A mathematical framework for finite strain elastoplastic consolidation
Part 1: Balance laws, variational formulation, and linearization

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Abstract

A mathematical formulation for finite strain elasto plastic consolidation of fully saturated soil media is presented. Strong and weak forms of the boundary-value problem are derived using both the material and spatial descriptions. The algorithmic treatment of finite strain elastoplasticity for the solid phase is based on multiplicative decomposition and is coupled with the algorithm for fluid flow via the Kirchhoff pore water pressure. Balance laws are written for the soil–water mixture following the motion of the soil matrix alone. It is shown that the motion of the fluid phase only affects the Jacobian of the solid phase motion, and therefore can be characterized completely by the motion of the soil matrix. Furthermore, it is shown from energy balance consideration that the effective, or intergranular, stress is the appropriate measure of stress for describing the constitutive response of the soil skeleton since it absorbs all the strain energy generated in the saturated soil–water mixture. Finally, it is shown that the mathematical model is amenable to consistent linearization, and that explicit expressions for the consistent tangent operators can be derived for use in numerical solutions such as those based on the finite element method.

1. Introduction

Non-linear responses of geotechnical structures typically result from plastic yielding and finite deformation of the soil skeleton. There are many classical geotechnical applications where non-linear effects due to these two factors could critically influence the outcome of a numerical analysis. Two examples are large movements of slopes and tilting of a tower due to $P - \delta$ effect. The impact of finite deformation and elastoplastic response is most evident in soft clays where movements develop with time due to so-called hydrodynamic lag, or consolidation, a phenomenon which involves transient interaction between the solid and fluid phases of a soil–water mixture. Unfortunately, mathematical models capable of handling the problem of coupled fluid flow and finite deformation of the soil matrix are not developed well enough to be useful for routine analysis of prototype geotechnical structures.

The mathematical structure and numerical analysis of elastic as well as elasto-plastic consolidation at infinitesimal strains are fairly well developed and adequately documented [1–13]. The general approach is to write the linear momentum and mass balance equations in terms of the solid displacement and fluid potential (or pore water pressure), and then solve them simultaneously via a two-field mixed formulation. The infinitesimal strain assumption simplifies the linear momentum balance equation since it produces an additive form of elastic and plastic deformations. In the context of finite element analysis, the infinitesimal strain assumption also simplifies the mass conservation equation since the volume change of the mixture becomes a linear function of the nodal solid displacements.

Extensions of the infinitesimal formulation of the classical consolidation equations to finite deformation are based primarily on the use of rate-constitutive equations [9, 10, 14]. In addition to the restriction of small elastic strains imposed by this hypoelastic formulation, it also obscures a proper definition of 'mean gradients' and 'average volume changes' necessary for imposing the mass conservation equation at finite increments. Consequently, second-order terms in the hypoelastic extension are ignored, particularly in the mass conservation equation, which leads to a degradation of accuracy when the load increment is large.

An alternative formulation for finite strain elastoplasticity is based on the multiplicative decomposition of the deformation gradient [15, 16]. This method completely circumvents the 'rate issue' in finite deformation analysis [17–23], and allows for the development of large elastic strains. In particular, a more recent development [24, 25] indicates that the multiplicative decomposition technique can be exploited to such an extent that the resulting algorithm may inherit all the features of the classical models of infinitesimal plasticity.

The appropriateness of the multiplicative decomposition technique for soils may be justified from the particulate nature of this material, much like for metals from its crystal microstructure [26–28]. From a micromechanical standpoint, plastic deformation in soils arises from slipping, crushing, yielding, and (for plate-like clay particles) plastic bending of granules comprising the assembly [29]. While it can be argued that deformations in soils are predominantly plastic, reversible deformations also could develop from the elasticity of the solid grains, and could be measurably large when the particles are 'locked' because of the initially high density of the assembly. The local multiplicative decomposition of the deformation gradient provides a means for describing mathematically the relationships between the reference configuration, the current configuration, and the unloaded, stress-free intermediate configuration of a soil assembly subjected to finite deformation in the macroscopic sense.

