A computer program for the analysis of the dynamic bending–torsion coupling in bridges using a mini-computer

R. PICON
ETS Ingenieros Industriales, Universidad de Sevilla, Seville, Spain
E. ALARCON
ETS Ingenieros Industriales, Universidad Politécnica de Madrid, Madrid 3, Spain

The analysis of modes and natural frequencies is of primary interest in the computation of the response of bridges. In this article the transfer matrix method is applied to this problem to provide a computer code to calculate the natural frequencies and modes of bridge-like structures. The Fortran computer code is suitable for running on small computers and results are presented for a railway bridge.

INTRODUCTION

The analysis of modes and natural frequencies is of primary interest in the computation of the response of bridges. In the case of simple beams in which the load is applied eccentrically the torsion modes are very important (Fig. 1). If the applied loads are through the torsion centre and parallel to one of the principal inertia axes of the section, there is a displacement in the force direction and vibrations can be easily studied.

If the loads have a general direction then the displacement can be obtained adding the components in the two principal directions. In general the load is not at the torsion centre and then, there is a complex phenomenon of bending and torsion (Fig. 2). The method of working is the use of an equivalent system: a force through the torsion centre and a torsion moment in it.

COUPLING THROUGH SHAPE FUNCTIONS

The energy method is very well adapted to the use of orthogonal shape functions. If y and θ are the displacement and twist in the torsion centre, the response can be approached setting:

$$y(x, t) = \sum_{1}^{m} Y_i(x) \cdot q_i(t) \quad (1)$$

$$\theta(x, t) = \frac{1}{e_0} \sum_{m+1}^{m} Y_i(x) \cdot q_i(t) \quad (2)$$

This leads to a matrix system as:

$$\mathbf{A}\ddot{\mathbf{q}} + \mathbf{B}\mathbf{q} = 0 \quad (3)$$

in which typical elements are:

$$a_{ri} = \int_0^L m Y_r Y_i \, dx \quad (4)$$

$$(k_{ri})_f = \int_0^L EI Y_r'' Y_i'' \, dx \quad (5)$$

Figure 1.

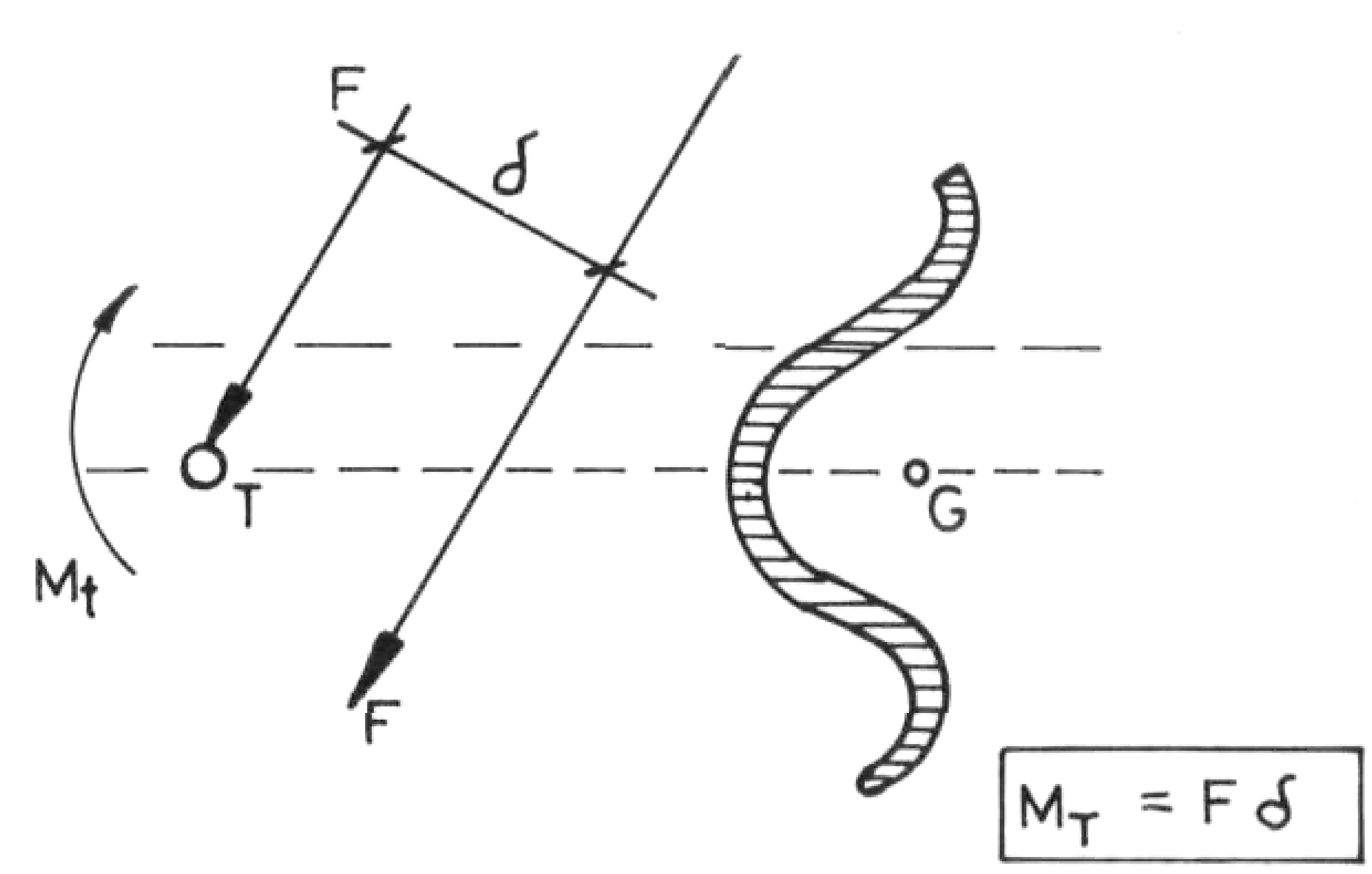

Figure 2.

