arXiv:1104.2710v1 [math-ph] 14 Apr 2011

Diffeomorphism-invariant Covariant Hamiltonians of a pseudo-Riemannian Metric and a Linear Connection

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Abstract

Let $M \to N$ (resp. $C \to N$) be the fibre bundle of pseudo-Riemannian metrics of a given signature (resp. the bundle of linear connections) on an orientable connected manifold N. A geometrically defined class of first-order Ehresmann connections on the product fibre bundle $M \times_N C$ is determined such that, for every connection γ belonging to this class and every $\mathbf{Diff}\,N$-invariant Lagrangian density Λ on $J^1(M \times_N C)$, the corresponding covariant Hamiltonian Λ^γ is also $\mathbf{Diff}\,N$-invariant. The case of $\mathbf{Diff}\,N$-invariant second-order Lagrangian densities on $J^2 M$ is also studied and the results obtained are then applied to Palatini and Einstein-Hilbert Lagrangians.

1   Introduction

In Mechanics, the Hamiltonian function attached to a Lagrangian density $\Lambda = L(t, q^i, \dot{q}^i)dt$ on $\mathbb{R} \times TQ$ is given by $H = \dot{q}^i \partial L/\partial \dot{q}^i - L$, but—as it was early observed in [**16**]—this is not an invariant definition if an arbitrary fibred manifold $t\colon E \to \mathbb{R}$ is considered (thus generalizing the notion of an absolute time) instead of the direct product bundle $\mathbb{R} \times Q \to \mathbb{R}$; e.g., see [**7**], [**23**], [**25**] for this point of view. In this case, an Ehresmann connection is needed in order to lift the vector field $\partial/\partial t$ from $\mathbb{R}$ to E, and the Hamiltonian is then defined by applying the Poincaré-Cartan form attached to Λ to the horizontal lift of $\partial/\partial t$.

In the field theory—where no distinguished vector field exists on the base manifold—the need of an Ehresmann connection is even greater, in order to attach a covariant Hamiltonian to each Lagrangian density; e.g., see [**24**, 4.1], [**23**], and the definitions below.

Let $p\colon E \to N$ be an arbitrary fibred manifold over a connected manifold N, $n = \dim N$, $\dim E = m+n$, oriented by $v_n = dx^1 \wedge \cdots \wedge dx^n$. Throughout this paper, Latin (resp. Greek) indices run from 1 to n (resp. m). An Ehresmann connection on a fibred manifold $p\colon E \to N$ is a differential 1-form γ on E taking values in the vertical sub-bundle $V(p)$ such that $\gamma(X) = X$ for every $X \in V(p)$ (e.g., see [**23**], [**24**], [**32**], [**34**]). Once an Ehresmann connection γ is given, a decomposition of vector bundles holds $T(E) = V(p) \oplus \ker \gamma$, where $\ker \gamma$ is called the horizontal sub-bundle determined by γ. In a fibred coordinate system (x^j, y^α) for p, an Ehresmann connection can be written as

$$\gamma = (dy^\alpha + \gamma_j^\alpha dx^j) \otimes \frac{\partial}{\partial y^\alpha}, \quad \gamma_j^\alpha \in C^\infty(E).$$

According to [**24**], the covariant Hamiltonian Λ^γ associated to a Lagrangian density on J^1E, $\Lambda = Lv_n$, $L \in C^\infty(J^1E)$, with respect to γ is the Lagrangian density defined by,

$$(1) \qquad \Lambda^\gamma = \left((p_0^1)^*\gamma - \theta\right) \wedge \omega_\Lambda - \Lambda,$$

where, $p_0^1 \colon J^1E \to J^0E = E$ is the projection mapping, $\theta = \theta^\alpha \otimes \partial/\partial y^\alpha$, $\theta^\alpha = dy^\alpha - y_i^\alpha dx^i$ is the $V(p)$-valued 1-form on J^1E associated with the contact structure, written on a fibred coordinate system (x^i, y^α), and ω_Λ is the Legendre form attached to Λ, i.e., the $V^*(p)$-valued p^1-horizontal $(n-1)$-form on J^1E given by

$$\omega_\Lambda = (-1)^{i-1}\frac{\partial L}{\partial y_i^\alpha} i_{\partial/\partial x^i} v_n \otimes dy^\alpha,$$

where $(x^i, y^\alpha; y_i^\alpha)$ is the coordinate system induced from (x^i, y^α) on the 1-jet bundle and $p^1 \colon J^1E \to N$ is the projection on the base manifold. Locally,

$$(2) \qquad \Lambda^\gamma = \left((\gamma_i^\alpha + y_i^\alpha)\frac{\partial L}{\partial y_i^\alpha} - L\right)dx^1 \wedge \cdots \wedge dx^n.$$

From (1) we obtain the following decomposition of the Poincaré-Cartan form attached to Λ (e.g., see [**17**], [**23**], [**27**]): $\Theta_\Lambda = \theta \wedge \omega_\Lambda + \Lambda = (p_0^1)^*\gamma \wedge \omega_\Lambda - \Lambda^\gamma$.

A diffeomorphism $\Phi \colon E \to E$ is said to be an automorphism of p if there exists $\phi \in \mathrm{Diff}\, N$ such that $p \circ \Phi = \phi \circ p$. The set of such automorphisms is denoted by $\mathrm{Aut}(p)$ and its Lie algebra is identified to the space $\mathrm{aut}(p) \subset \mathfrak{X}(E)$ of p-projectable vector fields on E. Given a subgroup $\mathcal{G} \subseteq \mathrm{Aut}(p)$, a Lagrangian density Λ is said to be $\mathcal{G}$-invariant if $(\Phi^{(1)})^*\Lambda = \Lambda$ for every $\Phi \in \mathcal{G}$, where $\Phi^{(1)} \colon J^1E \to J^1E$ denotes the 1-jet prolongation of Φ. Infinitesimally, the $\mathcal{G}$-invariance equation can be reformulated as $L_{X^{(1)}}\Lambda = 0$ for every $X \in \mathrm{Lie}(\mathcal{G})$, $X^{(1)}$ denoting the 1-jet prolongation of the vector field X.

When a group $\mathcal{G}$ of transformations of E is given, a natural question arises:

- Determine a class—as small as possible— of Ehresmann connections γ such that Λ^γ is $\mathcal{G}$-invariant for every $\mathcal{G}$-invariant Lagrangian density Λ.

Below we tackle this question in the framework of General Relativity, i.e., the group $\mathcal{G}$ is the group of all diffeomorphisms of the ground manifold N acting in a natural way either on the bundle of pseudo-Riemannian metrics $p_M \colon M = M(N) \to N$ of a given signature (n^+, n^-), $n^+ + n^- = n$, or on the product bundle $p \colon M \times_N C \to N$, where $p_C \colon C = C(N) \to N$ is the bundle of linear connections on N. Namely, we solve the following two problems:

(P): Determine a class—as small as possible— of Ehresmann connections γ such that for every $\operatorname{Diff}N$-invariant first-order Lagrangian density Λ on the bundle $J^1(M \times_N C)$, the corresponding covariant Hamiltonian Λ^γ is also $\operatorname{Diff}N$-invariant.

Similarly to the problem **(P)**, we formulate the corresponding problem on $J^2 M$ as follows:

(P2): Determine a class of second-order Ehresmann connections γ^2 on M such that for every $\operatorname{Diff}N$-invariant second-order Lagrangian density Λ on the bundle $J^2 M$, the corresponding covariant Hamiltonian Λ^{γ^2}—defined in (42)—is also $\operatorname{Diff}N$-invariant.

Essentially, a class of first-order Ehresmann connections on the bundle $M \times_N C$ is obtained, defined by the conditions (C_M) and (C_C) below (see Propositions 3.4 and 3.5), solving the problem **(P)**. This class of connections also helps to solve **(P2)** by means of a natural isomorphism between $J^1 M$ and $M \times_N C^{\mathrm{sym}}$, where C^{sym} denotes the sub-bundle of symmetric connections on N (cf. Theorem 4.1). Finally, this approach is applied to Palatini and Einstein-Hilbert Lagrangians ([**3**], [**4**]), obtaining results compatible with their usual Hamiltonian formalisms.

2   Invariance under diffeomorphisms

2.1   Preliminaries

2.1.1   Jet-bundle notations

Let $p^k \colon J^k E \to N$ be the k-jet bundle of local sections of an arbitrary fibred manifold $p \colon E \to N$, with projections $p_l^k \colon J^k E \to J^l E$, $p_l^k(j_x^k s) = j_x^l s$, for $k \geq l$, $j_x^k s$ denoting the k-jet at x of a section s of p defined on a neighbourhood of $x \in N$.

A fibred coordinate system (x^i, y^α) on V induces a coordinate system (x^i, y_I^α), $I = (i_1, \ldots, i_N) \in \mathbb{N}^n$, $0 \leq |I| = i_1 + \cdots + i_N \leq r$, on $(p_0^r)^{-1}(V) = J^r V$ as follows: $y_I^\alpha(j_x^r s) = (\partial^{|I|}(y^\alpha \circ s)/\partial x^I)(x)$, with $y_0^\alpha = y^\alpha$.

Every morphism $\Phi \colon E \to E'$ whose associated map $\phi \colon N \to N'$ is a diffeomorphism, induces a map

(3)
$$\Phi^{(r)} \colon J^r E \to J^r E',$$
$$\Phi^{(r)}(j_x^r s) = j_{\phi(x)}^r(\Phi \circ s \circ \phi^{-1}).$$

If Φ_t is the flow of a vector field $X \in \mathrm{aut}(p)$, then $\Phi_t^{(r)}$ is the flow of a vector field $X^{(r)} \in \mathfrak{X}(J^r E)$, called the infinitesimal contact transformation of order r associated to the vector field X. The mapping

$$\mathrm{aut}(p) \ni X \mapsto X^{(r)} \in \mathfrak{X}(J^r E)$$

is an injection of Lie algebras, namely, one has

$$(\lambda X + \mu Y)^{(r)} = \lambda X^{(r)} + \mu Y^{(r)},$$
$$[X, Y]^{(r)} = [X^{(r)}, Y^{(r)}],$$
$$\forall \lambda, \mu \in \mathbb{R}, \ \forall X, Y \in \mathrm{aut}(p).$$

In particular, for $r = 1$,

$$X = u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha}, \quad u^i \in C^\infty(N), v^\alpha \in C^\infty(E),$$

$$X^{(1)} = u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha} + v_i^\alpha \frac{\partial}{\partial y_i^\alpha}, \quad v_i^\alpha = \frac{\partial v^\alpha}{\partial x^i} + y_i^\beta \frac{\partial v^\alpha}{\partial y^\beta} - y_k^\alpha \frac{\partial u^k}{\partial x^i}.$$

2.1.2 Coordinates on $M(N)$, $F(N)$, $C(N)$

Every coordinate system (x^i) on an open domain $U \subseteq N$ induces the following coordinate systems:

1) (x^i, y_{jk}) on $(p_M)^{-1}(U)$, where $p_M \colon M \to N$ is the bundle of metrics of a given signature, and the functions $y_{jk} = y_{kj}$ are defined by,

(4) $$g_x = \sum_{i \le j} y_{ij}(g_x)(dx^i)_x \otimes (dx^j)_x, \ \forall g_x \in (p_M)^{-1}(U).$$

2) (x^i, x_j^i) on $(p_F)^{-1}(U)$, where $p_F \colon F(N) \to N$ is the bundle of linear frames on N, and the functions x_j^i are defined by,

$$u = \left((\partial/\partial x^1)_x, \ldots, (\partial/\partial x^n)_x \right) \cdot \left(x_j^i(u) \right), \ x = p_F(u), \forall u \in (p_F)^{-1}(U),$$

or equivalently,

(5) $$u = (X_1, \ldots, X_N) \in F_x(N), \ X_j = x_j^i(u) \left(\frac{\partial}{\partial x^i} \right)_x, \quad 1 \le j \le n.$$

3) (x^i, A_{kl}^j) on $(p_C)^{-1}(U)$, where $p_C \colon C \to N$ is the bundle of linear connections on N, and the functions A_{kl}^j are defined as follows. We first recall some basic facts. Connections on $F(N)$ (i.e., linear connections of N) are the splittings of the Atiyah sequence (cf. [**2**]),

$$0 \to \mathrm{ad} F(N) \to T_{Gl(n,\mathbb{R})} F(N) \xrightarrow{(p_F)_*} TN \to 0,$$

where
a) $\mathrm{ad} F(N) = T^* N \otimes TN$ is the adjoint bundle,

b) $T_{Gl(n,\mathbb{R})}(F(N)) = T(F(N))/Gl(n,\mathbb{R})$, and

c) $\mathrm{gau}F(N) = \Gamma(N, \mathrm{ad}F(N))$ is the gauge algebra of $F(N)$.