The volume constraint imposed by a fluid is another issue of long-standing in finite deformation consolidation analysis. Central to the formulation presented in this paper are the key role played by the Jacobian [30, 31], as well as the proper characterization of fluid flow. To describe the latter, we employ the classical theory of mixtures [32–38] and view the soil–water mixture as a two-phase continuum. In contrast to previous formulations of the mixture theory, however, we opt to follow the motion of the solid phase alone and write the generalized Darcy's law spatially in terms of the relative motion of the fluid with respect to that of the solid [39–41]. Whether or not the generalized Darcy's law can indeed be written spatially for the case where the solid-phase motion is finite, is a subject for further experimental studies, much like the required verifications for the spatial nature of many rate-constitutive models for soils. However, the spatial form of the generalized Darcy's law provides a mathematically complete constitutive flow theory. Furthermore, the use of the relative fluid motion also results in the intrinsic fluid motion dropping naturally from the mathematical formulation.

Finally, we develop herein the variational forms of the boundary-value problems and derive the linearization expressions of the relevant continuum equations. The variational forms of the boundary-value problem serve to motivate the finite element implementation of the non-linear consolidation theory, while the linearization provides a key link between the linear and non-linear consolidation theories. Linearization is also a crucial step in deriving explicit expressions for the consistent tangent operator that is used in non-linear finite element consolidation analysis [4–6].

The flow of presentation goes very much in the same order as indicated in the title: balance laws are presented in Section 2, variational equations in Section 3, and linearization in Section 4. As for notations and symbols, bold-face letters denote matrices and vectors; the symbol '$\cdot$' denotes an inner product of two vectors (e.g. $\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$), or a single contraction of adjacent indices of two tensors (e.g. $\boldsymbol{c} \cdot \boldsymbol{d} = c_{ij} d_{jk}$); the symbol ':' denotes an inner product of two second-order tensors (e.g. $\boldsymbol{c} : \boldsymbol{d} = c_{ij} d_{ij}$), or a double contraction of adjacent indices of tensors of rank two and higher (e.g. $\boldsymbol{D} : \boldsymbol{C} = D_{IJKL} C_{KL}$); upper-case subscripts refer to material coordinates while lower-case subscripts refer to spatial coordinates.

2. Balance laws

This section presents the balance principles that govern the interaction between the solid and fluid constituents of a two-phase saturated soil–water mixture. In the derivation of the balance laws we shall consider the motion of the solid and fluid phases separately. Then, we will use the mixture theory [32–38] to combine the field equations, choosing the intrinsic motion of the solid phase as the reference motion relative to which the motion of the fluid phase is described.

2.1. Balance of linear momentum

Let $\phi_t : \mathcal{B} \rightarrow R^{n_{sd}}$ be the motion, or set of configurations, of a water-saturated simple soil body $\mathcal{B} \subset R^{n_{sd}}$, and let $\mathcal{U}$ be any open set with piecewise C^1 boundary such that $\mathcal{U} \subset \mathcal{B}$. Further, let $\phi_t^{w} : \mathcal{B}^{w} \rightarrow R^{n_{sd}}$ be the motion of the fluid, which could be distinct from ϕ_t if seepage takes place in the saturated region $\mathcal{B} \subset \mathcal{B}^{w}$. Now, let $\boldsymbol{\sigma}^{s}$ and $\boldsymbol{\sigma}^{w}$ be the Cauchy partial stress tensors [32–38] arising from the intergranular and fluid stresses, respectively, and let $\boldsymbol{n}$ denote the unit normal vector to the surface $\partial\phi_t(\mathcal{U})$ of the deformed region $\phi_t(\mathcal{U})$. The Cauchy total stress tensor $\tilde{\boldsymbol{\sigma}}$ is obtained from the sum

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^{s} + \boldsymbol{\sigma}^{w} . \tag{2.1}$$