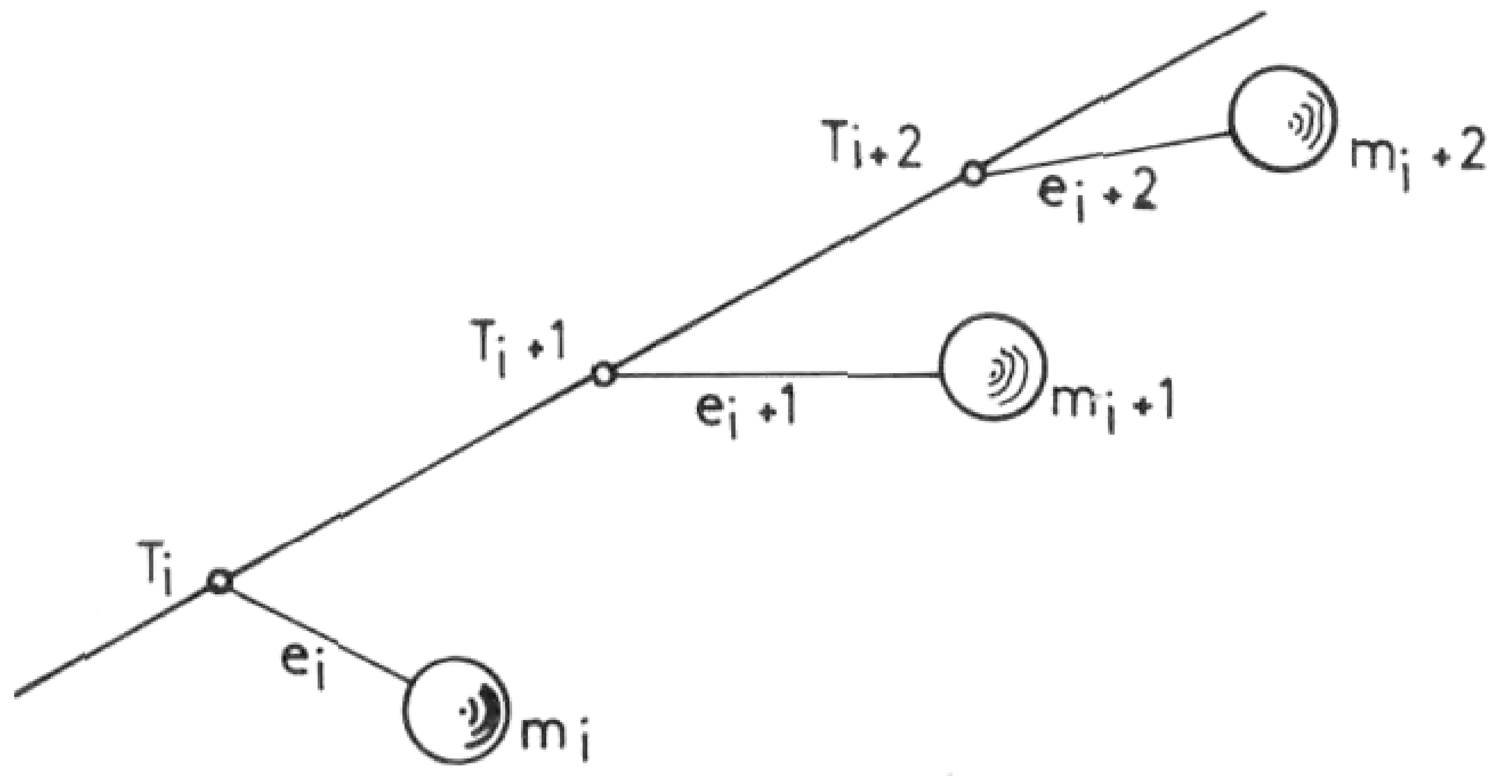

Figure 3.

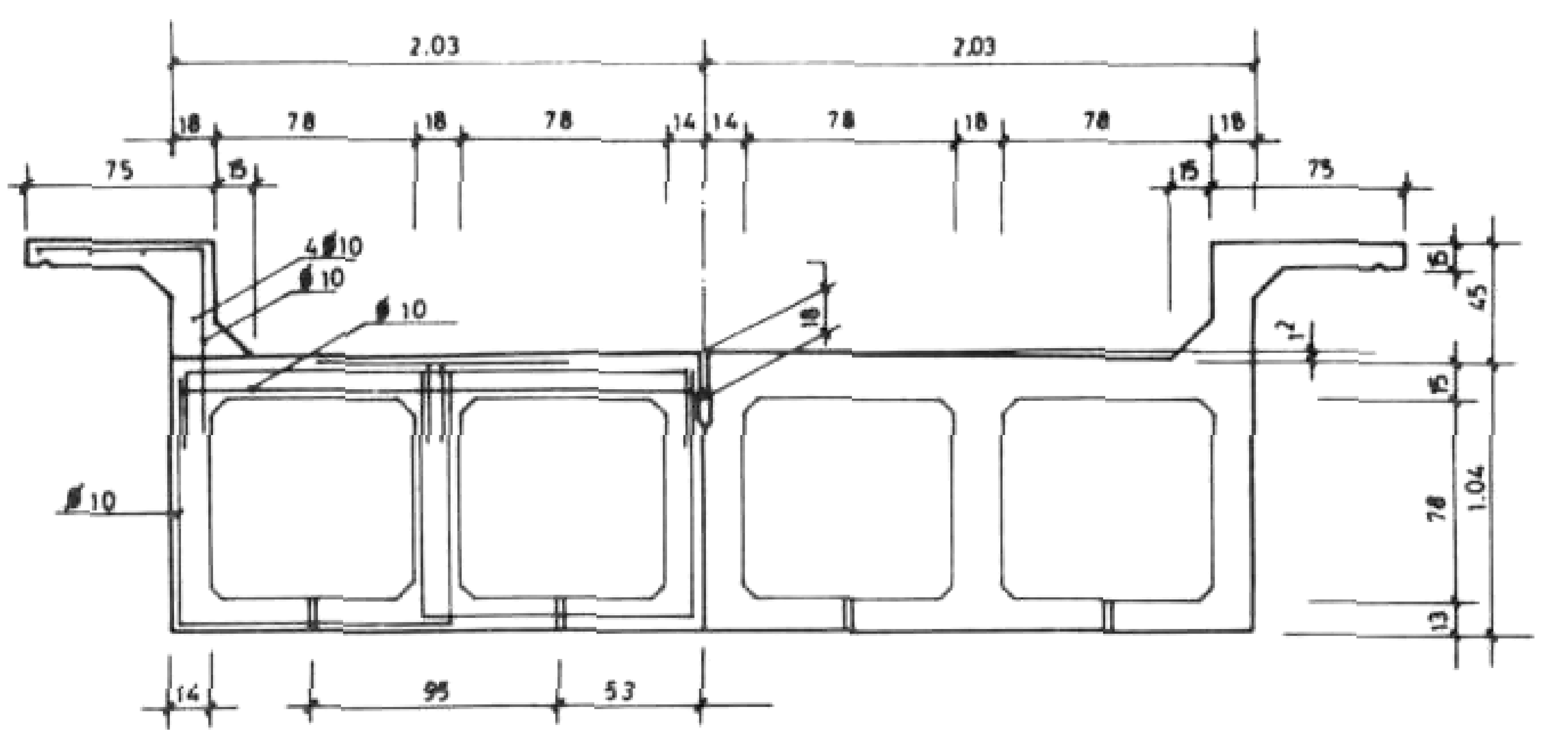

Figure 4.

$$(k_{ri})_t = \int_0^L \frac{GJ}{e_0^2} Y'_r Y'_i \mathrm{d}x \qquad (6)$$

The analytical evaluation of integrals is difficult and the derivatives make worse approximation than the original functions.