We think of $\mathrm{gau}F(N)$ as the 'Lie algebra' of the gauge group $\mathrm{Gau}F(N)$. Moreover, $p_C\colon C \to N$ is an affine bundle modelled over the vector bundle $\otimes^2 T^*N \otimes TN$. The section of p_C induced tautologically by the linear connection Γ is denoted by $s_\Gamma\colon N \to C$. Every $B \in \mathfrak{gl}(n,\mathbb{R})$ defines a one-parameter group $\varphi_t^B\colon U \times Gl(n,\mathbb{R}) \to U \times Gl(n,\mathbb{R})$ of gauge transformations by setting (cf. [**5**]), $\varphi_t^B(x, \Lambda) = (x, \exp(tB)\cdot\Lambda)$. Let us denote by $\bar{B} \in \mathrm{gau}(p_F)^{-1}(U)$ the corresponding infinitesimal generator. If (E_j^i) is the standard basis of $\mathfrak{gl}(n,\mathbb{R})$, then $\bar{E}_j^i = \sum_{h=1}^n x_h^j \partial/\partial x_h^i$, for $i, j = 1, \ldots, n$, is a basis of $\mathrm{gau}(p_F)^{-1}(U)$. Let $\tilde{E}_j^i = \bar{E}_j^i \bmod G$ be the class of $\bar{E}_j^i$ on $\mathrm{ad}F(N)$. Unique smooth functions A_{jk}^i on $(p_C)^{-1}(U)$ exist such that,

(6)
$$s_\Gamma\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^j} - (A_{jk}^i \circ \Gamma)\tilde{E}_k^i$$
$$= \frac{\partial}{\partial x^j} - (A_{jk}^i \circ \Gamma)x_h^k \frac{\partial}{\partial x_h^i},$$

for every s_Γ and $A_{jk}^i(\Gamma_x) = \Gamma_{jk}^i(x)$, where Γ_{jk}^i are the Christoffel symbols of the linear connection Γ in the coordinate system (x^i), see [**20**, III, Poposition 7.4].

2.2   Natural lifts

Let $f_M\colon M \to M$, cf. [**30**] (resp. $\tilde{f}\colon F(N) \to F(N)$, cf. [**20**, p. 226]) be the natural lift of $f \in \mathrm{Diff}N$ to the bundle of metrics (resp. linear frame bundle); namely $f_M(g_x) = (f^{-1})^*g_x$ (resp. $\tilde{f}(X_1, \ldots, X_N) = (f_*X_1, \ldots, f_*X_N)$, where $(X_1, \ldots, X_N) \in F_x(N)$); hence $p_M \circ f_M = f \circ p_M$ (resp. $p_F \circ \tilde{f} = f \circ p_F$), and $f_M\colon M \to M$ (resp. $\tilde{f}\colon F(N) \to F(N)$) have a natural extension to jet bundles $f_M^{(r)}\colon J^r(M) \to J^r(M)$ (resp. $\tilde{f}^{(r)}\colon J^r(FN) \to J^r(FN)$) as defined in the formula (3), i.e.,

$$f_M^{(r)}(j_x^r g) = j_{f(x)}^r(f_M \circ g \circ f^{-1}) \quad (\text{resp. } \tilde{f}^{(r)}(j_x^r s) = j_{f(x)}^r(\tilde{f} \circ s \circ f^{-1})).$$

As $\tilde{f}$ is an automorphism of the principal $Gl(n,\mathbb{R})$-bundle $F(N)$, it acts on linear connections by pulling back connection forms, i.e., $\Gamma' = \tilde{f}(\Gamma)$ where $\omega_{\Gamma'} = (\tilde{f}^{-1})^*\omega_\Gamma$ (see [**20**, II, Proposition 6.2-(b)], [**5**, 3.3]). Hence, there exists a unique diffeomorphism $\tilde{f}_C\colon C \to C$ such that,

1) $p_C \circ \tilde{f}_C = f \circ p_C$, and

2) $\tilde{f}_C \circ s_\Gamma = s_{\tilde{f}(\Gamma)}$ for every linear connection Γ.

If f_t is the flow of a vector field $X \in \mathfrak{X}(N)$, then the infinitesimal generator of $(f_t)_M$ (resp. $\tilde{f}_t$, resp. $(\tilde{f}_t)_C$) in $\mathrm{Diff}\, M$ (resp. $\mathrm{Diff}\, F(N)$, resp. $\mathrm{Diff}\, C$) is denoted by X_M (resp. $\tilde{X}$, resp. $\tilde{X}_C$) and the following Lie-algebra homomorphisms are obtained:

$$
\begin{cases}
\mathfrak{X}(N) \to \mathfrak{X}(M), & X \mapsto X_M \\
\mathfrak{X}(N) \to \mathfrak{X}(F(N)), & X \mapsto \tilde{X} \\
\mathfrak{X}(N) \to \mathfrak{X}(C), & X \mapsto \tilde{X}_C
\end{cases}
$$

If $X = u^i \partial/\partial x^i \in \mathfrak{X}(N)$ is the local expression for X, then

1) From [**30**, eqs. (2)–(4)] we know that the natural lift of X to M is given by,

$$
X_M = u^i \frac{\partial}{\partial x^i} - \sum_{i \le j} \left(\frac{\partial u^h}{\partial x^i} y_{hj} + \frac{\partial u^h}{\partial x^j} y_{ih} \right) \frac{\partial}{\partial y_{ij}} \in \mathfrak{X}(M).
$$

and its 1-jet prolongation,

$$
X_M^{(1)} = u^i \frac{\partial}{\partial x^i} - \sum_{i \le j} \left(\frac{\partial u^h}{\partial x^i} y_{hj} + \frac{\partial u^h}{\partial x^j} y_{hi} \right) \frac{\partial}{\partial y_{ij}}
$$

$$
- \sum_{i \le j} \left(\frac{\partial^2 u^h}{\partial x^i \partial x^k} y_{hj} + \frac{\partial^2 u^h}{\partial x^j \partial x^k} y_{hi} + \frac{\partial u^h}{\partial x^i} y_{hj,k} + \frac{\partial u^h}{\partial x^j} y_{hi,k} + \frac{\partial u^h}{\partial x^k} y_{ij,h} \right) \frac{\partial}{\partial y_{ij,k}}.
$$

2) From [**10**, Proposition 3] (also see [**20**, VI, Proposition 21.1]) we know that the natural lift of X to $F(N)$ is given by,

$$
\tilde{X} = u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^l} x^l_j \frac{\partial}{\partial x^i_j},
$$

and its 1-jet prolongation,

$$
\tilde{X}^{(1)} = u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^l} x^l_j \frac{\partial}{\partial x^i_j} + v^i_{jk} \frac{\partial}{\partial x^i_{j,k}},
$$

$$
v^i_{jk} = \frac{\partial u^i}{\partial x^l} x^l_{j,k} - \frac{\partial u^l}{\partial x^k} x^i_{j,l} + \frac{\partial^2 u^i}{\partial x^k \partial x^l} x^l_j.
$$

3) Finally,

$$
\tilde{X}_C = u^i \frac{\partial}{\partial x^i} - \left(\frac{\partial^2 u^i}{\partial x^j \partial x^k} - \frac{\partial u^i}{\partial x^l} A^l_{jk} + \frac{\partial u^l}{\partial x^k} A^i_{jl} + \frac{\partial u^l}{\partial x^j} A^i_{lk} \right) \frac{\partial}{\partial A^i_{jk}},
$$

$$\tilde{X}_C^{(1)} = u^i \frac{\partial}{\partial x^i} + w^i_{jk} \frac{\partial}{\partial A^i_{jk}} + w^i_{jkh} \frac{\partial}{\partial A^i_{jk,h}},$$

(7) $$w^i_{jk} = -\frac{\partial^2 u^i}{\partial x^j \partial x^k} + \frac{\partial u^i}{\partial x^l} A^l_{jk} - \frac{\partial u^l}{\partial x^k} A^i_{jl} - \frac{\partial u^l}{\partial x^j} A^i_{lk},$$

(8) $$w^i_{jkh} = -\frac{\partial^3 u^i}{\partial x^h \partial x^j \partial x^k} + \frac{\partial^2 u^i}{\partial x^h \partial x^l} A^l_{jk} - \frac{\partial^2 u^l}{\partial x^h \partial x^k} A^i_{jl} - \frac{\partial^2 u^l}{\partial x^h \partial x^j} A^i_{lk}$$
$$+ \frac{\partial u^i}{\partial x^l} A^l_{jk,h} - \frac{\partial u^l}{\partial x^k} A^i_{jl,h} - \frac{\partial u^l}{\partial x^j} A^i_{lk,h} - \frac{\partial u^l}{\partial x^h} A^i_{jk,l}.$$

Let $p \colon M \times_N C \to N$ be the natural projection.

We denote by $\bar{f} = (f_M, \tilde{f}_C)$ (resp. $\bar{X} = (X_M, \tilde{X}_C) \in \mathfrak{X}(M \times_N C)$) the natural lift of f (resp. X) to $M \times_N C$. The prolongation to the bundle $J^1(M \times_N C)$ of $\bar{X}$ is as follows:

(9)

$$\bar{X}^{(1)} = \left(X_M^{(1)}, \tilde{X}_C^{(1)} \right)$$
$$= u^i \frac{\partial}{\partial x^i} + \sum_{i \leq j} v_{ij} \frac{\partial}{\partial y_{ij}} + \sum_{i \leq j} v_{ijk} \frac{\partial}{\partial y_{ij,k}} + w^i_{jk} \frac{\partial}{\partial A^i_{jk}} + w^i_{jkh} \frac{\partial}{\partial A^i_{jk,h}},$$

where

(10) $$v_{ij} = -\frac{\partial u^h}{\partial x^i} y_{hj} - \frac{\partial u^h}{\partial x^j} y_{hi},$$

(11) $$v_{ijk} = -\frac{\partial^2 u^h}{\partial x^i \partial x^k} y_{hj} - \frac{\partial^2 u^h}{\partial x^j \partial x^k} y_{hi} - \frac{\partial u^h}{\partial x^i} y_{hj,k} - \frac{\partial u^h}{\partial x^j} y_{hi,k} - \frac{\partial u^h}{\partial x^k} y_{ij,h},$$

and w^i_{jk}, w^i_{jkh} are given in the formulas (7), (8), respectively.

2.3   Diff N- and $\mathfrak{X}(N)$-invariance

A differential form $\omega_r \in \Omega^r(J^1(M \times_N C))$, $r \in \mathbb{N}$, is said to be Diff N-invariant— or invariant under diffeomorphisms— (resp. $\mathfrak{X}(N)$-invariant) if the following equation holds: $(\bar{f}^{(1)})^* \omega_r = \omega_r$, $\forall f \in$ Diff N (resp. $L_{\bar{X}^{(1)}} \omega_r = 0$, $\forall X \in \mathfrak{X}(N)$). Obviously, "Diff N-invariance" implies "$\mathfrak{X}(N)$-invariance" and the converse is almost true (see [**14**], [**28**]). Because of this, below we consider $\mathfrak{X}(N)$-invariance only.

A linear frame $(X_1, \ldots, X_N)$ at x is said to be orthonormal with respect to $g_x \in M_x(N)$ (or simply g_x-orthonormal) if $g_x(X_i, X_j) = 0$ for $1 \leq i < j \leq n$, $g(X_i, X_i) = 1$ for $1 \leq i \leq n^+$, $g(X_i, X_i) = -1$ for $n^+ + 1 \leq i \leq n$.

As N is an oriented manifold, there exists a unique p-horizontal n-form $\mathbf{v}$ on $M \times_N C$ such that, $\mathbf{v}_{(g_x, \Gamma_x)}(X_1, \ldots, X_N) = 1$, for every g_x-orthonormal basis $(X_1, \ldots, X_N)$ belonging to the orientation of N. Locally $\mathbf{v} = \rho v_n$, where $\rho = \sqrt{(-1)^{n^-} \det(y_{ij})}$ and $v_n = dx^1 \wedge \cdots \wedge dx^n$. As proved in [**30**, Proposition 7], the form $\mathbf{v}$ is Diff N-invariant and hence $\mathfrak{X}(N)$-invariant. A Lagrangian density Λ on $J^1(M \times_N C)$ can be globally written as $\Lambda = \mathcal{L}\mathbf{v}$ for a unique function $\mathcal{L} \in C^\infty(J^1(M \times_N C))$ and Λ is $\mathfrak{X}(N)$-invariant if and only if the function $\mathcal{L}$ is. Therefore, the invariance of Lagrangian densities is reduced to that of scalar functions.