For the solid phase the momentum balance equation in the absence of inertia forces and momentum supplies due to chemical reactions with the fluid takes the form

$$\int_{\phi_t(\mathcal{U})} \rho_s(1-\varphi)\boldsymbol{g}\, dv + \int_{\phi_t(\mathcal{U})} \boldsymbol{h}^{s}\, dv + \int_{\partial\phi_t(\mathcal{U})} \boldsymbol{\sigma}^{s} \cdot \boldsymbol{n}\, da = \boldsymbol{0} \tag{2.2a}$$

with the following localization

$$\rho_s(1-\varphi)\boldsymbol{g} + \boldsymbol{h}^{s} + \operatorname{div} \boldsymbol{\sigma}^{s} = \boldsymbol{0} , \tag{2.2b}$$

where ρ_s is the mass density of the solid grains, φ is the porosity of the soil (defined as the macroscopic ratio of the volume of the void to the total volume of the soil–water mixture), $\boldsymbol{g}$ is the vector of gravity accelerations, $\boldsymbol{h}^{s}$ is the seepage force per unit volume arising from the frictional drag of the fluid phase on the solid matrix due to fluid flow (see e.g. [32]), and div is the spatial divergence operator.

For the fluid phase the momentum balance equation can be written in a similar fashion as follows

$$\int_{\phi_t(\mathcal{U})} \rho_w \varphi \boldsymbol{g}\, dv + \int_{\phi_t(\mathcal{U})} \boldsymbol{h}^{w}\, dv + \int_{\partial\phi_t(\mathcal{U})} \boldsymbol{\sigma}^{w} \cdot \boldsymbol{n}\, da = \boldsymbol{0} \tag{2.3a}$$

with the localization

$$\rho_w \varphi \boldsymbol{g} + \boldsymbol{h}^{w} + \operatorname{div} \boldsymbol{\sigma}^{w} = \boldsymbol{0} , \tag{2.3b}$$

where ρ_w is the mass density of the fluid phase and $\boldsymbol{h}^{w}$ is the reactive force per unit volume exerted by the solid matrix on the fluid phase as the fluid seeps through the soil. Note that since $\boldsymbol{h}^{s}$ and $\boldsymbol{h}^{w}$ are internal forces which naturally will not affect the soil-water mixture as a whole, we have $\boldsymbol{h}^{s} + \boldsymbol{h}^{w} = \boldsymbol{0}$, i.e. the seepage force exerted by the fluid on the solid matrix is the negative of the reactive force exerted by the solid matrix on the fluid. Consequently, the sum of (2.2a) and (2.3a) results in

$$\int_{\phi_t(\mathcal{U})} \rho \boldsymbol{g}\, dv + \int_{\partial\phi_t(\mathcal{U})} \tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{n}\, da = \boldsymbol{0} \tag{2.4a}$$

with the localization

$$\rho \boldsymbol{g} + \operatorname{div} \tilde{\boldsymbol{\sigma}} = \boldsymbol{0} , \tag{2.4b}$$

where

$$\rho = \rho_s(1-\varphi) + \rho_w \varphi \tag{2.5}$$

is the saturated mass density of the soil–water mixture.

Now, let $\boldsymbol{P}^{\mathrm{w}}$ and $\boldsymbol{P}^{\mathrm{s}}$ be the (non-symmetric) first Piola–Kirchhoff partial stress tensors arising from the fluid and intergranular stresses, respectively. Also, let $\boldsymbol{N}$ denote the unit normal vector to the surface $\partial\mathcal{U}$ of the undeformed region $\mathcal{U}$. The tensor $\boldsymbol{P}^{\mathrm{w}}$ is defined such that the product $\boldsymbol{P}^{\mathrm{w}}\cdot\boldsymbol{N}$ represents the resultant force exerted by the fluid per unit area of the solid matrix in the undeformed configuration; similarly, the product $\boldsymbol{P}^{\mathrm{s}}\cdot\boldsymbol{N}$ is the resultant net force exerted by the individual grains (which may include the partial effects of fluid pressures) over the same undeformed reference area. By the additive decomposition of the Cauchy partial stress tensors, we obtain a similar expression for the first Piola–Kirchhoff total stress tensor $\tilde{\boldsymbol{P}}$

$$\tilde{\boldsymbol{P}} = J\tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{F}^{-\mathrm{t}} = \boldsymbol{P}^{\mathrm{s}} + \boldsymbol{P}^{\mathrm{w}}\,, \tag{2.6}$$