Nevertheless, as it is well known, the use of locally defined shape functions leads to the finite element method (FEM).

COUPLING THROUGH TRANSFER MATRICES

The use of transfer matrices[1] was overpassed by the FEM boom. Nevertheless in some problems, just like this under study, its use can be very fruitful.

As is known, the masses of the beam are lumped in some nodes which are related by elastic bars. In free vibrations loads are only inertial, therefore the general transfer matrix is subdivided in a point matrix which relates forces between both sides of every mass and a field matrix, which refer to both ends of every bar.

If

$$\mathbf{Z} = \begin{Bmatrix} y \\ \alpha \\ M \\ Q \\ \hline \theta \\ M_t \end{Bmatrix} \qquad (7)$$

contains the characteristic features in every node, then

$$\mathbf{Z}_i^r = \mathbf{U}_{pi}\, \mathbf{Z}_i^l \qquad (8)$$

where $\mathbf{U}_{pi}$ is the point matrix and the superior index refer to the 'right' and 'left' words.

In every field:

$$\mathbf{Z}_{i+1}^l = \mathbf{U}_{fi}\mathbf{Z}_i^r = \mathbf{U}_{fi}\mathbf{U}_{pi}\mathbf{Z}_i^l \qquad (9)$$

If one defines the transfer matrix in section i as:

$$\mathbf{Z}_{i+1}^l = \mathbf{U}_i\mathbf{Z}_i^l \qquad (10)$$

then

$$\mathbf{U}_i = \mathbf{U}_{fi}\mathbf{U}_{pi} \qquad (11)$$

For the simple beam case, the field matrix is:

$$\mathbf{U}_i^F = \begin{pmatrix} 1 & L_i & L_1^2/2EI_i & L_i^3/6EI_i \\ 0 & 1 & L_i/EI_i & L_i^2/2EI_i \\ 0 & 0 & 1 & L_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (12)$$

If there is only torsion:

$$\mathbf{U}_i^T = \begin{pmatrix} 1 & L_i/GJ_i \\ 0 & 1 \end{pmatrix}$$

where G is the shear modulus and J the polar moment of inertia of the bar cross-section.

As the coupling is a dynamic one[2] it will only appear in the point matrices, so in general:

$$U_{ci} = \left(\begin{array}{cccc|cc} 1 \; L_i & L_i^2/2EI_i & L_i^3/6EI_i & & 0 & 0 \\ 0 \; 1 & L_i/EI_i & L_i^2/2EI_i & & 0 & 0 \\ 0 \; 0 & 1 & L_i & & 0 & 0 \\ 0 \; 0 & 0 & 1 & & 0 & 0 \\ \hline 0 \; 0 & 0 & 0 & & 1 & L_i/GJ_i \\ 0 \; 0 & 0 & 0 & & 0 & 0 \end{array}\right) \qquad (13)$$

and

$$\begin{Bmatrix} y \\ \alpha \\ M \\ Q \\ \theta \\ M_t \end{Bmatrix}_{i+1}^{1} = U_{ci} \begin{Bmatrix} y \\ \alpha \\ M \\ \theta \\ M_t \end{Bmatrix}_{i}^{r} \qquad (14)$$

To establish the point matrix it is worth remembering that coupling appears because of the lack of coincidence of the torsion centre and the centroid.

The equilibrium is:

$$Q_i^r = Q_i^l + p^2 m_i y_i^m \qquad (15)$$

$$M_i^r = M_i^l - p^2 m_i y_i^m e_i - p^2 J_i \theta_i \quad (16)$$

where y_i^m = centroid displacement, p = natural circular frequency, e_i = eccentricity, J_i = polar mass moment.

The torsion centre displacement is:

$$y_i = y_i^m - e_i - \theta_i \quad (17)$$

and then

$$Q_i^r = Q_i^l + p^2 m_i y_i + p^2 m_i e_i \theta_i$$

$$M_i^r = M_i^l - p^2 m_i e_i y_i - p^2 J_i \theta_i - p^2 m_i e_i^2 \theta_i$$

The point matrix is:

$$\mathbf{U}_{pi} = \left(\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ p^2 m_i & 0 & 0 & 1 & p^2 m_i e_i & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ (-p^2 m_i e_i) & 0 & 0 & 0 & -p^2(m_i e_i^2 + J_i) & 1 \end{array} \right) \quad (18)$$

The $\mathbf{U}_{ci}$ and $\mathbf{U}_{pi}$ products is the transfer matrix $\mathbf{U}_i$ for the i knot (shown in equation 19).

$$\mathbf{u}_i = \left(\begin{array}{cccc|cc} \left(1 + \frac{p^2 m_i L_i^2}{6EI_i}\right) & L_i & \frac{L_i^2}{2EI_i} & \frac{L_i^3}{6EI_i} & \frac{p^2 m_i e_i L_i^3}{6EI_i} & 0 \\ \frac{p^2 m_i L_i^2}{2EI_i} & 1 & \frac{L_i}{EI_i} & \frac{L_i^2}{2EI_i} & \frac{p^2 m_i e_i L_i^2}{2EI_i} & 0 \\ p^2 m_i L_i & 0 & 1 & L_i & p^2 m_i e_i L_i & 0 \\ p^2 m_i & 0 & 0 & 1 & p^2 m_i e_i & 0 \\ \hline -\frac{p^2 m_i e_i L_i}{GJ_i'} & 0 & 0 & 0 & 1 - \frac{p^2(m_i e_i^2 + J_i) L_i}{GJ_i} & \frac{L_i}{GJ_i} \\ -p^2 m_i e_i & 0 & 0 & 0 & -p^2(m_i e_i^2 + J_i) & 1 \end{array} \right) \quad (19)$$

It is worth noting that if $e_i = 0$ there is no coupling and both phenomenon are independent.

COMPUTER PROGRAM

Using the previous theory a computer program to be used in a Hewlett–Packard 21 MX has been prepared.

There are a variable number of masses, with a maximum of 200. The program allows $\begin{pmatrix} \mathrm{K1}=1 \\ \mathrm{K1}=2 \end{pmatrix}$ different elements of uniform cross-section. (Numbers in brackets refer to program variables.)

Automatically masses are lumped at the ends of every element. Then mass and constant characteristics of every element are read (D1, D2, D3, D4, D5, D6).

Using the data, transfer matrices are formed, starting with the last mass-element unity (a fictitious element of zero length is added for preventing the symmetry of the procedure). Matrix products gives the transfer matrix which relates the ends of every element.

The transfer matrices formation is completed by the HAMA subroutine, whose argument define the mass-element unity.