Proposition 2.1. *A function $\mathcal{L} \in C^\infty(J^1(M \times_N C))$ is $\mathfrak{X}(N)$-invariant if and only if the following system of partial differential equations hold:*

(12)
$$
\begin{aligned}
0 &= X^i(\mathcal{L}), && \forall i, \\
0 &= X_h^i(\mathcal{L}), && \forall h, i, \\
0 &= X_h^{ik}(\mathcal{L}), && \forall h, i \leq k, \\
0 &= X_i^{jkh}(\mathcal{L}), && \forall i, j \leq k \leq h,
\end{aligned}
$$

where

$$X^i = \frac{\partial}{\partial x^i}, \ \forall i,$$

$$X_h^i = -y_{hi}\frac{\partial}{\partial y_{ii}} - y_{hj}\frac{\partial}{\partial y_{ij}} - y_{ih,k}\frac{\partial}{\partial y_{ii,k}} - y_{hj,k}\frac{\partial}{\partial y_{ij,k}} - \sum_{s \leq j} y_{sj,h}\frac{\partial}{\partial y_{sj,i}}$$

$$+ A_{jk}^i \frac{\partial}{\partial A_{jk}^h} - A_{jh}^r \frac{\partial}{\partial A_{ji}^r} - A_{hk}^r \frac{\partial}{\partial A_{ik}^r}$$

$$+ A_{jk,s}^i \frac{\partial}{\partial A_{jk,s}^h} - A_{jh,r}^s \frac{\partial}{\partial A_{ji,r}^s} - A_{hk,r}^s \frac{\partial}{\partial A_{ik,r}^s} - A_{jk,h}^r \frac{\partial}{\partial A_{jk,i}^r}, \ \forall h, i,$$

(13) $\quad X_h^{ik} = -y_{ih}\dfrac{\partial}{\partial y_{ii,k}} - y_{kh}\dfrac{\partial}{\partial y_{kk,i}} - y_{hj}\dfrac{\partial}{\partial y_{ij,k}} - y_{hj}\dfrac{\partial}{\partial y_{kj,i}} - \dfrac{\partial}{\partial A_{ik}^h} - \dfrac{\partial}{\partial A_{ki}^h}$

$$+ A_{js}^k \frac{\partial}{\partial A_{js,i}^h} - A_{jh}^s \frac{\partial}{\partial A_{jk,i}^s} - A_{hr}^s \frac{\partial}{\partial A_{kr,i}^s}$$

$$+ A_{js}^i \frac{\partial}{\partial A_{js,k}^h} - A_{jh}^s \frac{\partial}{\partial A_{ji,k}^s} - A_{hr}^s \frac{\partial}{\partial A_{ir,k}^s}, \ \forall h, i \leq k,$$

(14) $X_i^{jkh} = \dfrac{\partial}{\partial A_{jk,h}^i} + \dfrac{\partial}{\partial A_{jh,k}^i} + \dfrac{\partial}{\partial A_{hk,j}^i} + \dfrac{\partial}{\partial A_{hj,k}^i} + \dfrac{\partial}{\partial A_{kj,h}^i} + \dfrac{\partial}{\partial A_{kh,j}^i}, \ \forall i, h \leq j \leq k.$

Moreover, the vector fields $X^i, X_h^i, X_h^{ik}, X_i^{jkh}$ are linearly independent and they span an involutive distribution on $J^1(M \times_N C)$ of rank $n\binom{n+3}{3}$. Hence, the number of functionally invariant Lagrangians on $J^1(M \times_N C)$ is

$$\tfrac{1}{6}\left(5n^4 + 3n^3 - 5n^2 + 3n\right).$$

Proof. According to the formula (9), $\mathcal{L}$ is invariant if and only if,

$$u^i \frac{\partial \mathcal{L}}{\partial x^i} + \sum_{i \leq j} v_{ij} \frac{\partial \mathcal{L}}{\partial y_{ij}} + \sum_{i \leq j} v_{ijk} \frac{\partial \mathcal{L}}{\partial y_{ij,k}} + w^i_{jk} \frac{\partial \mathcal{L}}{\partial A^i_{jk}} + w^i_{jkh} \frac{\partial \mathcal{L}}{\partial A^i_{jk,h}} = 0,$$

$$\forall u^i \in C^\infty(N),$$

and expanding on this equation by using the formulas (10), (11), (7), and (8) we obtain

$$
\begin{aligned}
0 = {}& u^i \frac{\partial \mathcal{L}}{\partial x^i} \\
&+ \frac{\partial u^h}{\partial x^i} \left(-y_{hi} \frac{\partial \mathcal{L}}{\partial y_{ii}} - y_{hj} \frac{\partial \mathcal{L}}{\partial y_{ij}} - y_{ih,k} \frac{\partial \mathcal{L}}{\partial y_{ii,k}} - y_{hj,k} \frac{\partial \mathcal{L}}{\partial y_{ij,k}} \right. \\
&- \sum_{s \leq j} y_{sj,h} \frac{\partial \mathcal{L}}{\partial y_{sj,i}} + A^i_{jk} \frac{\partial \mathcal{L}}{\partial A^h_{jk}} - A^r_{jh} \frac{\partial \mathcal{L}}{\partial A^r_{ji}} - A^r_{hk} \frac{\partial \mathcal{L}}{\partial A^r_{ik}} \\
&\left. + A^i_{jk,s} \frac{\partial \mathcal{L}}{\partial A^h_{jk,s}} - A^s_{jh,r} \frac{\partial \mathcal{L}}{\partial A^s_{ji,r}} - A^s_{hk,r} \frac{\partial \mathcal{L}}{\partial A^s_{ik,r}} - A^r_{jk,h} \frac{\partial \mathcal{L}}{\partial A^r_{jk,i}} \right) \\
&+ \frac{\partial^2 u^h}{\partial x^i \partial x^k} \left(-y_{ih} \frac{\partial \mathcal{L}}{\partial y_{ii,k}} - y_{hj} \frac{\partial \mathcal{L}}{\partial y_{ij,k}} - \frac{\partial \mathcal{L}}{\partial A^h_{ik}} \right. \\
&\left. + A^k_{js} \frac{\partial \mathcal{L}}{\partial A^h_{js,i}} - A^s_{jh} \frac{\partial \mathcal{L}}{\partial A^s_{jk,i}} - A^r_{hr} \frac{\partial \mathcal{L}}{\partial A^r_{kr,i}} \right) \\
&- \frac{\partial^3 u^i}{\partial x^h \partial x^k \partial x^j} \frac{\partial \mathcal{L}}{\partial A^i_{jk,h}}.
\end{aligned}
$$

This equation is equivalent to the system of the statement as the values for u^h, $\partial u^h / \partial x^i$, $\partial^2 u^h / \partial x^i \partial x^j$ ($i \leq j$), and $\partial^3 u^h / \partial x^i \partial x^j \partial x^k$ ($i \leq j \leq k$) at a point $x \in N$ can be taken arbitrarily. Moreover, assume a linear combination holds

(15)
$$\lambda_a X^a + \lambda^a_b X^b_a + \sum_{b \leq c} \lambda^a_{bc} X^{bc}_a + \sum_{b \leq c \leq d} \lambda^a_{bcd} X^{bcd}_a = 0,$$
$$\lambda_a, \lambda^a_b, \lambda^a_{bc}, \lambda^a_{bcd} \in C^\infty(J^1(M \times_N C)).$$

By applying (15) to x^a (resp. y_{ab}) we obtain $\lambda_a = 0$ (resp. $\lambda^a_b = 0$); again by applying (15) to A^a_{bc}, $b \leq c$ (resp. A^a_{bc}, $c \leq b$) and taking the expressions of the vector fields (13) and (14) into account, we obtain $\lambda^a_{bc} = 0$, $b \leq c$ (resp. $\lambda^a_{bc} = 0$, $c \leq b$). Hence, (15) reads $\sum_{b \leq c \leq d} \lambda^a_{bcd} X^{bcd}_a = 0$, and by applying it to $A^a_{bc,d}$ and taking the expressions of the vector fields (14) into account, we finally obtain $\lambda^a_{bcd} = 0$. The distribution

$$\mathcal{D}_{M \times_N C} = \left\{ \bar{X}^{(1)}_{(j^1_x g, j^1_x s_\Gamma)} : X \in \mathfrak{X}(N), (j^1_x g, j^1_x s_\Gamma) \in J^1(M \times_N C) \right\}$$

in $T\left(J^1(M \times_N C)\right)$, where $\bar{X}^{(1)}$ is defined in (9), is involutive as

$$\left[\bar{X}^{(1)}, \bar{Y}^{(1)}\right] = \overline{[X,Y]}^{(1)}, \quad \forall X, Y \in \mathfrak{X}(N),$$

and it is spanned by $X^i, X_h^i, X_h^{ik}, X_i^{jkh}$, as proved by the formulas above. The rest of the statement follows from the following identities:

$$\# \left\{ X^i; X_h^i; X_h^{ik}, i \leq k; X_i^{jkh}, h \leq j \leq k : h, i, j, k = 1, \ldots, n \right\}$$
$$= n + n^2 + n\binom{n+1}{2} + n\binom{n+2}{3} = n\binom{n+3}{3},$$

$$\dim J^1\left(M \times_N C\right) - n\binom{n+3}{3} = \tfrac{1}{6}\left(5n^4 + 3n^3 - 5n^2 + 3n\right).$$

$\square$

3 Invariance of covariant Hamiltonians

3.1 Position of the problem

On the bundle $E = M \times_N C$, an Ehresmann connection can locally be written as follows:

(16)
$$\gamma = \sum_{i \leq j} \left(dy_{ij} + \gamma_{ijk}dx^k\right) \otimes \frac{\partial}{\partial y_{ij}} + \left(dA_{jk}^i + \gamma_{jkl}^i dx^l\right) \otimes \frac{\partial}{\partial A_{jk}^i},$$
$$\gamma_{ijk}, \gamma_{jkl}^i \in C^\infty(M \times_N C).$$

In particular, for a Lagrangian density Λ on $J^1(M \times_N C)$ we obtain

$$\Lambda^\gamma = \left(\sum_{i \leq j}\left(\gamma_{ijk} + y_{ij,k}\right)\frac{\partial L}{\partial y_{ij,k}} + \left(\gamma_{jkl}^i + A_{jk,l}^i\right)\frac{\partial L}{\partial A_{jk,l}^i} - L\right) dx^1 \wedge \cdots \wedge dx^n,$$

or equivalently, $\mathcal{L}^\gamma = D^\gamma(\mathcal{L}) - \mathcal{L}$, where

$$D^\gamma = \sum_{i \leq j}\left(\gamma_{ijk} + y_{ij,k}\right)\frac{\partial}{\partial y_{ij,k}} + \left(\gamma_{jkl}^i + A_{jk,l}^i\right)\frac{\partial}{\partial A_{jk,l}^i}.$$

REMARK 3.1. The horizontal form $(p_0^1)^*\gamma - \theta = (\gamma_i^\alpha + y_i^\alpha)\, dx^i \otimes \partial/\partial y^\alpha$ can also be viewed as the p_0^1-vertical vector field

(17)
$$D^\gamma = (\gamma_i^\alpha + y_i^\alpha)\,\frac{\partial}{\partial y_i^\alpha},$$

taking the natural isomorphism $V(p_0^1) \cong (p_0^1)^*(p^*T^*N \otimes V(p))$ into account (cf. [**23**], [**24**], [**32**], [**34**]).

According to the previous formulas, this means: If the system (12) holds for a Lagrangian function $\mathcal{L}$, then it also holds for the covariant Hamiltonian $\mathcal{L}^\gamma$.

If $X \in \{X^i, X^i_h, X^{ik}_h, X^{jkh}_i\}$, then $X(\mathcal{L}^\gamma) = X(D^\gamma(\mathcal{L}))$, as $\mathcal{L}$ is assumed to be invariant and hence $X(\mathcal{L}) = 0$. Therefore

$$X(\mathcal{L}^\gamma) = X(D^\gamma(\mathcal{L}))$$
$$= [X, D^\gamma](\mathcal{L}),$$

and we conclude the following:

Proposition 3.2. *The property* **(P)** *holds for an Ehresmann connection* γ *on* $M \times_N C$ *if and only if the vector field* D^γ *transforms the sections of the distribution* $\mathcal{D}_{M \times_N C}$ *into themselves, namely,* $[D^\gamma, \Gamma(\mathcal{D}_{M \times_N C})] \subseteq \Gamma(\mathcal{D}_{M \times_N C})$.