where $\boldsymbol{P}^{\mathrm{s}} = J\boldsymbol{\sigma}^{\mathrm{s}}\cdot\boldsymbol{F}^{-\mathrm{t}}$ and $\boldsymbol{P}^{\mathrm{w}} = J\boldsymbol{\sigma}^{\mathrm{w}}\cdot\boldsymbol{F}^{-\mathrm{t}}$ are the first Piola–Kirchhoff partial stress tensors arising from the solid and fluid stresses, respectively, and

$$J = \det(\boldsymbol{F})\,; \qquad \boldsymbol{F} = \frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{X}}\,; \qquad \boldsymbol{\phi} = \boldsymbol{X} + \boldsymbol{u}\,. \tag{2.7}$$

In (2.7), J is the Jacobian, $\boldsymbol{F}$ is the deformation gradient, $\boldsymbol{\phi}$ are the coordinates of the motion, $\boldsymbol{X}$ are the coordinates of a point X in the undeformed configuration, and $\boldsymbol{u}$ is the macroscopic displacement field of the solid phase.

A more common decomposition of the tensor $\tilde{\boldsymbol{P}}$ is based on the use of so-called effective stresses and takes the form

$$\tilde{\boldsymbol{P}} = \boldsymbol{P} + \frac{\boldsymbol{P}^{\mathrm{w}}}{\varphi}\,, \tag{2.8}$$

where $\boldsymbol{P}$ is the first Piola–Kirchhoff effective stress tensor, and $\boldsymbol{P}^{\mathrm{w}}/\varphi$ is a non-symmetric tensor defined such that $(\boldsymbol{P}^{\mathrm{w}}/\varphi)\cdot\boldsymbol{N}$ represents the resultant force exerted by the fluid per unit area of void in the undeformed configuration. Note that the effective stress tensor $\boldsymbol{P}$ and the partial stress tensor $\boldsymbol{P}^{\mathrm{s}}$ are not the same, but instead are related by the equation

$$\boldsymbol{P}^{\mathrm{s}} = \boldsymbol{P} + \left(\frac{1}{\varphi} - 1\right)\boldsymbol{P}^{\mathrm{w}}\,. \tag{2.9}$$

Eq. (2.8) is borrowed from Terzaghi's effective stress equation [42], which states that the total stress can be expressed as the sum of the effective stress and the fluid pressure.

An integral equation similar to (2.4a) can be developed in terms of the tensor $\tilde{\boldsymbol{P}}$. With reference to the undeformed configuration, (2.4) can be written in the form

$$\int_{\mathcal{U}} \rho_0 \boldsymbol{G}\,\mathrm{d}V + \int_{\partial\mathcal{U}} \tilde{\boldsymbol{P}}\cdot\boldsymbol{N}\,\mathrm{d}A = \boldsymbol{0}\,, \tag{2.10}$$

where $\boldsymbol{G} \equiv \boldsymbol{g}$ is the vector of gravity accelerations associated with the undeformed configuration, and $\rho_0 = J\rho$ is a pull-back mass density of the soil–water mixture.

The above definitions are based on a transformation equation associated with the displacement $\boldsymbol{u}$ of the solid phase. Consequently, the fluid now occupying the void in a soil at a point $\phi(X,t)$ may not necessarily be the same fluid material that occupied the same void at a reference point $\phi(X,0)$ in the undeformed region $\mathcal{U}$. Mathematically, $\phi_t(X) = \phi_t^{\mathrm{w}}(Y)$, where $\phi_t^{\mathrm{w}}(Y)$ is the configuration of the fluid phase that is now coincident with $\phi_t(X)$, and $\phi_{t=0}^{\mathrm{w}}(Y)$ is the original configuration of the fluid material point Y that is now at ϕ_t (note that $Y \equiv \phi_{t=0}^{\mathrm{w}}(Y)$ does not necessarily have to be in $\mathcal{B}$, see Fig. 1). Hence, ρ_0 does not necessarily represent the true mass density in $\mathcal{B}$ of the soil mass which now occupies the volume $\phi_t(\mathcal{B})$, since fluid could migrate into or out from the soil matrix during the motion. In other words, the total mass of the soil–water mixture in $\mathcal{B}$ is not necessarily conserved in $\phi_t(\mathcal{B})$.