The only remaining task is to apply the boundary conditions from which the frequency determinant is obtained. The value of the last (RE) is the final product of the program and, when this value is zero, we know that it is a natural mode of vibration.

Though the program is for the analysis of natural modes with coupling, it is also possible to compute bending frequencies (by the residuals 1 to 6) or torsion frequencies (by residuals 7 to 9).

In these cases it is only necessary to put the eccentricity (EX) equal to zero and to define with a non-zero number (1, for instance) the necessary data to circumvect a divide by zero.

If one is studying non-circular sections it will be necessary to define the adequate connection factor (FACT).

In the circular case FACT = 1.

As an application we have considered the following boundary conditions $i_p e$: hinged ends for bending and built-in ends for torsion. It is a simple matter to establish other kinds of conditions.

The following step in the program is the change of sign detection in (RE) to get the number of modes established in the input (NFN).

The INPUT–OUTPUT units are defined by an internal subroutine (RMPAR) which establishes the correspondence between the LOG and INPUT units and between the NPR and the OUTPUT. Variables to define input and output are provided through the keyboard in the same order as they are executed in the program.

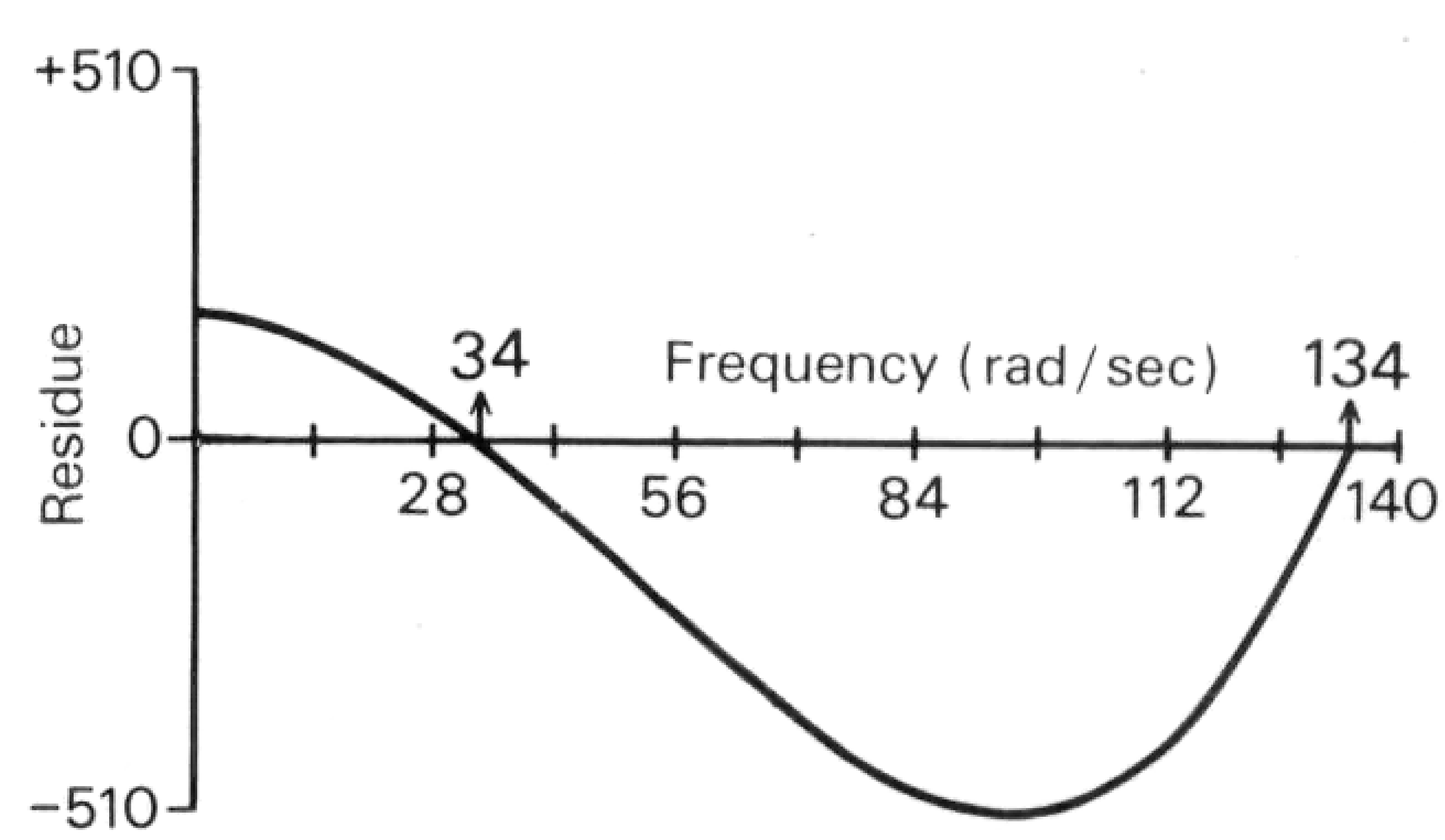

Figure 5. Single track bridge: bending

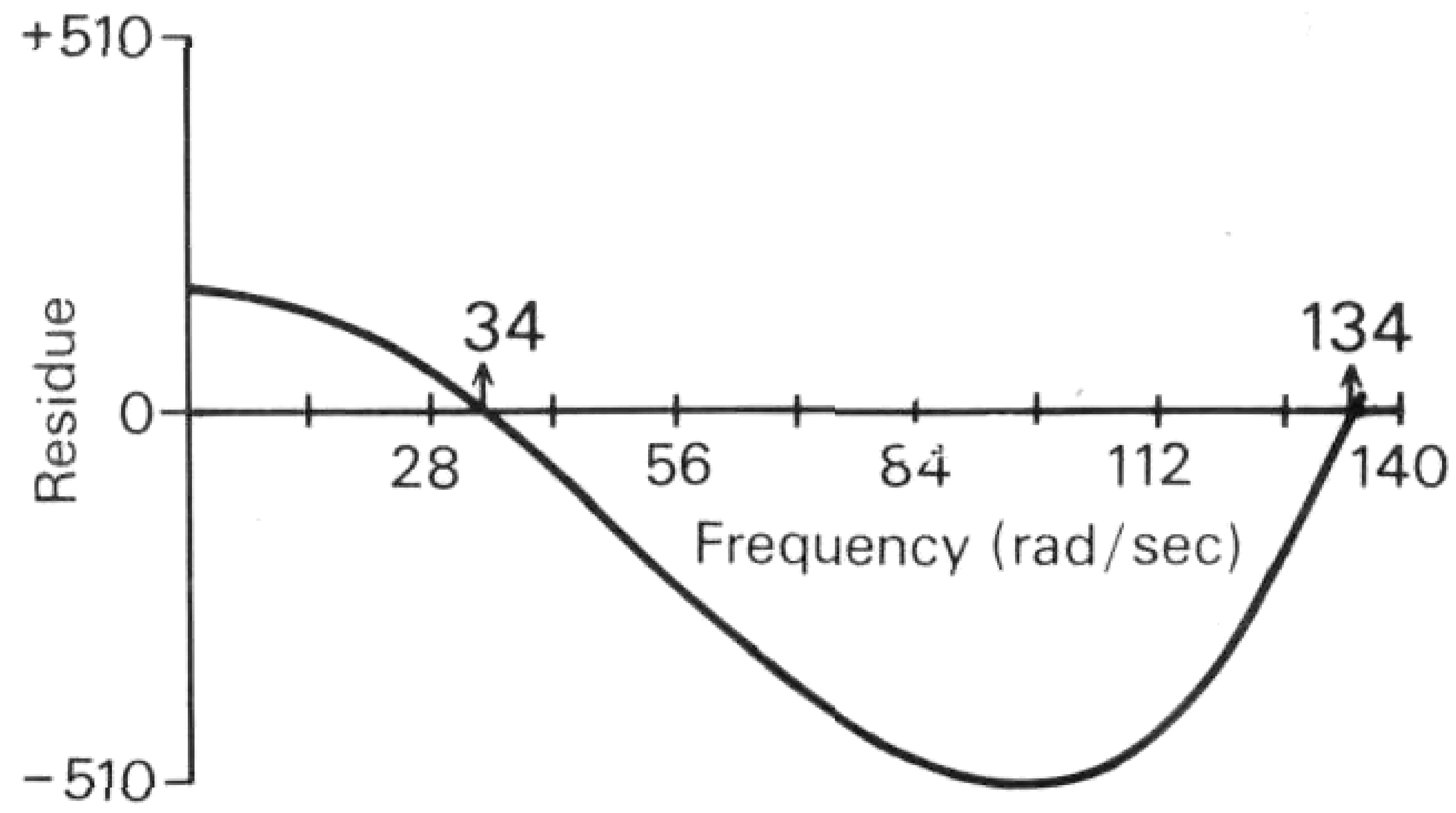

Figure 6. Double track bridge: two trains bending

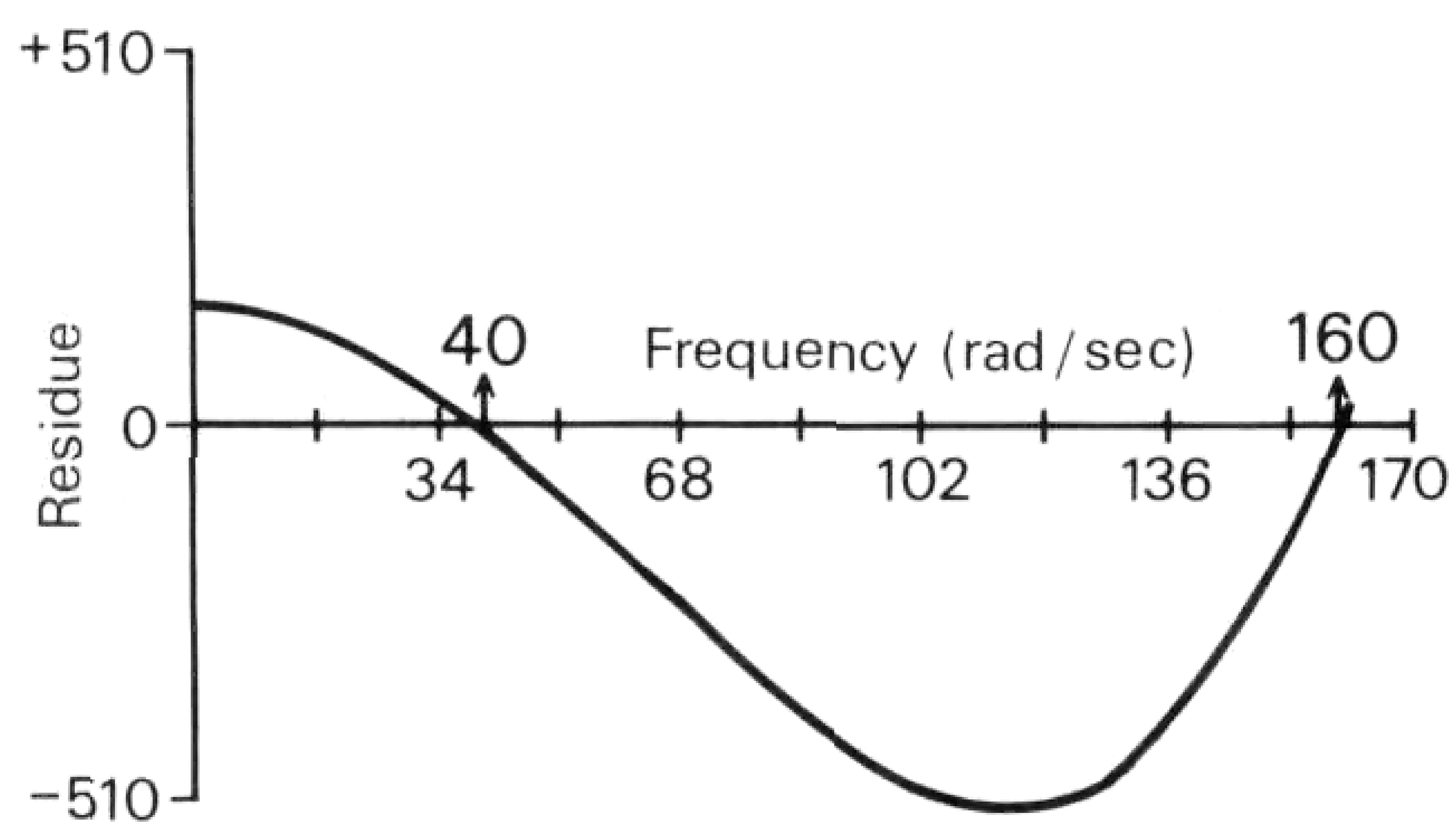

Figure 7. Double track bridge: one train bending (EX = 0)