The problem thus reduces to compute the brackets $[X^i, D^\gamma]$, $[X^i_h, D^\gamma]$, $[X^{ik}_h, D^\gamma]$, and $[X^{jkh}_i, D^\gamma]$. We have

$$(18) \qquad \left[X^h, D^\gamma\right] = \sum_{i \leq j} \frac{\partial \gamma_{ijk}}{\partial x^h} \frac{\partial}{\partial y_{ij,k}} + \frac{\partial \gamma^i_{jkl}}{\partial x^h} \frac{\partial}{\partial A^i_{jk,l}},$$

$$\left[X^{cda}_b, D^\gamma\right] = X^{cda}_b, \quad \forall b, c \leq d \leq a,$$

$$(19) \qquad \left[X^i_h, D^\gamma\right] = \sum_{a \leq b} Y^i_h(\gamma_{abk}) \frac{\partial}{\partial y_{ab,k}} + \sum_{i \leq h} \gamma_{ihk} \frac{\partial}{\partial y_{ii,k}} + \sum_{h < i} \gamma_{hik} \frac{\partial}{\partial y_{ii,k}}$$

$$+ \sum_{h \leq j} \gamma_{hjk} \frac{\partial}{\partial y_{ij,k}} + \sum_{j < h} \gamma_{jhk} \frac{\partial}{\partial y_{ij,k}} + \sum_{a \leq b} \gamma_{abh} \frac{\partial}{\partial y_{ab,i}}$$

$$+ \left(Y^i_h(\gamma^a_{bcr}) - \delta^h_a \gamma^i_{bcr} + \delta^c_i \gamma^a_{bhr} + \delta^b_i \gamma^a_{hcr} + \delta^r_i \gamma^a_{bch}\right) \frac{\partial}{\partial A^a_{bc,r}},$$

$$(20) \quad \left[X^{ik}_h, D^\gamma\right] = \sum_{a \leq b} Y^{ik}_h(\gamma_{abc}) \frac{\partial}{\partial y_{ab,c}} + Y^{ik}_h(\gamma^d_{abc}) \frac{\partial}{\partial A^d_{ab,c}} + X^{ik}_h - Y^{ik}_h,$$

where

$$Y^i_h = -y_{hi} \frac{\partial}{\partial y_{ii}} - y_{hj} \frac{\partial}{\partial y_{ij}} + A^i_{jk} \frac{\partial}{\partial A^h_{jk}} - A^r_{jh} \frac{\partial}{\partial A^r_{ji}} - A^r_{hk} \frac{\partial}{\partial A^r_{ik}},$$

$$Y^{ik}_h = -\frac{\partial}{\partial A^h_{ik}} - \frac{\partial}{\partial A^h_{ki}},$$

and the following formula has been used:

$$\frac{\partial y_{rs,k}}{\partial y_{ij,h}} = \delta_h^k \left(\delta_i^r \delta_j^s + \delta_j^r \delta_i^s - \delta_j^i \delta_r^i \delta_s^j \right).$$

3.2 The class of the Ehresmann connections defined

Let $p \colon M \times_N C \to N$, $\mathrm{pr}_1 \colon M \times_N C \to M$, $\mathrm{pr}_2 \colon M \times_N C \to C$ be the natural projections. By taking the differential of pr_1 and pr_2, a natural identification is obtained $T(M \times_N C) = TM \times_{TN} TC$. Hence

$$V(p) = V(p_M) \times_N V(p_C)$$
$$= \mathrm{pr}_1^* V(p_M) \oplus \mathrm{pr}_2^* V(p_C)$$

and two unique vector-bundle homomorphisms exist

$$\gamma_M \colon \mathrm{pr}_1^* TM \to \mathrm{pr}_1^* V(p_M), \quad \gamma_C \colon \mathrm{pr}_2^* TC \to \mathrm{pr}_2^* V(p_C),$$

such that,

$$\gamma(X) = (\gamma_M (\mathrm{pr}_{1*} X), \gamma_C (\mathrm{pr}_{2*} X)), \quad \forall X \in T(M \times_N C),$$
$$\gamma_M(Y) = Y, \qquad \qquad \forall Y \in \mathrm{pr}_1^* V(p_M),$$
$$\gamma_C(Z) = Z, \qquad \qquad \forall Z \in \mathrm{pr}_2^* V(p_C).$$

If γ is given by the local expression of the formula (16), then

$$\gamma_M = \sum_{i \leq j} \left(dy_{ij} + \gamma_{ijk} dx^k \right) \otimes \frac{\partial}{\partial y_{ij}}, \ \gamma_C = \left(dA_{jk}^i + \gamma_{jkl}^i dx^l \right) \otimes \frac{\partial}{\partial A_{jk}^i},$$

$$\gamma_{ijk}, \gamma_{jkl}^i \in C^\infty(M \times_N C).$$

3.2.1 The first geometric condition on γ

Let $q \colon F(N) \to M$ be the projection given by

(21) $$q(X_1, \ldots, X_N) = g_x$$
$$= \varepsilon_h w^h \otimes w^h,$$

where $(w^1, \ldots, w^n)$ is the dual coframe of $(X_1, \ldots, X_N) \in F_x(N)$, i.e., g_x is the metric for which $(X_1, \ldots, X_N)$ is a g_x-orthonormal basis and $\varepsilon_h = 1$ for $1 \leq h \leq n^+$, $\varepsilon_h = -1$ for $n^+ + 1 \leq h \leq n$. As readily seen, q is a principal G-bundle with $G = O(n^+, n^-)$.

Given a linear connection Γ and a tangent vector $X \in T_x N$, for every u in $p^{-1}(x)$ there exists a unique Γ-horizontal tangent vector $X_u^{h_\Gamma} \in T_u(FN)$

such that, $(p_F)_* X_u^{h_\Gamma} = X$. The local expression for the horizontal lift is known to be ([**20**, Chapter III, Proposition 7.4]),

$$(22) \qquad \left(\frac{\partial}{\partial x^j}\right)^{h_\Gamma} = \frac{\partial}{\partial x^j} - \Gamma_{jk}^i x_l^k \frac{\partial}{\partial x_l^i}.$$

Lemma 3.3. *Given a metric* $g_x \in p_M^{-1}(x)$, *let* $u \in p_F^{-1}(x)$ *be a linear frame such that* $q(u) = g_x$. *The projection* $q_*(X_u^{h_{\Gamma_x}})$ *does not depend on the linear frame* u *chosen over* g_x.

Proof. In fact, any other linear frame projecting onto g_x can be written as $u \cdot A$, $A \in G$. As the horizontal distribution is invariant under right translations (see [**20**, II, Proposition 1.2]), the following equation holds: $(R_A)_* \left(X_u^{h_\Gamma}\right) = X_{u \cdot A}^{h_\Gamma}$. Hence

$$\begin{aligned} q_* \left(X_{u \cdot A}^{h_\Gamma}\right) &= q_* \left((R_A)_* \left(X_u^{h_\Gamma}\right)\right) \\ &= (q \circ R_A)_* \left(X_u^{h_\Gamma}\right) \\ &= q_* \left(X_u^{h_\Gamma}\right). \end{aligned}$$

$\square$

Proposition 3.4. *An Ehresmann connection* γ *on* $M \times_N C$ *satisfies the following condition:*

$$(C_M)\text{: } \gamma_M \left((g_x, \Gamma_x), X\right) = X - q_* \left(((p_M)_*(X))_u^{h_{\Gamma_x}}\right),$$

$\forall X \in T_{g_x} M$, $u \in q^{-1}(g_x)$, *(which does not depend on the linear frame* $u \in q^{-1}(g_x)$ *chosen, according to* Lemma 3.3*) if and only if the following equations hold:*

$$(23) \qquad \gamma_{klj} = - \left(y_{al} A_{jk}^a + y_{ak} A_{jl}^a\right),$$

where the functions γ_{klj} *(resp.* y_{ij}, *resp.* A_{jk}^i) *are defined in the formula* (16) *(resp.* (4), *resp.* (6)).

Proof. Letting $(\chi_j^i)_{i,j=1}^n = \left((x_j^i)_{i,j=1}^n\right)^{-1}$, the dual coframe of the linear frame $u = (X_1, \ldots, X_N) \in F_x(N)$ given in (5) is $(w^1, \ldots, w^n)$, $w^h = \chi_k^h(u) \left(dx^k\right)_x$, $1 \le h \le n$, and the projection q is given by

$$q(u) = g_x$$
$$= \sum_{h=1}^n \varepsilon_h \chi_k^h(u) \chi_l^h(u) \left(dx^k\right)_x \otimes \left(dx^l\right)_x.$$

Therefore the equations of the projection (21) are as follows:

$$x^i \circ q = x^i,$$
$$y_{kl} \circ q = \sum_{h=1}^{n} \varepsilon_h \chi_k^h \chi_l^h.$$

Hence

$$q_* \left(\frac{\partial}{\partial x_b^a} \right)_u = \sum_{k \leq l} \varepsilon_h \left\{ \frac{\partial \chi_k^h}{\partial x_b^a} \chi_l^h + \chi_k^h \frac{\partial \chi_l^h}{\partial x_b^a} \right\} (u) \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x}.$$

Taking derivatives with respect to x_b^a on the identity $\chi_r^h x_i^r = \delta_i^h$, multiplying the outcome by χ_k^i, and summing up over the index i, the following formula is obtained: $\partial \chi_k^h / \partial x_b^a = -\chi_a^h \chi_k^b$. Replacing this equation into the expression for $q_* (\partial/\partial x_b^a)_u$ above, we have

$$q_* \left(\frac{\partial}{\partial x_b^a} \right)_u = - \sum_{k \leq l} \left\{ \chi_k^b(u) y_{al} (g_x) + \chi_l^b(u) y_{ak} (g_x) \right\} \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x}.$$

From (22), evaluated at $u \in q^{-1}(g_x)$, we deduce

$$q_* \left(\frac{\partial}{\partial x^j} \right)_u^{h_\Gamma} = \left(\frac{\partial}{\partial x^j} \right)_{g_x} - \Gamma_{jc}^a(x) x_b^c(u) q_* \left(\frac{\partial}{\partial x_b^a} \right)_{g_x}$$

$$= \left(\frac{\partial}{\partial x^j} \right)_{g_x}$$

$$+ \sum_{k \leq l} \Gamma_{jc}^a(x) x_b^c(u) \left\{ \chi_k^b(u) y_{al} (g_x) + \chi_l^b(u) y_{ak} (g_x) \right\} \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x}$$

$$= \left(\frac{\partial}{\partial x^j} \right)_{g_x} + \sum_{k \leq l} \left\{ \Gamma_{jk}^a(x) y_{al} (g_x) + \Gamma_{jl}^a(x) y_{ak} (g_x) \right\} \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x}.$$

The condition (C_M) holds automatically whenever $X \in V(p_M)$. Hence, (C_M) holds if and only if it holds for $X = (\partial/\partial x^j)_{g_x}$, namely,

$$\sum_{k \leq l} \gamma_{klj}(g_x, \Gamma_x) \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x} = \gamma_M \left((g_x, \Gamma_x), \left(\frac{\partial}{\partial x^j} \right)_{g_x} \right)$$

$$= \left(\frac{\partial}{\partial x^j} \right)_{g_x} - q_* \left(\frac{\partial}{\partial x^j} \right)_u^{h_{\Gamma_x}}$$

$$= - \sum_{k \leq l} \left\{ \Gamma_{jk}^a(x) y_{al} (g_x) + \Gamma_{jl}^a(x) y_{ak} (g_x) \right\} \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x},$$

thus proving the formula (23) in the statement. $\qquad \square$

3.2.2   The canonical covariant derivative

As is known (e.g., see [**20**, III, section 1], [**23**, pp. 157–158]) every connection Γ on a principal G-bundle $P \to N$ induces a covariant derivative ∇^Γ on the vector bundle associated to P under a linear representation $\rho\colon G \to Gl(m, \mathbb{R})$ with standard fibre $\mathbb{R}^m$. In particular, this applies to the principal bundle of linear frames, thus proving that every linear connection Γ on N induces a covariant derivative ∇^Γ on every tensorial vector bundle $E \to N$.

The bundles $(p_C)^* E$, where E is a tensorial vector bundle, are endowed with a canonical covariant derivative ∇^E completely determined by the formula:

$$(24) \qquad \left(\left(\nabla^E \right)_X (f\xi) \right) (\Gamma_x) = \left((Xf)\,\xi \right) (\Gamma_x) + f\,(\Gamma_x) \left(\nabla^{\Gamma_x}_{(p_C)_* X} \xi \right)(x),$$

for all $X \in T_{\Gamma_x} C$, $f \in C^\infty(C)$, and every local section ξ of E defined on a neighbourhood of x. The uniqueness of ∇^E follows from (24) as the sections of E span the sections of $(p_C)^* E$ over $C^\infty(C)$, see [**8**, 0.3.6]. Below, we are specially concerned with the cases $E = TN$ and $E = \wedge^2 T^* N \otimes TN$.