To further understand the implications of the diffusion effects on mass densities, consider the following simple phase relationship analysis. Let $\varphi_0 = \varphi_0(X, t=0)$ be the initial porosity of the point X in $\mathcal{B}$. Then, the initial volume of the voids in an elementary volume $\mathrm{d}V$ is $\varphi_0\,\mathrm{d}V$, while the initial volume of the solid region is $(1-\varphi_0)\,\mathrm{d}V$. As the solid matrix deforms, its volume changes to $\mathrm{d}v = J\,\mathrm{d}V$. Now, assume that the solid phase is incompressible. Since $\boldsymbol{u}$ is the displacement of the solid phase, then

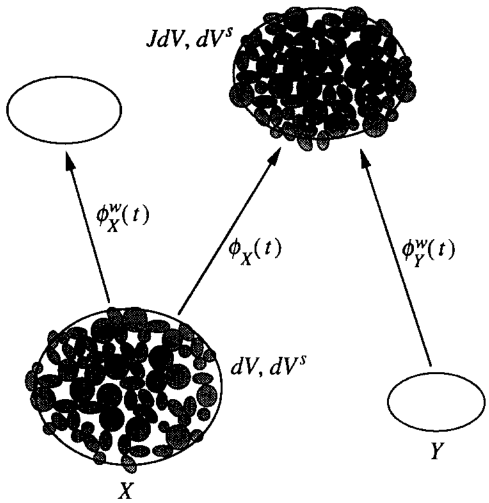

Fig. 1. Balance of mass: solid phase motion is described by the trajectory $\phi_X(t)$ which conserves dV^s but not dV. Fluid initially at macroscopic point X follows the trajectory $\phi^w_X(t)$ and leaves dV while fluid initially at Y enters $J\,dV$ at time t. Hence, total mass of soil–water mixture in dV is not conserved in $J\,dV$.

its volume is conserved at $(1-\varphi_0)\,dV$ in dv, while the volume of the voids changes to $dv-(1-\varphi_0)\,dV$. Consequently, the porosity varies according to

$$\varphi=\frac{J\,dV-(1-\varphi_0)\,dV}{J\,dV}=1-(1-\varphi_0)J^{-1}\,. \tag{2.11}$$

Hence, the total mass density and porosity of the soil vary with deformation through the Jacobian J.

Now, the localization of (2.10) results in the following partial statement of the boundary-value problem: Find the solid phase motion ϕ_t: $\mathscr{B}\rightarrow R^{n_{sd}}$ such that

$$\mathrm{DIV}\,\tilde{\boldsymbol{P}}+\rho_0\boldsymbol{G}=\boldsymbol{0}\,, \tag{2.12}$$

subject to the following boundary conditions: the motion ϕ is prescribed to be ϕ_d on a portion $\partial\mathscr{B}^d$ of the boundary $\partial\mathscr{B}$, and the traction $\tilde{\boldsymbol{P}}\cdot\boldsymbol{N}=\boldsymbol{t}$ is prescribed on the remainder $\partial\mathscr{B}^t$; and subject further to the constraint imposed by the balance of mass. In (2.12), DIV is the material divergence operator and $\boldsymbol{G}$ is the vector of gravity accelerations.

2.2. Balance of mass

Let the total masses of solid and fluid in an arbitrary deformed configuration $\phi_t(\mathscr{U})$ be denoted by m_s and m_w, respectively. In terms of densities, these masses are given by the volume integrals

$$m_s=\int_{\phi_t(\mathscr{U})}\rho_s(1-\varphi)\,dv\,;\qquad m_w=\int_{\phi_t(\mathscr{U})}\rho_w\varphi\,dv\,. \tag{2.13}$$