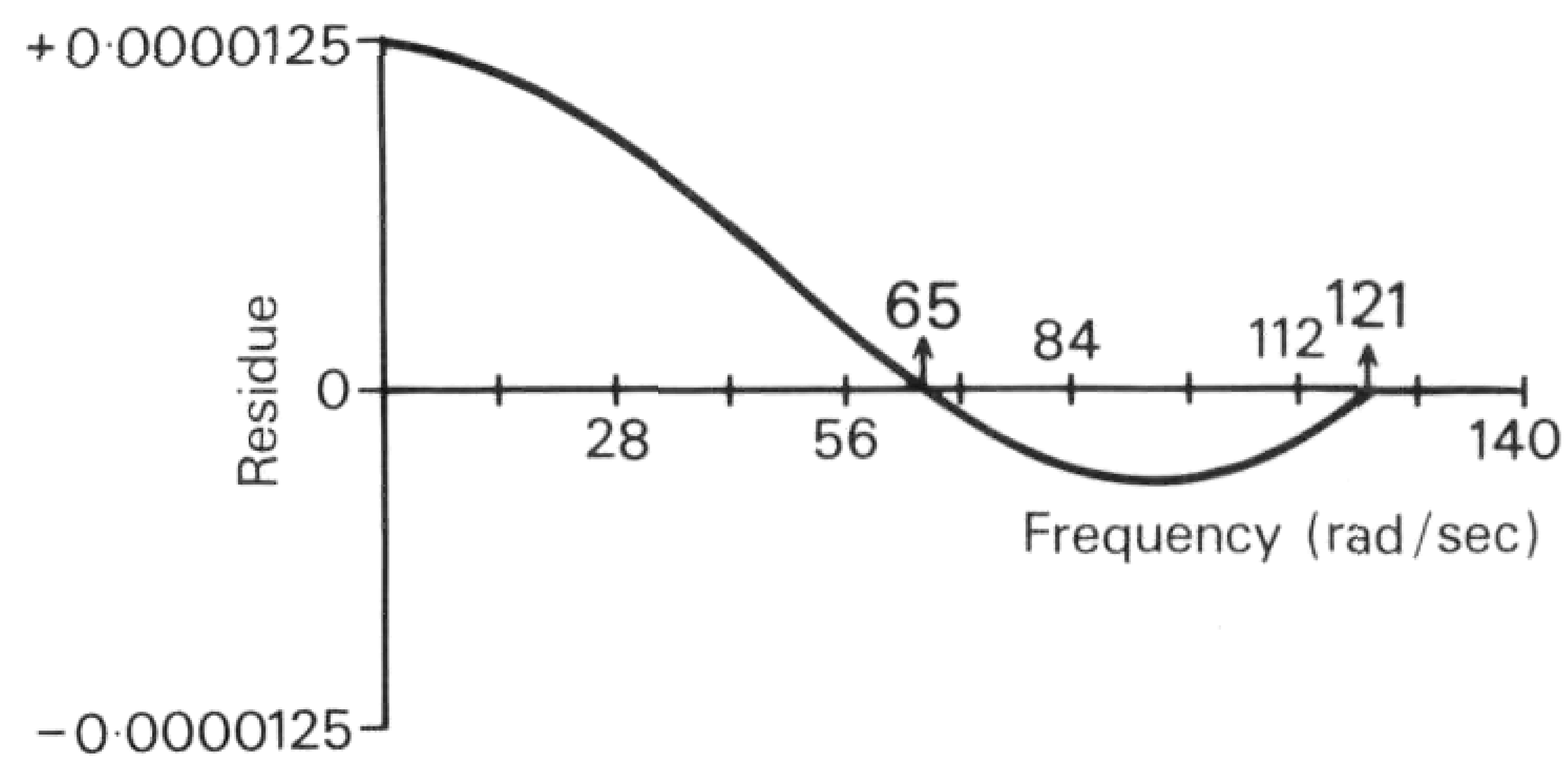

Figure 8. Double track bridge: one train torsion (EX = 0)

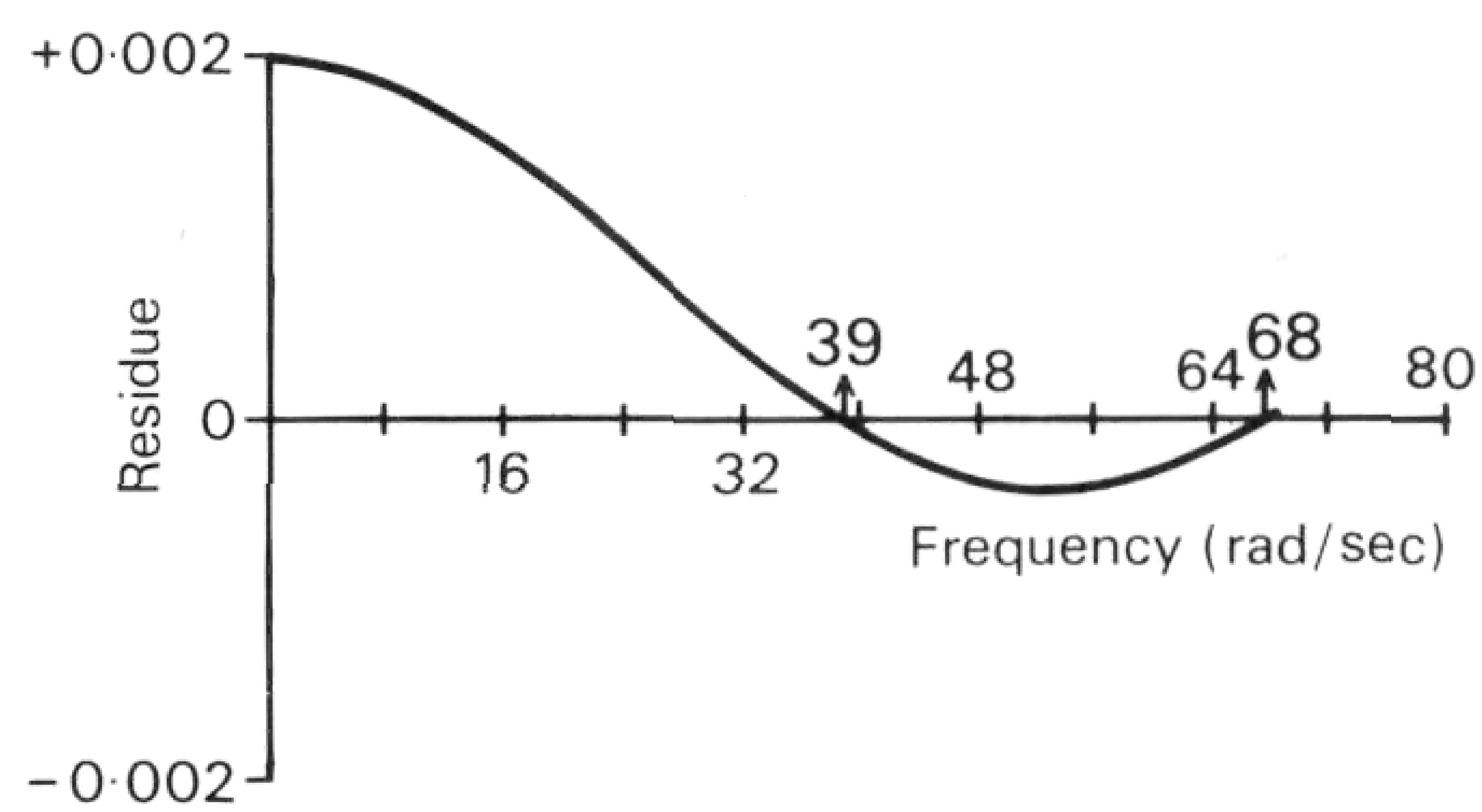

Figure 9. Double track bridge = one bending–torsion coupling (EX = 0)

Input data

If K1 = 1

1. Header card, columns 1–80
2. Column 5: K1(1)
   Columns 9–10: number of the sought residue, after boundary conditions (see program listing)
3. Columns 8–9–10: number of masses
   Columns 10–20: number of sought natural modes.
4. Beam length (LT), beam mass (MW), frequency increment (INW), initial frequency (WINI) and correction factor for the inertia polar moment (FACT) in 10 columns fields with point.
5. Inertia bending moment (1), radius of gyration of the masses (RGIR), called D2 in the reading of data), polar moment (MIP). Young's modulus (*E*), shear modulus (*G*) and eccentricity (EX) in 10 columns fields with point.