3.2.3   The 2-form associated with γ_C

As $p_C\colon C \to N$ is an affine bundle modelled over $\otimes^2 T^* N \otimes TN$, there is a natural identification

$$V(p_C) \cong (p_C)^* \left(\otimes^2 T^* N \otimes TN \right)$$

and consequently, an Ehresmann connection γ_C on C can also be viewed as a homomorphism $\gamma_C\colon TC \to \otimes^2 T^* N \otimes TN$. If γ_C is locally given by

$$(25) \qquad \gamma_C = \left(dA^i_{jk} + \gamma^i_{jkl} dx^l \right) \otimes \frac{\partial}{\partial A^i_{jk}}, \quad \gamma^i_{jkl} \in C^\infty(C),$$

then

$$\gamma_C = (dA^i_{jk} + \gamma^i_{jkl} dx^l) \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i},$$

and γ_C induces a 2-form $\tilde{\gamma}_C$ taking values in $(p_C)^* (T^* N \otimes TN)$ as follows:

$$\tilde{\gamma}_C(X, Y) = c^1_1 \left((p_C)_*(Y) \otimes \gamma_C(X) \right) - c^1_1 \left((p_C)_*(X) \otimes \gamma_C(Y) \right),$$
$$\forall X, Y \in T_{\Gamma_x} C,$$

where

$$c^1_1\colon TN \otimes T^* N \otimes T^* N \otimes TN \to T^* N \otimes TN,$$
$$c^1_1 \left(X_1 \otimes w_1 \otimes w_2 \otimes X_2 \right) = w_1(X_1) w_2 \otimes X_2,$$
$$X_1, X_2 \in T_x N, \; w_1, w_2 \in T^*_x N.$$

If γ_C is given by (25), then from the very definition of $\tilde{\gamma}_C$ the following local expression is obtained:

$$\tilde{\gamma}_C = (dA^c_{lh} + (\gamma^c_{lha} - \gamma^c_{ahl})\,dx^a) \wedge dx^l \otimes dx^h \otimes \frac{\partial}{\partial x^c}.$$

3.2.4 The second geometric condition on γ

Let $\mathrm{alt}_{12}\colon \otimes^2 T^*N \otimes TN \to \wedge^2 T^*N \otimes TN$ be the operator alternating the two covariant arguments.

The vector bundle $(p_C)^* \left(\wedge^2 T^*N \otimes TN \right)$ admits a canonical section

$$\tau_N \colon C \to \wedge^2 T^*N \otimes TN,$$
$$\tau_N\left(\Gamma_x\right) = T^{\Gamma_x}, \ \forall \Gamma_x \in C,$$

where T^{Γ_x} is the torsion of Γ_x. Locally,

$$\tau_N = \sum_{j<k}(A^i_{jk} - A^i_{kj})dx^j \wedge dx^k \otimes \frac{\partial}{\partial x^i}.$$

From the previous formulas the next result follows:

Proposition 3.5. *Let γ be an Ehresmann connection on $M\times_N C$, let $\nabla^{(1)} = \nabla^{E_1}$ with $E_1 = TN$, let $R^{\nabla^{(1)}}$ be its curvature form, and finally, let $\nabla^{(2)} = \nabla^{E_2}$ with $E_2 = \wedge^2 T^*N \otimes TN$.*

(C_C) *Assume the component γ_C of γ is defined on C. Then, the equations*

$$\text{(26)} \qquad\qquad\qquad \tilde{\gamma}_C = R^{\nabla^{(1)}},$$

$$\text{(27)} \qquad\qquad\qquad \mathrm{alt}_{12} \circ \gamma_C = \nabla^{(2)}\tau_N,$$

are locally equivalent to the following ones:

$$\text{(28)} \qquad\qquad \gamma^h_{str} - \gamma^h_{rts} = A^h_{rm}A^m_{st} - A^h_{sm}A^m_{rt},$$

$$\text{(29)} \qquad \gamma^h_{rst} - \gamma^h_{srt} = A^h_{tm}\left(A^m_{rs} - A^m_{sr}\right) + A^m_{ts}\left(A^h_{mr} - A^h_{rm}\right)$$
$$+ A^m_{tr}\left(A^h_{sm} - A^h_{ms}\right).$$

3.3 Solution to the problem (P)

Theorem 3.6. *If the connection γ on $M\times_N C$ satisfies the conditions (C_M) and (C_C) introduced above, then the vector field D^γ satisfies the property stated in* Proposition 3.2 *and, accordingly the covariant Hamiltonian with respect to γ of every $\mathfrak{X}(N)$-invariant Lagrangian is also $\mathfrak{X}(N)$-invariant.*

Proof. When γ_M satisfies the condition (C_M) the brackets (18), (19), and (20) are respectively given by

$$(30) \qquad \left[X^h, D^\gamma\right] = \frac{\partial \gamma^i_{jkl}}{\partial x^h} \frac{\partial}{\partial A^i_{jk,l}},$$

$$(31) \quad \left[X^i_h, D^\gamma\right] = \left(Y^i_h\left(\gamma^a_{bcr}\right) - \delta^h_a \gamma^i_{bcr} + \delta^c_i \gamma^a_{bhr} + \delta^b_i \gamma^a_{hcr} + \delta^r_i \gamma^a_{bch}\right) \frac{\partial}{\partial A^a_{bc,r}},$$

$$\left[X^{ik}_h, D^\gamma\right] = \left(-\frac{\partial \gamma^d_{abc}}{\partial A^h_{ik}} + \delta^c_i \left(\delta^h_d A^k_{ab} - \delta^k_b A^d_{ah} - \delta^k_a A^d_{hb}\right)\right.$$
$$\left. -\frac{\partial \gamma^d_{abc}}{\partial A^h_{ki}} + \delta^c_k \left(\delta^h_d A^i_{ab} - \delta^i_b A^d_{ah} - \delta^i_a A^d_{hb}\right)\right) \frac{\partial}{\partial A^d_{ab,c}}.$$

In addition, if γ_C satisfies the condition (C_C), then taking derivatives with respect to x^h in (28) and (29) we obtain

$$\frac{\partial \gamma^i_{klj}}{\partial x^h} = \frac{\partial \gamma^i_{jlk}}{\partial x^h}, \quad \frac{\partial \gamma^i_{jkl}}{\partial x^h} = \frac{\partial \gamma^i_{kjl}}{\partial x^h},$$

and renaming indices we deduce

$$\frac{\partial \gamma^i_{jjk}}{\partial x^h} = \frac{\partial \gamma^i_{jkj}}{\partial x^h} = \frac{\partial \gamma^i_{kjj}}{\partial x^h} \ (j < k),$$

$$\frac{\partial \gamma^i_{kkj}}{\partial x^h} = \frac{\partial \gamma^i_{kjk}}{\partial x^h} = \frac{\partial \gamma^i_{jkk}}{\partial x^h} \ (j < k),$$

$$\frac{\partial \gamma^i_{jkl}}{\partial x^h} = \frac{\partial \gamma^i_{klj}}{\partial x^h} = \frac{\partial \gamma^i_{ljk}}{\partial x^h} = \frac{\partial \gamma^i_{kjl}}{\partial x^h} = \frac{\partial \gamma^i_{lkj}}{\partial x^h} = \frac{\partial \gamma^i_{jlk}}{\partial x^h} \ (j < k < l).$$

From (30) we obtain

$$\left[X^h, D^\gamma\right] = \sum_{j<k<l} \frac{\partial \gamma^i_{jkl}}{\partial x^h} X^{jkl}_i + \tfrac{1}{2} \sum_{j<k} \frac{\partial \gamma^i_{jjk}}{\partial x^h} X^{jjk}_i$$
$$+ \tfrac{1}{2} \sum_{j<k} \frac{\partial \gamma^i_{kkj}}{\partial x^h} X^{kkj}_i + \tfrac{1}{6} \frac{\partial \gamma^i_{jjj}}{\partial x^h} X^{jjj}_i,$$

and consequently the values of $\left[X^h, D^\gamma\right]$ belong to the distribution $\mathcal{D}_{M \times_N C}$.

Moreover, as γ_C is assumed to be defined on C, we have

$$Y^i_h\left(\gamma^a_{bcr}\right) = \left(\delta^s_h A^i_{jk} - \delta^i_k A^s_{jh} - \delta^i_j A^s_{hk}\right) \frac{\partial \gamma^a_{bcr}}{\partial A^s_{jk}}.$$

For the sake of simplicity, below we set

$$\left(T^i_h\right)^a_{bcr} = A^i_{jk} \frac{\partial \gamma^a_{bcr}}{\partial A^h_{jk}} - A^s_{jh} \frac{\partial \gamma^a_{bcr}}{\partial A^s_{ji}} - A^s_{hk} \frac{\partial \gamma^a_{bcr}}{\partial A^s_{ik}} - \delta^h_a \gamma^i_{bcr} + \delta^b_i \gamma^a_{hcr} + \delta^c_i \gamma^a_{bhr} + \delta^r_i \gamma^a_{bch}.$$

Taking derivatives with respect to A_{jk}^s, the equations (28) y (29) yield

$$\frac{\partial \gamma_{bcr}^a}{\partial A_{jk}^s} - \frac{\partial \gamma_{rcb}^a}{\partial A_{jk}^s} = \delta_r^j \delta_s^a A_{bc}^k - \delta_b^j \delta_s^a A_{rc}^k + \delta_b^j \delta_C^k A_{rs}^a - \delta_r^j \delta_C^k A_{bs}^a,$$

$$\frac{\partial \gamma_{rbc}^a}{\partial A_{jk}^s} - \frac{\partial \gamma_{brc}^a}{\partial A_{jk}^s} = \delta_C^j \delta_s^a A_{rb}^k - \delta_s^a \delta_C^j A_{br}^k - \delta_s^a \delta_b^k A_{cr}^j - \delta_s^a \delta_r^j A_{cb}^k + \delta_s^a \delta_r^k A_{cb}^j + \delta_s^a \delta_b^j A_{cr}^k$$

$$+ \delta_C^j \delta_b^k A_{sr}^a - \delta_C^j \delta_r^k A_{sb}^a + \delta_r^j \delta_b^k A_{cs}^a - \delta_b^j \delta_r^k A_{cs}^a + \delta_C^j \delta_r^k A_{bs}^a - \delta_C^j \delta_b^k A_{rs}^a.$$

From these expressions, the following symmetries of indices are obtained:

$(T_h^i)_{bbc}^a = (T_h^i)_{bcb}^a = (T_h^i)_{cbb}^a \ (b < c),$

$(T_h^i)_{bcc}^a = (T_h^i)_{cbc}^a = (T_h^i)_{ccb}^a \ (b < c),$

$(T_h^i)_{bcd}^a = (T_h^i)_{dbc}^a = (T_h^i)_{cdb}^a = (T_h^i)_{bdc}^a = (T_h^i)_{dcb}^a = (T_h^i)_{cbd}^a \ (b < c < d),$

and from (31) we obtain

$$\left[X_h^i, D^\gamma\right] = \sum_{b<c<d} (T_h^i)_{bcd}^a X_a^{bcd} + \tfrac{1}{2} \sum_{b<c} (T_h^i)_{bbc}^a X_a^{bbc}$$

$$+ \tfrac{1}{2} \sum_{b<c} (T_h^i)_{ccb}^a X_a^{ccb} + \tfrac{1}{6} (T_h^i)_{bbb}^a X_a^{bbb}.$$

Hence $\left[X_h^i, D^\gamma\right]$ also takes values into the distribution $\mathcal{D}_{M\times_N C}$.

