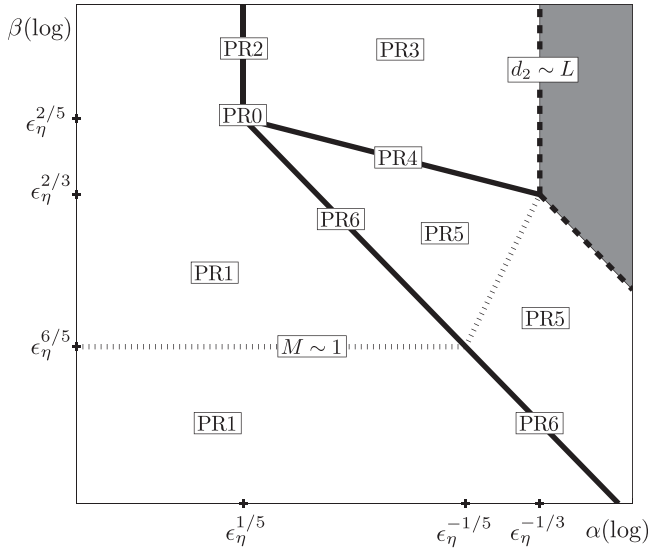
FIG. 3. Sketch of the locations of the seven asymptotic parametric regions (from PR0 to PR6) covering both the subsonic and supersonic ranges of the tearing instability. The dotted line $M \sim 1$ corresponds to the transonic range. The hypersonic or cold limit corresponds to $\beta \to 0$. The grey region, bounded by $d_2 \sim L$, is excluded from the present analysis.

$M \geq 1$, which is a very high upper bound on the Hall parameter, unlikely to be reached in situations of interest.

Finally, as a consistency check, we can verify that the last term of Eq. (14), which was omitted in the subsonic analysis of Ref. 15, was negligible then but becomes relevant in the supersonic regime. A comparison under the parametric orderings of PR6 between that last term and the first, which is the leading one for $M \leq O(1)$, yields

$$\frac{\gamma \mu_0 B'_{0y} B_{0y} \xi}{\eta B_{0z} \nabla^2 B_{1z}} \sim \frac{\alpha \epsilon_\gamma^{3/4}}{\epsilon_\eta^{1/4}(kL_B)^{1/2}} \sim M.$$

IV. CONCLUDING DISCUSSION

The overall diagram that summarizes the Hall-MHD resistive tearing results, including both the subsonic and supersonic regimes is shown in Fig. 3. The dispersion relations (42), (40), and (43) apply, respectively, in the regions PR1, PR6, and PR5, for any value of the Mach number M. For the sake of completeness, it is worth recalling the dispersion relations that apply in the strictly subsonic regions ($M \ll 1$), namely, the electron-MHD region PR3 where[6]

$$\epsilon_\gamma = \epsilon_\eta^{1/2} \alpha^{1/2} \left(\frac{\Delta'^2}{C^2 k^3 L_B} \right)^{1/2}, \qquad (44)$$

the high-β intermediate region PR2 where[15]

$$\epsilon_\gamma = \epsilon_\eta^{3/5} \left(\frac{\Delta'^2}{C^2 k^3 L_B} \right)^{2/5} f_2^{-4/5} \left(\frac{\epsilon_\gamma^{1/2} \alpha}{\epsilon_\eta^{1/2}} \right), \qquad (45)$$

with f_2 the analytic function given in Ref. 15, and the intermediate region PR4 between the d_i and d_s two-fluid regimes where[14]

$$\epsilon_\gamma = \epsilon_\eta^{1/2} \alpha^{1/2} \left(\frac{\Delta'^2}{C^2 k^3 L_B} \right)^{1/2} f_4^{-1} \left(\frac{\epsilon_\gamma kL_B}{\alpha \beta} \right), \qquad (46)$$

with f_4 the analytic function given in Refs. 14 and 15.

The diagram of Fig. 3 indicates also the general validity limit of our analysis, set by the condition that, when there is an ion inertial sublayer broader than the diffusive sublayer, its width d_2 must remain much smaller than the macroscopic equilibrium width L. This excludes the regions $kd_i \geq O(\epsilon_\eta^{-1/3})$ in the subsonic electron-MHD PR3 and $kd_s = kd_i \beta^{1/2} \geq O(1)$ in the supersonic PR5, which are not likely to be reached in situations of interest. Consideration of these excluded regions would necessitate an "outer" treatment of the ion inertial terms together with the ideal ones and a modification of the Δ' definition. The extension of the present analysis to include the finite electron inertia effect is immediate, as shown in Ref. 15.

As discussed above, the validity of our result extends all the way to the hypersonic regime $M \to \infty$ or $\beta \to 0$. In particular, for strictly $\beta = 0$ and any finite α, we are always in the domain of the single-fluid dispersion relation (42) as shown in Fig. 3, reflecting the fact that the argument of the function f_6 of the general dispersion relation (40) vanishes in this case. This hypersonic or zero-β behavior of our result disagrees with the discussion and the corresponding part of the diagram shown in Ref. 17, purporting a transition from single-fluid to electron-MHD through some intermediate regimes for $\beta < S^{-2}$. It disagrees also with Ref. 8, which finds a similar transition from single-fluid to electron-MHD at $\beta = 0$, working with a strictly cold plasma model where the coupling between magnetic pressure and density perturbations is ignored without justification. Our finding is consistent with the observation in Ref. 14 that, before the zero-β tearing layer equations could transition from the single-fluid to the electron-MHD regime, the mode ion inertial width would become comparable to the macroscopic length scale.

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