If K1 = 2, everything is the same but the 5th and following cards in which we define the properties of masses and bars in 10 columns fields with point, with a maximum of 7 fields every card.

The input order is: inertia bending moments (*I*), polar mass moments (MIMPA), polar cross-section moments (MIP), elasticity modulus (*E*), shear modulus (*G*), eccentricity (EX), element length (*L*) and mass (*M*).

If there are 20 masses there will be 3 cards (7, 7 and 6) with bending inertias, then 3 polar mass inertias, etc.

Output

The program produces the following output:
Echo-check of first card and number of masses,
listing of frequencies and residuals.

Example

The example presented here (Fig. 2) corresponds to the characteristics of a prestressed precast bridge of Barton[3]. The Figure is of one-track bridge but a double one will be built with two of them.

The span is 13 m and we have taken the train-mass as 10 t/ml. There is also a ballast layer of 40 cm.

The residuals for the first two frequencies have been plotted in Figs. 5, 6, 7, 8, 9. Number of lumped masses was 5 and it is worth noting that an increase to 20 masses affects results very slightly (about 32).

REFERENCES

1 Pestel and Leckie, Matric Methods in Elastomechanics — McGraw-Hill, New York.
2 Hurty and Rubinstein, Dynamics of Structures, Prentice-Hall, Englewood Cliffs.
3 R. M. Barton, Prestressed precast concrete railroad bridges, *J. Struct. Div. ASCE*, 1968, (12).