The proof for the third bracket is similar to the previous two cases but longer. Letting

$$\left(T_h^{ik}\right)_{rbc}^a = -\frac{\partial \gamma_{rbc}^a}{\partial A_{ik}^h} - \frac{\partial \gamma_{rbc}^a}{\partial A_{ki}^h} + \delta_i^c \left(\delta_a^h A_{rb}^k - \delta_b^k A_{rh}^a - \delta_r^k A_{hb}^a\right)$$

$$+ \delta_k^c \left(\delta_a^h A_{rb}^i - \delta_b^i A_{rh}^a - \delta_r^i A_{hb}^a\right),$$

the following symmetries are obtained:

$(T_h^{ik})_{bbc}^a = (T_h^{ik})_{bcb}^a = (T_h^{ik})_{cbb}^a \ (b < c),$

$(T_h^{ik})_{bcc}^a = (T_h^{ik})_{cbc}^a = (T_h^{ik})_{ccb}^a \ (b < c),$

$(T_h^{ik})_{bcd}^a = (T_h^{ik})_{dbc}^a = (T_h^{ik})_{cdb}^a = (T_h^{ik})_{bdc}^a = (T_h^{ik})_{dcb}^a = (T_h^{ik})_{cbd}^a \ (b < c < d).$

Hence

$$\left[X_h^{ik}, D^\gamma\right] = \sum_{b<c<d} \left(T_h^{ik}\right)_{bcd}^a X_a^{bcd} + \tfrac{1}{2} \sum_{b<c} \left(T_h^{ik}\right)_{bbc}^a X_a^{bbc}$$

$$+ \tfrac{1}{2} \sum_{b<c} \left(T_h^{ik}\right)_{ccb}^a X_a^{ccb} + \tfrac{1}{6} \left(T_h^{ik}\right)_{bbb}^a X_a^{bbb},$$

and the proof is complete. □

Theorem 3.7. *The Ehresmann connections on C satisfying the equations (26) and (27) are the sections of an affine bundle over C modelled over the vector bundle $(p_C)^* \left(S^3 T^* N \otimes TN \right)$. Consequently, there always exist Ehresmann connections on $M \times_N C$ fulfilling the conditions (C_M) and (C_C) introduced above.*

Proof. If two Ehresmann connections γ_C, γ'_C satisfy the equations (26) and (27), then the difference tensor field $t = \gamma'_C - \gamma_C$, which is a section of the bundle $(p_C)^* \left(\otimes^3 T^* N \otimes TN \right)$, satisfies the following symmetries:

$$(32) \qquad\qquad t(X_1, X_2, X_3) = t(X_3, X_2, X_1),$$

$$(33) \qquad\qquad t(X_1, X_2, X_3) = t(X_2, X_1, X_3),$$

according to (28), (29), respectively, for all $X_1, X_2, X_3 \in T_x N$, $\Gamma_x \in C_x(N)$. Hence

$$t(X_1, X_3, X_2) \overset{(32)}{=} t(X_2, X_3, X_1) \overset{(33)}{=} t(X_3, X_2, X_1) \overset{(32)}{=} t(X_1, X_2, X_3),$$

thus proving that t is totally symmetric. The second part of the statement thus follows from the fact that an affine bundle always admits global sections, e.g., see [**20**, I, Theorem 5.7]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

REMARK 3.8. The results obtained above also hold if the bundle of linear connections is replaced by the subbundle $C^{\text{sym}} = C^{\text{sym}}(N) \subset C$ of symmetric linear connections; the only difference to be observed between both bundles is that in the symmetric cases the equation (27), or equivalently (29), holds automatically.

4   The second-order formalism

In this section we consider the problem of invariance of covariant Hamiltonians for second-order Lagrangians defined on the bundle of metrics, i.e., for functions $\mathcal{L} \in C^\infty(J^2 M)$, where M denotes, as throughout this paper, the bundle of pseudo-Riemannian metrics of a given signature (n^+, n^-) on N.

4.1   Second-order Ehresmann connections

A second-order Ehresmann connection on $p \colon E \to N$ is a differential 1-form γ^2 on $J^1 E$ taking values in the vertical sub-bundle $V(p^1)$ such that $\gamma^2(X) = X$ for every $X \in V(p^1)$. (We refer the reader to [**29**] for the basics on Ehresmann connections of arbitrary order.) Once a connection γ^2 is given, we have a decomposition of vector bundles $T(J^1 E) = V(p^1) \oplus \ker \gamma^2$, where $\ker \gamma^2$ is called the horizontal sub-bundle determined by γ^2. In the

coordinate system on J^1E induced from a fibred coordinate system (x^j, y^α) for p, a connection form can be written as

$$(34) \quad \gamma^2 = (dy^\alpha + \gamma_j^\alpha dx^j) \otimes \frac{\partial}{\partial y^\alpha} + (dy_i^\alpha + \gamma_{ij}^\alpha dx^j) \otimes \frac{\partial}{\partial y_i^\alpha}, \quad \gamma_j^\alpha, \gamma_{ij}^\alpha \in C^\infty(J^1E).$$

As in the first-order case, the action of the group $\mathrm{Aut}(p)$ on the space of second-order connections is defined by the formula

$$\Phi \cdot \gamma^2 = \left(\Phi^{(1)} \right)_* \circ \gamma^2 \circ \left(\Phi^{(1)} \right)_*^{-1}, \quad \forall \Phi \in \mathrm{Aut}(p).$$

As $\Phi^{(1)} \colon J^1M \to J^1M$ is a morphism of fibred manifolds over N, $(\Phi^{(1)})_*$ transforms the vertical subbundle $V(p^1)$ into itself; hence the previous definition makes sense.

4.2 A remarkable isomorphism

Theorem 4.1. *Let Γ^g be the Levi-Civita connection of a pseudo-Riemannian metric g on N. The mapping $\zeta_N \colon J^1M \to M \times_N C^{\mathrm{sym}}$, $\zeta_N(j_x^1 g) = (g_x, \Gamma_x^g)$ is a diffeomorphism. There is a natural one-to-one correspondence between first-order Ehresmann connections on the bundle $p \colon M \times_N C^{\mathrm{sym}} \to N$ and second-order Ehresmann connections on the bundle $p_M \colon M \to N$, which is explicitly given by,*

$$(35) \quad \gamma^2 = ((\zeta_N^v)_*)^{-1} \circ \gamma \circ (\zeta_N)_* ,$$

where $\gamma \colon T(M \times_N C^{\mathrm{sym}}) \to V(p)$ is a first-order Ehresmann connection,

$$(\zeta_N)_* \colon T(J^1M) \to T(M \times_N C^{\mathrm{sym}})$$

is the Jacobian mapping induced by ζ_N, and $(\zeta_N^v)_* \colon V(p_M^1) \to V(p)$ is its restriction to the vertical bundles.

Proof. As a computation shows, the equations of ζ_N in the coordinate systems introduced in the section 2.1.2, are as follows:

$$x^i \circ \zeta_N = x^i,$$

$$y_{ij} \circ \zeta_N = y_{ij},$$

$$(36) \qquad A_{ij}^h \circ \zeta_N = \tfrac{1}{2} y^{hk}(y_{ik,j} + y_{jk,i} - y_{ij,k}), \quad i \le j,$$

where $(y^{ij})_{i,j=1}^n$ is the inverse mapping of the matrix $(y_{ij})_{i,j=1}^n$ and the functions y_{ij} are defined in (4). Hence

$$x^i \circ \zeta_N^{-1} = x^i,$$

$$y_{ij} \circ \zeta_N^{-1} = y_{ij},$$

$$(37) \qquad y_{ij,k} \circ \zeta_N^{-1} = y_{hi} A_{jk}^h + y_{hj} A_{ik}^h, \quad i \le j.$$

As the diffeomorphism ζ_N induces the identity on the ground manifold N, it follows that the definition of γ^2 in (35) makes sense and the following formulas are obtained:

$$\gamma^2\left(\frac{\partial}{\partial x^r}\right) = \sum_{a \leq b} (\gamma_{abr} \circ \zeta_N) \frac{\partial}{\partial y_{ab}} + \sum_{i \leq j} \gamma_{ijkr} \frac{\partial}{\partial y_{ij,k}},$$

$$\gamma_{ijkr} = \frac{1}{2} \sum_{a \leq b} \frac{\delta_{ah}\delta_{bi} + \delta_{ai}\delta_{bh}}{1 + \delta_{hi}} (\gamma_{abr} \circ \zeta_N) y^{hl}(y_{jl,k} + y_{kl,j} - y_{jk,l})$$

$$+ \frac{1}{2} \sum_{a \leq b} \frac{\delta_{ah}\delta_{bj} + \delta_{aj}\delta_{bh}}{1 + \delta_{hj}} (\gamma_{abr} \circ \zeta_N) y^{hl}(y_{il,k} + y_{kl,i} - y_{ik,l})$$

$$+ \sum_{j \leq a} \frac{\delta_{ak}}{1 + \delta_{jk}} \left(\gamma^h_{jar} \circ \zeta_N\right) y_{hi} + \sum_{a \leq j} \frac{\delta_{ak}}{1 + \delta_{jk}} \left(\gamma^h_{ajr} \circ \zeta_N\right) y_{hi}$$

$$+ \sum_{i \leq a} \frac{\delta_{ak}}{1 + \delta_{ik}} \left(\gamma^h_{iar} \circ \zeta_N\right) y_{hj} + \sum_{a \leq i} \frac{\delta_{ak}}{1 + \delta_{ik}} \left(\gamma^h_{air} \circ \zeta_N\right) y_{hj},$$

where

$$\gamma = \sum_{i \leq j} \left(dy_{ij} + \gamma_{ijk} dx^k\right) \otimes \frac{\partial}{\partial y_{ij}} + \sum_{j \leq k} \left(dA^i_{jk} + \gamma^i_{jkl} dx^l\right) \otimes \frac{\partial}{\partial A^i_{jk}},$$

or equivalently,

$$\gamma = \frac{1}{2 - \delta_{ij}} \left(dy_{ij} + \gamma_{ijk} dx^k\right) \otimes \frac{\partial}{\partial y_{ij}} + \frac{1}{2 - \delta_{jk}} \left(dA^i_{jk} + \gamma^i_{jkl} dx^l\right) \otimes \frac{\partial}{\partial A^i_{jk}},$$

assuming $\gamma_{hir} = \gamma_{ihr}$ for $h > i$, and $\gamma^h_{jkr} = \gamma^h_{kjr}$ for $j > k$. Taking the symmetry $A^i_{jk} = A^i_{kj}$ into account, we obtain

$$\gamma_{ijkr} = \frac{1}{2} (\gamma_{hir} \circ \zeta_N) y^{hl}(y_{jl,k} + y_{kl,j} - y_{jk,l})$$

$$+ \frac{1}{2} (\gamma_{hjr} \circ \zeta_N) y^{hl}(y_{il,k} + y_{kl,i} - y_{ik,l})$$

$$+ \left(\gamma^h_{jkr} \circ \zeta_N\right) y_{hi} + \left(\gamma^h_{ikr} \circ \zeta_N\right) y_{hj}.$$

Hence

(38) $\qquad \gamma_{ijkr} \circ \zeta_N^{-1} = \gamma_{hir} A^h_{jk} + \gamma_{hjr} A^h_{ik} + \gamma^h_{jkr} y_{hi} + \gamma^h_{ikr} y_{hj}, \; i \leq j.$

Permuting the indices i, j, k cyclically on the previous equation, we have

(39) $\quad \gamma^s_{ijr} = -\gamma_{hkr} A^h_{ij} y^{ks} - \frac{1}{2} \left(\gamma_{ijkr} \circ \zeta_N^{-1} - \gamma_{jkir} \circ \zeta_N^{-1} - \gamma_{kijr} \circ \zeta_N^{-1}\right) y^{ks},$

thus proving that the mapping $\gamma \mapsto \gamma^2$ defined in the statement, is bijective. $\qquad\qquad\square$

4.3 Covariant Hamiltonians for second-order Lagrangians

The Legendre form of a second-order Lagrangian density $\Lambda = Lv_n$ on the bundle $p \colon E \to N$ is the $V^*(p^1)$-valued p^3-horizontal $(n-1)$-form ω_Λ on J^3E locally given by (e.g., see [**17**], [**26**], [**35**]),

$$\omega_\Lambda = i_{\partial/\partial x^i} v_n \otimes \left(L_\alpha^{i0} dy^\alpha + L_\alpha^{ij} dy_j^\alpha \right),$$

where

(40) $$L_\alpha^{ij} = \frac{1}{2-\delta_{ij}} \frac{\partial L}{\partial y_{ij}^\alpha},$$

(41) $$L_\alpha^i = \frac{\partial L}{\partial y_i^\alpha} - \sum_j \frac{1}{2-\delta_{ij}} D_j \left(\frac{\partial L}{\partial y_{ij}^\alpha} \right),$$

and

$$D_j = \frac{\partial}{\partial x^j} + \sum_{I \in \mathbb{N}^n, |I|=0}^{\infty} y_{I+(j)}^\alpha \frac{\partial}{\partial y_I^\alpha}$$

denotes the total derivative with respect to the variable x^j.

The Poincaré-Cartan form attached to Λ is then defined to be the ordinary n-form on J^3E given by, $\Theta_\Lambda = (p_2^3)^* \theta^2 \wedge \omega_\Lambda + \Lambda$, where θ^2 is the second-order structure form (cf. [**33**, (0.36)]) and the exterior product of $(p_2^3)^* \theta^2$ and the Legendre form, is taken with respect to the pairing induced by duality, $V(p^1) \times_{J^1E} V^*(p^1) \to \mathbb{R}$. The most outstanding difference with the first-order case is that the Legendre and Poincaré-Cartan forms associated with a second-order Lagrangian density are generally defined on J^3E, thus increasing by one the order of the density.