FIMTAI T=00004 IS ON CR00018 USING 00029 BLKS R=0256

```
FTN4,L
      PROGRAM MTR
C
C
C    *************************************************************
C    *  PROGRAM TO COMPUTE THE NATURAL FRECUENCIES OF A          *
C    *  CONTINUOUS SYSTEM USING THE TRANSFER MATRIX METHOD.      *
C    *************************************************************
C
C          LEGEND
C          ******
C
C      E: MODULUS OF ELASTICITY.
C      G: SHEAR MODULUS.
C      EX: EXCENTRICITY. DISTANCE FROM THE CENTRE OF GRAVITY TO
C          THE CENTRE OF TORSION.
C      MIP: POLAR MOMENT OF INERTIA OF THE MEMBER CROSS SECTION.
C      MIPMA: POLAR MOMENT OF INERTIA OF THE MASSES.
C      RGIR: RADIOUS OF INERTIA OF THE MASSES.(D2 WHEN DATA INFUT)
C      I: FLEXURAL MOMENT OF INERTIA OF THE MEMBER CROSS SECTION.
C      RE: RESIDUE.
C      LT: TOTAL LENGTH OF THE BEAM.
C      LT: TOTAL MASS OF THE BEAM.
C      W: CIRCULAR FRECUENCY.
C      INW: INCREMENT OF FRECUENCY.
C      WINI: INITIAL FRECUENCY.
C      M: LUMPED MASSES.
C      L: LENGTH OF THE BEAM SEGMENTS.
C      NM: NUMBER OF LUMFED MASSES.
C      FACT: CORRECTING FACTOR FOR THE MIP. FACT=1 WHEN CIRCULAR
C            CROSS SECTION.
C      K1: PARAMETER THAT DEPENDS ON THE INPUT FORM.
C      K2: PARAMETER TO SELECT THE RESIDUE.
C      NFN: NUMBER OF MODES TO BE COMPUTED.
C      LOG: INPUT UNIT.
C      NPR: OUTPUT UNIT.
C
C
C
      INTEGER A(40)
      REAL  E(200),EX(200),MIP(200),I(200),RE,LT,MV
     1,INW,M(200),L(200),G(200),UN(6,6),U(6,6)
     2,UMU(6,6),MT(6,6),MIPMA(200)
      COMMON W,M,L,E,MIP,EX,I,G,UN,MIPMA,FACT
C
C   SET INPUT AND OUTFUT PARAMETERS.
C
      DIMENSION IPAR(5)
      CALL RMPAR(IPAR)
      LOG=IPAR(1)
      NPR=IPAR(2)
      IF(LOG .LE. 0) LOG=1
      IF(NPR .LE. 0) NPR=6
C
C   INITIAL VALUES OF THE CONSTANTS.
C
      C=1.
      NMO=0
```

```
C
C  READ TITLE.
C
      READ(LOG,102)A
  102 FORMAT(40A2)
C
C  READ CHARACTERISTICS AND SET SEGMENTS AND FARTIAL MASSES.
C
      READ(LOG,43)K1,K2
   43 FORMAT(2I5)
      READ(LOG,10)NM,NFN,LT,MV,INW,WINI,FACT
   10 FORMAT(2I10/5F10.2)
      GO TO (40,41),K1
   41 READ(LOG,21) (I(J),J=1,NM)
      I(NM)=1.
      READ(LOG,21) (MIFMA(J),J=1,NM)
      READ(LOG,21) (MIP(J),J=1,NM)
      MIP(NM)=1.
      READ(LOG,21) (E(J),J=1,NM)
      READ(LOG,21) (G(J),J=1,NM)
      READ(LOG,21) (EX(J),J=1,NM)
      READ(LOG,21) (L(J),J=1,NM)
   21 FORMAT(7F10.2)
      L(NM)=0.
      READ(LOG,21) (M(J),J=1,NM)
      GO TO 800
   40 NN=NM-1
      DO 3 J=1,NN
    3 L(J)=LT/(NM-1)
      L(NM)=0.
      M(1)=MV/(2.*(NM-1))
      M(NM)=M(1)
      DO 4 J=2,NN
    4 M(J)=MV/(NM-1)
      READ(LOG,21)D1,D2,D3,D4,D5,D6
      DO 70 J=1,NM
      I(J)=D1
      MIPMA(J)=M(J)*D2**2
      MIP(J)=D3
      E(J)=D4
      G(J)=D5
   70 EX(J) = D6
C
C  PRINT TITLE AND HEAD LINES.
C
  800 WRITE(NPR,20)A,NM
   20 FORMAT(2X,40A2,///,2X,"NUMBER OF MASSES= ",I3//////,7X,"FRECUEN
     1(RDS/SG)",9X,"RESIDUE"/7X,19("*"),5X,16("*"))
C
C  FORM THE FIRST TRANSFER MATRIX USING THE INITIAL FRECUENCY.
C
  42  W=WINI
   19 N=NM
      CALL HAMA(N)
      DO 5 II=1,6
      DO 6 JJ=1,6
    6 U(II,JJ)=UN(II,JJ)
    5 CONTINUE
C
C  FORM THE NEXT MATRIX. MULTIPLY MATRICES AND TRANSFER PRODUCT TO
```

```
C   MATRIX "U".REPEAT UNTIL THE FINAL MATRIX IS OBTAINED.
C
          DO 7 LL=2,NM
          NN=NM-LL+1
          CALL HAMA(NN)
          DO 8 J=1,6
          DO 9 K=1,6
        9 UMU(J,K)=UN(J,K)
        8 CONTINUE
          DO 11 J=1,6
          DO 12 K=1,6
          P=0.
          DO 13 N=1,6
          PP=U(J,N)*UMU(N,K)
       13 P=P+PP
          MT(J,K)=P
       12 CONTINUE
       11 CONTINUE
          DO 14 J=1,6
          DO 15 K=1,6
       15 U(J,K)=MT(J,K)
       14 CONTINUE
        7 CONTINUE
C
C   COMPUTE RESIDUE DEPENDING ON BOUNDARY CONDITIONS.
C   PRINT FRECUENCY AND RESIDUE.
C
          GO TO (47,48,49,50,51,52,53,55,56,57),K2
C
C   FLEXURAL CANTILEVER BEAM. -1-
C
       47 RE=U(3,3)*U(4,4)-U(4,3)*U(3,4)
          GO TO 97
C
C   FLEXURAL SIMPLY-SUPPORTED BEAM. -2--2-
C
       48 RE=U(1,2)*U(3,4)-U(3,2)*U(1,4)
          GO TO 97
C
C   FLEXURAL FIXED ENDS BEAM. -3-
C
       49 RE=U(1,3)*U(2,4)-U(2,3)*U(1,4)
          GO TO 97
C
C   FLEXURAL FREE ENDS BEAM. -4--
C
       50 RE=U(3,1)*U(4,2)-U(4,1)*U(3,2)
          GO TO 97
C
C   FLEXURAL SIMPLY-SUPPORTED AND FIXED BEAM. -5-
C
       51 RE=U(1,3)*U(3,4)-U(3,3)*U(1,4)
          GO TO 97
C
C   FLEXURAL SIMPLY-SUPPORTED AND FREE BEAM. -6-
C
       52 RE=U(3,2)*U(4,4)-U(4,2)*U(3,4)
          GO TO 97
C
C   TORSIONAL FREE ENDS BEAM. -7-
```

```
C
   53 RE=U(6,5)
      GO TO 97
C
C  TORSIONAL FIXED ENDS BEAM. -8-
C
   55 RE=U(5,6)
      GO TO 97
C
C  TORSIONAL CANTILEVER BEAM. -9-
C
   56 RE=U(6,6)
      GO TO 97
C
C  COUPLING. FLEXURAL SIMPLY-SUPPORTED. TORSIONAL FIXED ENDS. -10-
C
   57 RE=U(1,2)*U(3,4)*U(5,6)+U(1,6)*U(3,2)*U(5,4)+U(1,4)*U(3,6)*U(5,2)-
     1U(1,6)*U(3,4)*U(5,2)-U(1,4)*U(3,2)*U(5,6)-U(1,2)*U(3,6)*U(5,4)
   97 WRITE(NPR,500)W,RE
  500 FORMAT(8X,E13.6,10X,E13.6)
C
C  DETECT CHANGES OF SIGN. INCREASE FRECUENCY AND GO BACK IN THE
C  ITERATING PROCESS.
C
      B=RE/C
      IF(B)82,84,83
   83 W=W+INW
      C=RE
      GO TO 19
   82 NMO=NMO+1
      IF(NMO-NFN) 83,18,18
   84 RE=1.
      GO TO 82
   18 STOP
      END
C
C  FORM THE TRANSFER MATRICES.
C
      SUBROUTINE HAMA(N)
      REAL M(200),L(200),MIP(200),I(200),MIPMA(200)
      COMMON W,M,L,E(200),MIP,EX(200),I,G(200),UN(6,6),MIPMA,FACT
      UN(1,1)=1.+(W**2*M(N)*L(N)**3)/(6.*E(N)*I(N))
      UN(1,2)=L(N)
      UN(1,3)=(L(N)**2)/(2.*E(N)*I(N))
      UN(1,4)=(L(N)**3)/(6.*E(N)*I(N))
      UN(1,5)=(W**2*M(N)*EX(N)*L(N)**3)/(6.*E(N)*I(N))
      UN(1,6)=0.
      UN(2,1)=(W**2*M(N)*L(N)**2)/(2.*E(N)*I(N))
      UN(2,2)=1.
      UN(2,3)=(L(N))/ (E(N)*I(N))
      UN(2,4)=(L(N)**2)/(2.*E(N)*I(N))
      UN(2,5)=(W**2*M(N)*EX(N)*L(N)**2)/(2.*E(N)*I(N))
      UN(2,6)=0.
      UN(3,1)=(W**2*M(N)*L(N))
      UN(3,2)=0.
      UN(3,3)=1.
      UN(3,4)=L(N)
      UN(3,5)=(W**2*M(N)*EX(N)*L(N))
      UN(3,6)=0.
      UN(4,1)=(W**2*M(N))
```

```
      UN(4,2)=0.
      UN(4,3)=0.
      UN(4,4)=1.
      UN(4,5)=(W**2*M(N)*EX(N))
      UN(4,6)=0.
      UN(5,1)=(-1.*W**2*M(N)*EX(N)*L(N))/(FACT*G(N)*MIP(N))
      UN(5,2)=0.
      UN(5,3)=0.
      UN(5,4)=0.
      UN(5,5)= -((W**2*L(N))*((M(N)*EX(N)**2)
     1+MIPMA(N)))/(FACT*G(N)*MIP(N))+1.
      UN(5,6)=L(N)/(FACT*G(N)*MIP(N))
      UN(6,1)=(-1.*W**2*M(N)*EX(N))
      UN(6,2)=0.
      UN(6,3)=0.
      UN(6,4)=0.
      UN(6,5)=(-1.*W**2)*(M(N)*EX(N)**2+MIPMA(N))
      UN(6,6)=1.
      RETURN
      END
  $
```