Similarly to the first-order case (see [**11**], [**24**]), given a second-order Lagrangian density Λ on $p \colon E \to N$ and a second-order connection γ^2 on $p \colon E \to N$, by subtracting $(p_2^3)^* \theta^2$ from $(p_1^3)^* \gamma^2$ we obtain a p^3-horizontal form, and we can define the corresponding covariant Hamiltonian to be the Lagrangian density Λ^{γ^2} of third order,

(42) $$\Lambda^{\gamma^2} = \left((p_1^3)^* \gamma^2 - (p_2^3)^* \theta^2 \right) \wedge \omega_\Lambda - \Lambda.$$

Expanding on the right-hand side of the previous equation, we obtain a decomposition of Θ_Λ that generalizes the classical formula for the Hamiltonian in Mechanics; namely, $\Theta_\Lambda = (p_1^3)^* \gamma^2 \wedge \omega_\Lambda - \Lambda^{\gamma^2}$. With the same notations as in the formulas (34), (40), (41) the following formula is deduced:

(43) $$L^{\gamma^2} = (\gamma_i^\alpha + y_i^\alpha) L_\alpha^{i0} + (\gamma_{hi}^\alpha + y_{hi}^\alpha) L_\alpha^{ih} - L.$$

Because of the equation (41), Θ_Λ and L^{γ^2} are generally defined on J^3E.

4.4   Invariant covariant Hamiltonians on J^2M

Lemma 4.2. *If γ is a first-order Ehresmann connection on $M \times_N C^{\mathrm{sym}}$ satisfying the conditions (C_M), then the following equation holds for the second-order Ehresmann connection γ^2 on M given in the formula (35):*

$$\gamma_{abr} \circ \zeta_N = -y_{ab,r}.$$

Proof. Actually, from the formulas (23) and (36) we obtain

$$\gamma_{abr} \circ \zeta_N = -\left(y_{mb}\left(A^m_{ra} \circ \zeta_N\right) + y_{ma}\left(A^m_{rb} \circ \zeta_N\right)\right)$$
$$= -\tfrac{1}{2}\left\{y_{mb}y^{mk}(y_{rk,a}+y_{ak,r}-y_{ra,k}) + y_{ma}y^{mk}(y_{rk,b}+y_{bk,r}-y_{rb,k})\right\}$$
$$= -y_{ab,r}.$$

$\square$

Lemma 4.3. *If a first-order connection γ on $M \times_N C^{\mathrm{sym}}$ satisfies the condition (C_C) introduced above, then the following formulas for its components hold:*

$$(44) \qquad\qquad \gamma^h_{rts} - \gamma^h_{rst} = A^h_{sm}A^m_{rt} - A^h_{tm}A^m_{rs}.$$

Proof. As the bundle under consideration is that of symmetric connections, the following symmetry holds: $\gamma^h_{abc} = \gamma^h_{bac}$, and we have

$$\begin{aligned}
\gamma^h_{rts} &= \gamma^h_{str} - \left(A^h_{rm}A^m_{st} - A^h_{sm}A^m_{rt}\right) \quad \text{[by virtue of (28)]}\\
&= \gamma^h_{tsr} - \left(A^h_{rm}A^m_{st} - A^h_{sm}A^m_{rt}\right)\\
&= \left(\gamma^h_{rst} + A^h_{rm}A^m_{st} - A^h_{tm}A^m_{rs}\right) \quad \text{[by virtue of (28)]}\\
&\quad - \left(A^h_{rm}A^m_{st} - A^h_{sm}A^m_{rt}\right)\\
&= \gamma^h_{rst} + \left(A^h_{sm}A^m_{rt} - A^h_{tm}A^m_{rs}\right)
\end{aligned}$$

$\square$

Proposition 4.4. *Let*

$$\zeta^2_N = \zeta^{(1)}_N\Big|_{J^2M} : J^2M \to J^1(M \times_N C^{\mathrm{sym}})$$

be the restriction to the closed submanifold $J^2M \subset J^1(J^1M)$ of the prolongation $\zeta^{(1)}_N \colon J^1(J^1M) \to J^1(M \times_N C^{\mathrm{sym}})$ of the mapping ζ_N defined in Theorem 4.1. *For every $(j^1_x g, j^1_x \Gamma) \in J^1(M \times_N C^{\mathrm{sym}})$ there exists a unique $j^2_x g' \in J^2M$ such that, $j^1_x g' = j^1_x g$ and $j^1_x \Gamma^{g'} = j^1_x \Gamma$ and the mapping $\varkappa \colon J^1(M \times_N C^{\mathrm{sym}}) \to J^2M$ defined by $\varkappa(j^1_x g, j^1_x \Gamma) = j^2_x g'$ is a $\mathrm{Diff}\,N$-equivariant rectract of ζ^2_N.*

Proof. From the formulas (36) and (37) we obtain

$$\frac{\partial g'_{ij}}{\partial x^k} = g'_{hi} \left(\Gamma^{g'}\right)^h_{jk} + g'_{hj} \left(\Gamma^{g'}\right)^h_{ik},$$

$$\left(\Gamma^{g'}\right)^h_{ij} = \tfrac{1}{2} g'^{hk} \left(\frac{\partial g'_{ik}}{\partial x^j} + \frac{\partial g'_{jk}}{\partial x^i} - \frac{\partial g'_{ij}}{\partial x^k}\right)$$

for every non-singular metric g' on N. Hence the second partial derivatives of g'_{ij} are completely determined, namely

$$\frac{\partial^2 g'_{ij}}{\partial x^k \partial x^l} = \frac{\partial g_{hi}}{\partial x^l}\Gamma^h_{jk} + g_{hi}\frac{\partial \Gamma^h_{jk}}{\partial x^l} + \frac{\partial g_{hj}}{\partial x^l}\Gamma^h_{ik} + g_{hj}\frac{\partial \Gamma^h_{ik}}{\partial x^l}.$$

Moreover, the Levi-Civita connection of a metric depends functorially on the metric, i.e., $\phi \cdot \Gamma^g = \Gamma^{\phi \cdot g}$ for every $\phi \in \operatorname{Diff} N$. Hence, by transforming the equations $j^1_x g' = j^1_x g$ and $j^1_x \Gamma^{g'} = j^1_x \Gamma^g$ by ϕ we can conclude. $\qquad\square$

Theorem 4.5. *If a first-order Ehresmann connection γ on $M \times_N C^{\mathrm{sym}}$ satisfies the conditions (C_M) and (C_C) introduced above, then the covariant Hamiltonian Λ^{γ^2} attached to every $\operatorname{Diff} N$-invariant second-order Lagrangian density Λ on M with respect to the second-order Ehresmann connection γ^2 on M defined in the formula (35), is defined on $J^2 M$ and it is also $\operatorname{Diff} N$-invariant.*

Proof. Given a $\operatorname{Diff} N$-invariant second-order Lagrangian density $\Lambda = \mathcal{L}\mathbf{v}$ on M, let $\Lambda' = \mathcal{L}'\mathbf{v}$ be the first-order Lagrangian density on $M \times_N C^{\mathrm{sym}}$ given by $\Lambda' = \varkappa^* \Lambda$, which is also $\operatorname{Diff} N$-invariant as $\varkappa$ is a $\operatorname{Diff} N$-equivariant mapping according to Proposition 4.4. Moreover, as $\varkappa$ is a retract of ζ^2_N, we have $\left(\zeta^2_N\right)^* \Lambda' = \left(\zeta^2_N\right)^* \varkappa^* \Lambda = (\varkappa \circ \zeta^2_N)^* \Lambda = \Lambda$, i.e., $\Lambda = \left(\zeta^2_N\right)^* \Lambda'$. This formula is equivalent to saying $\mathcal{L} = \mathcal{L}' \circ \zeta^2_N$, as the n-form $\mathbf{v}$ is $\operatorname{Diff} N$-invariant, and it is even equivalent to $L = L' \circ \zeta^2_N$ because ζ^2_N induces the identity on N.

We claim $\mathcal{L}^{\gamma^2} = (\mathcal{L}')^\gamma \circ \zeta^2_N$. This formula will end the proof as the mapping ζ^2_N is $\operatorname{Diff} N$-equivariant and $(\mathcal{L}')^\gamma$ is $\operatorname{Diff} N$-invariant by virtue of Theorem 3.6.

To start with, we observe that the formula (40) for Λ can be written, in the present case, as follows:

$$L^{abij} = \tfrac{1}{2-\delta_{ij}} \frac{\partial L}{\partial y_{ab,ij}},$$

or equivalently, letting $\mathcal{L}^{abij} = \rho^{-1} L^{abij}$,

(45) $$\mathcal{L}^{abij} = \frac{1}{2 - \delta_{ij}} \frac{\partial \mathcal{L}}{\partial y_{ab,ij}}.$$

Taking the formula in Lemma 4.2 into account, the formula (43) for Λ reads as $L^{\gamma^2} = \sum_{a \le b} (\gamma_{abij} + y_{ab,ij}) L^{abij} - L$, or even

$$\mathcal{L}^{\gamma^2} = \sum_{a \le b} (\gamma_{abij} + y_{ab,ij}) \mathcal{L}^{abij} - \mathcal{L},$$

where $\mathcal{L}^{\gamma^2} = \rho^{-1} L^{\gamma^2}$. Hence $\mathcal{L}^{\gamma^2}$ is defined over $J^2 M$. As $y_{ab,ij} = y_{ab,ji}$, we obtain

$$\mathcal{L}^{\gamma^2} = \sum_{a \le b} \sum_{i \le j} \left(\tfrac{1}{2} \left(\gamma_{abij} + \gamma_{abji} \right) + y_{ab,ij} \right) \frac{\partial \left(\mathcal{L}' \circ \zeta_N^2 \right)}{\partial y_{ab,ij}} - \mathcal{L}' \circ \zeta_N^2$$

$$= \sum_{a \le b} \sum_{i \le j} \sum_{k \le l} \left(\tfrac{1}{2} \left(\gamma_{abij} + \gamma_{abji} \right) + y_{ab,ij} \right) \left(\frac{\partial \mathcal{L}'}{\partial A_{kl,q}^h} \circ \zeta_N^2 \right) \frac{\partial (A_{kl,q}^h \circ \zeta_N^2)}{\partial y_{ab,ij}} - \mathcal{L}' \circ \zeta_N^2$$

$$= \sum_{k \le l} \tfrac{1}{4} y^{hm} \left(\gamma_{kmql} + \gamma_{kmlq} + \gamma_{lmqk} + \gamma_{lmkq} - \gamma_{klqm} - \gamma_{klmq} \right) \left(\frac{\partial \mathcal{L}'}{\partial A_{kl,q}^h} \circ \zeta_N^2 \right)$$

$$+ \sum_{k \le l} \tfrac{1}{2} y^{hm} \left(y_{km,ql} + y_{lm,qk} - y_{kl,qm} \right) \left(\frac{\partial \mathcal{L}'}{\partial A_{kl,q}^h} \circ \zeta_N^2 \right) - \mathcal{L}' \circ \zeta_N^2.$$

Moreover, we have

$$\left(\mathcal{L}' \right)^\gamma = \sum_{a \le b} \left(\gamma_{abc} + y_{ab,c} \right) \frac{\partial \mathcal{L}'}{\partial y_{ab,c}} + \sum_{a \le b} \left(\gamma_{abl}^i + A_{ab,l}^i \right) \frac{\partial \mathcal{L}'}{\partial A_{ab,l}^i} - \mathcal{L}'.$$

Hence

$$\left(\mathcal{L}' \right)^\gamma \circ \zeta_N^2 = \sum_{k \le l} \left(\gamma_{klq}^h \circ \zeta_N + A_{kl,q}^h \circ \zeta_N \right) \left(\frac{\partial \mathcal{L}'}{\partial A_{kl,q}^h} \circ \zeta_N^2 \right) - \mathcal{L}' \circ \zeta_N^2$$

$$= \sum_{k \le l} \left\{ -\tfrac{1}{2} \left(\gamma_{klrq} - \gamma_{lrkq} - \gamma_{rklq} \right) y^{rh} \right.$$

$$\left. + \tfrac{1}{2} \left(y_{kr,lq} + y_{lr,kq} - y_{kl,rq} \right) y^{hr} \right\} \left(\frac{\partial \mathcal{L}'}{\partial A_{kl,q}^h} \circ \zeta_N^2 \right) - \mathcal{L}' \circ \zeta_N^2.$$

Consequently, the proof reduces to state that the following equation

$$\tfrac{1}{4} \left(\gamma_{krql} + \gamma_{krlq} + \gamma_{lrqk} + \gamma_{lrkq} - \gamma_{klqr} - \gamma_{klrq} \right) = -\tfrac{1}{2} \left(\gamma_{klrq} - \gamma_{lrkq} - \gamma_{rklq} \right)$$

holds true, or equivalently,

(46) $$0 = \left(\gamma_{ijkr} - \gamma_{ijrk} \right) + \left(\gamma_{irjk} - \gamma_{irkj} \right) + \left(\gamma_{rjki} - \gamma_{rjik} \right).$$

According to the formulas (38) and (23) we obtain

$$\gamma_{ijkr} \circ \zeta_N^{-1} = \left(\gamma_{jkr}^h - A_{ra}^h A_{jk}^a\right) y_{hi} + \left(\gamma_{ikr}^h - A_{ra}^h A_{ik}^a\right) y_{hj}$$
$$- \left(A_{rj}^h A_{ik}^a + A_{ri}^h A_{jk}^a\right) y_{ah}.$$

The third term on the right-hand side of this equation is symmetric in the indices k and r, as $A_{bc}^a = A_{cb}^a$. Hence

$$(\gamma_{ijkr} - \gamma_{ijrk}) \circ \zeta_N^{-1} = \left(\gamma_{jkr}^h - \gamma_{jrk}^h - A_{ra}^h A_{jk}^a + A_{ka}^h A_{jr}^a\right) y_{hi}$$
$$+ \left(\gamma_{ikr}^h - \gamma_{irk}^h - A_{ra}^h A_{ik}^a + A_{ka}^h A_{ir}^a\right) y_{hj}.$$

By composing the right-hand side of the equation (46) and ζ_N^{-1}, and taking the previous formula and the formulas (28) and (44) into account, we conclude that this expression vanishes indeed. □

5   Palatini and Einstein-Hilbert Lagrangians

Let us compute the covariant Hamiltonian density attached to the Palatini Lagrangian. Following the notations in [**20**], the Ricci tensor field attached to the symmetric connection Γ is given by $S^\Gamma(X, Y) = \text{tr}(Z \mapsto R^\Gamma(Z, X)Y)$, where R^Γ denotes the curvature tensor field of the covariant derivative ∇^Γ associated to Γ on the tangent bundle; hence $S^\Gamma = (R^\Gamma)_{jl} dx^l \otimes dx^j$, where

$$(R^\Gamma)_{jl} = (R^\Gamma)_{jkl}^k,$$
$$(R^\Gamma)_{jkl}^i = \partial \Gamma_{jl}^i / \partial x^k - \partial \Gamma_{jk}^i / \partial x^l + \Gamma_{jl}^m \Gamma_{km}^i - \Gamma_{jk}^m \Gamma_{lm}^i.$$

The Lagrangian is the function on $J^1(M \times_N C^{\text{sym}})$ thus given by,

$$\mathcal{L}_P(j_x^1 g, j_x^1 \Gamma) = g^{ij}(x)(R^\Gamma)_{ij}(x)$$

and local expression

$$\mathcal{L}_P = y^{ij}(A_{ij,k}^k - A_{ik,j}^k + A_{ij}^m A_{km}^k - A_{ik}^m A_{jm}^k).$$

As a computation shows, for every first-order connection γ on $M \times_N C^{\text{sym}}$ satisfying (44) and taking the formula (2) into account, we obtain $\mathcal{L}_P^\gamma = 0$. This result is essentially due to the fact that the P-C form of the P density $\Lambda_P = \mathcal{L}_P \mathbf{v} = L_P v_n$ projects onto $M \times_N C^{\text{sym}}$. In fact, the following general characterization holds:

Proposition 5.1. *Let $p \colon E \to N$ be an arbitrary fibred manifold and let γ be a first-order Ehresmann connection on E. The equation $L^\gamma = 0$ holds true for a Lagrangian $L \in C^\infty(J^1 E)$ if and only if, i) the Poincaré-Cartan form of the density $\Lambda = L v_n$ projects onto $J^0 E$ and, ii) $L = \left\langle (p_0^1)^* \gamma - \theta, dL|_{V(p_0^1)} \right\rangle$.*

Proof. The equation $L^\gamma = 0$ is equivalent to the equation $D^\gamma L = L$, where D^γ is the p_0^1-vertical vector field defined in the formula (17), and the general solution to the latter is $L = f(x^i, y^\alpha, \gamma_i^\alpha + y_i^\alpha)$, $f(x^i, y^\alpha, y_i^\alpha)$ being a homogeneous smooth function of degree one in the variables (y_i^α), $1 \leq \alpha \leq m$, $1 \leq i \leq n$, according to Euler's homogeneous function theorem. As f is defined for all values of the variables (y_i^α), $1 \leq \alpha \leq m$, $1 \leq i \leq n$, we conclude that the functions $L_\alpha^i = \partial L/\partial y_i^\alpha$ must be defined on E. Hence L is written as $L = L_\alpha^i(x^j, y^\beta)y_i^\alpha + L_0(x^j, y^\beta)$, but this is exactly the condition for the P-C form of Λ to be projectable onto $J^0E = E$, as follows from the local expression of this form, namely,

$$
\Theta_\Lambda = \frac{\partial L}{\partial y_i^\alpha} \theta^\alpha \wedge i_{\partial/\partial x^i} v_n + L v_n
$$

$$
= \frac{\partial L}{\partial y_i^\alpha} dy^\alpha \wedge i_{\partial/\partial x^i} v_n + \left(L - y_i^\alpha \frac{\partial L}{\partial y_i^\alpha} \right) v_n.
$$

Moreover, by imposing the condition $D^\gamma L = L$ we obtain $L_0 = L_\alpha^i \gamma_i^\alpha$, or in other words $L = (\gamma_i^\alpha + y_i^\alpha)\partial L/\partial y_i^\alpha$, which is equivalent to the equation ii) in the statement. $\square$

The corresponding result for the second-order formalism is similar but the computations are more cumbersome. Let us compute the covariant Hamiltonian density attached to the Einstein-Hilbert Lagrangian. As a matter of notation, we set $S^g(X, Y) = S^{\Gamma^g}(X, Y)$ for the metric g, Γ^g being its Levi-Civita connection, and similarly, $(R^g)_{jkl}^i = (R^{\Gamma^g})_{jkl}^i$.

The E-H Lagrangian is thus given by $\mathcal{L}_{EH} \circ j^2 g = (y^{ij} \circ g)(R^g)_{ihj}^h$. As the Levi-Civita connection Γ^g depends functorially on g, $\mathcal{L}_{EH}$ is readily seen to be $\mathrm{Diff}\,N$-invariant; it is in addition linear in the second-order variables $y_{ij,kl}$. By using the third formula in (36) the following local expression for $\mathcal{L}_{EH}$ is obtained:

$$
\mathcal{L}_{EH} = \tfrac{1}{2}y^{ij}y^{hd}\left(y_{dj,hi} - y_{ij,dh} - y_{dh,ij} + y_{hi,dj} \right) + \mathcal{L}'_{EH},
$$

$$
\begin{aligned}
\mathcal{L}'_{EH} = \tfrac{1}{2}y^{ij} \Big\{ & y^{hm}y_{mr,j}y^{rd}\left(y_{id,h} + y_{hd,i} - y_{ih,d} \right) \\
& - y^{hm}y_{mr,h}y^{rd}\left(y_{id,j} + y_{jd,i} - y_{ij,d} \right) \\
& + \tfrac{1}{2}y^{hr}y^{md}\left(y_{id,j} + y_{jd,i} - y_{ij,d} \right)\left(y_{hr,m} + y_{mr,h} - y_{hm,r} \right) \\
& - \tfrac{1}{2}y^{hr}y^{md}\left(y_{id,h} + y_{hd,i} - y_{ih,d} \right)\left(y_{jr,m} + y_{mr,j} - y_{jm,r} \right) \Big\}.
\end{aligned}
$$

According to (45), for every first-order connection form γ on $M \times_N C^{\text{sym}}$ satisfying the conditions (C_M) and (C_C) above, we have

$$\mathcal{L}_{EH}^{\gamma^2} = \sum_{a \leq b} \tfrac{1}{2-\delta_{ij}}(\gamma_{abij} + y_{ab,ij})\frac{\partial \mathcal{L}_{EH}}{\partial y_{ab,ij}} - \mathcal{L}_{EH},$$

and as a computation shows,

$$\begin{aligned}
\mathcal{L}_{EH}^{\gamma^2} = {} & \tfrac{1}{2}y^{ij}\left(\gamma_{idjh} + \gamma_{jdih} - \gamma_{ijdh} - \gamma_{idhj} - \gamma_{hdij} + \gamma_{ihdj}\right)y^{hd} \\
& + \tfrac{1}{2}y^{ij}\Big\{y^{hm}y_{mr,h}y^{rd}\left(y_{id,j} + y_{jd,i} - y_{ij,d}\right) \\
& - y^{hm}y_{mr,j}y^{rd}\left(y_{id,h} + y_{hd,i} - y_{ih,d}\right) \\
& - \tfrac{1}{2}y^{hr}y^{md}\left(y_{id,j} + y_{jd,i} - y_{ij,d}\right)\left(y_{hr,m} + y_{mr,h} - y_{hm,r}\right) \\
& + \tfrac{1}{2}y^{hr}y^{md}\left(y_{id,h} + y_{hd,i} - y_{ih,d}\right)\left(y_{jr,m} + y_{mr,j} - y_{jm,r}\right)\Big\} \\
= {} & 0,
\end{aligned}$$

where the formulas (39), (44), (36), and Lemma 4.3 have been used. In this case, the P-C form of the E-H density $\Lambda_{EH} = \mathcal{L}_{EH}\mathbf{v} = L_{EH}v_n$,

$$(47) \qquad \Theta_{\Lambda_{EH}} = \sum_{k \leq l}\left(L_{EH}^{i,kl}dy_{kl} + L_{EH}^{ij,kl}dy_{kl,j}\right) \wedge i_{\partial/\partial x^i}v_n + Hv_n,$$

$$H = L'_{EH} - \sum_{k \leq l}L_{EH}^{i,kl}y_{kl,i},$$

$$L_{EH}^{i,kl} = \frac{\partial L'_{EH}}{\partial y_{kl,i}} - \frac{1}{2-\delta_{ij}}y_{ab,j}\frac{\partial^2 L_{EH}}{\partial y_{ab}\partial y_{kl,ij}},$$

$$L_{EH}^{ij,kl} = \frac{1}{2-\delta_{ij}}\frac{\partial L_{EH}}{\partial y_{kl,ij}},$$

(cf. (40), (41)) is not only projectable onto $J^2 M$ but also on $J^1 M$ (e.g., see [**13**]), although there is no first-order Lagrangian on $J^1 M$ admitting (47) as its P-C form. This fact is strongly related to a classical result by Hermann Weyl ([**39**, Appendix II], also see [**22**], [**18**]) according to which the only Diff N-invariant Lagrangians on $J^2 M$ depending linearly on the second-order coordinates $y_{ab,ij}$ are of the form $\lambda \mathcal{L}_{EH} + \mu$, for scalars λ, μ. This also explains why a true first-order Hamiltonian formalism exists in the Einstein-Cartan gravitation theory, e.g., see [**37**], [**38**]. In fact, if

$$L_{EH}^i = \frac{1}{2-\delta_{ij}}\frac{\partial L_{EH}}{\partial y_{kl,ij}}y_{kl,j} \quad \left(\text{hence } L_{EH}^{ij,kl} = \frac{\partial L_{EH}^i}{\partial y_{kl,j}}\right)$$

and the momentum functions are defined as follows:

$$p_{kl,i} = L_{EH}^{i,kl} - \frac{\partial L_{EH}^i}{\partial y_{kl}},$$

then

$$d\Theta_{\Lambda_{EH}} = dp_{kl,i} \wedge dy_{kl} \wedge i_{\partial/\partial x^i}v_n + dH \wedge v_n,$$

and from the Hamilton-Cartan equation (e.g., see [**13**, (1)]) we conclude that a metric g is an extremal for Λ_{EH} if and only if,

$$0 = \frac{\partial(p_{ab,i} \circ j^1 g)}{\partial x^i} - \frac{\partial H}{\partial y_{ab}} \circ j^1 g,$$

$$0 = \frac{\partial(y_{ab} \circ g)}{\partial x^i} + \frac{\partial H}{\partial y_{ab,i}} \circ j^1 g.$$

On the other hand, it is no longer true that the covariant Hamiltonians of the non-linear Lagrangians of the form $f(\mathcal{L}_{EH})$, $f'' \neq 0$, considered in some cosmological models (e.g., see [**1**], [**6**], [**9**], [**12**], [**19**], [**21**], [**31**]) and those in higher dimensions (e.g., see [**15**], [**36**]) vanish. In fact, as a computation shows, one has $f(\mathcal{L}_{EH})^{\gamma^2} = f'(\mathcal{L}_{EH})\mathcal{L}_{EH} - f(\mathcal{L}_{EH})$, $\forall f \in C^\infty(\mathbb{R})$.

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