By the law of conservation of mass the material time derivatives of these masses vanish individually. For the solid phase, we have

$$\frac{d(m_s)}{dt}=\int_{\phi_t(\mathscr{U})}\left\{\frac{\partial[\rho_s(1-\varphi)]}{\partial t}+\mathrm{div}[\rho_s(1-\varphi)\boldsymbol{v}]\right\}dv=0 \tag{2.14a}$$

with the localization

$$\frac{\partial[\rho_s(1-\varphi)]}{\partial t}+\mathrm{div}[\rho_s(1-\varphi)\boldsymbol{v}]=0\,, \tag{2.14b}$$

where $\boldsymbol{v}$ is the intrinsic velocity of the solid phase. Similarly, for the fluid phase, we have

$$\frac{\mathrm{d}(m_{\mathrm{w}})}{\mathrm{d}t} = \int_{\phi_t(\mathcal{U})} \left\{ \frac{\partial(\rho_{\mathrm{w}}\varphi)}{\partial t} + \mathrm{div}(\rho_{\mathrm{w}}\varphi \boldsymbol{v}^{\mathrm{w}}) \right\} \mathrm{d}v = 0 \tag{2.15a}$$

with the localization

$$\frac{\partial(\rho_{\mathrm{w}}\varphi)}{\partial t} + \mathrm{div}(\rho_{\mathrm{w}}\varphi \boldsymbol{v}^{\mathrm{w}}) = 0 \,, \tag{2.15b}$$

where $\boldsymbol{v}^{\mathrm{w}}$ is the intrinsic velocity of the fluid phase.

The conservation of mass for the soil–water mixture can be derived directly from (2.14) and (2.15). For example, adding (2.14b) and (2.15b) gives

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \bar{\boldsymbol{v}}) = 0 \,, \tag{2.16}$$

where $\bar{\boldsymbol{v}}$ is the volume-average velocity of the mixture, given explicitly by

$$\bar{\boldsymbol{v}} = \frac{\rho_{\mathrm{s}}(1-\varphi)\boldsymbol{v} + \rho_{\mathrm{w}}\varphi \boldsymbol{v}^{\mathrm{w}}}{\rho} \,. \tag{2.17}$$

Now, when the fluid and solid phases follow the same motion, we have $\bar{\boldsymbol{v}} = \boldsymbol{v}^{\mathrm{w}} = \boldsymbol{v}$, and the soil–water mixture undergoes a one-phase flow. Such flow prevails under a condition of undrained deformation.

For future reference, it is useful to define a superficial, or Darcy, velocity as

$$\tilde{\boldsymbol{v}} = \varphi(\boldsymbol{v}^{\mathrm{w}} - \boldsymbol{v}) \,. \tag{2.18}$$

The vector $\tilde{\boldsymbol{v}}$ represents the relative volumetric rate of flow of fluid per unit area of the deforming soil mass. In general, $\tilde{\boldsymbol{v}}$ is induced by a fluid potential Π in $\phi_t(\mathcal{B})$. The exact form for the potential Π as well as the constitutive relationship that governs between $\tilde{\boldsymbol{v}}$ and Π are given in Section 3.5. For now let us take this potential as unknown, just as the solid phase motion ϕ_t, and develop in the following paragraph a boundary-value problem complementary to (2.12).

Assume that both the fluid and solid phases are homogeneous and incompressible. Then, ρ_{s} and ρ_{w} can be factored out of the partial derivative operators and eliminated from (2.14a) and (2.15a). Adding the resulting expressions yields

$$\mathrm{div}[(1-\varphi)\boldsymbol{v}] + \mathrm{div}(\varphi \boldsymbol{v}^{\mathrm{w}}) = 0 \,. \tag{2.19}$$

Since $\varphi \boldsymbol{v}^{\mathrm{w}} = \tilde{\boldsymbol{v}} + \varphi \boldsymbol{v}$ from (2.18), we obtain the following statement of the boundary-value problem complementary to (2.12): Find the potential Π in $\phi_t(\mathcal{B})$ such that

$$\mathrm{div}\, \boldsymbol{v} + \mathrm{div}\, \tilde{\boldsymbol{v}} = 0 \,, \tag{2.20}$$

subject to the following boundary conditions: Π is prescribed to be Π_θ on a portion $\partial\phi_t^\theta$ of $\partial\phi_t(\mathcal{B})$, and the volumetric flow is $\tilde{\boldsymbol{v}} \cdot \boldsymbol{n} = -q$ on the remainder $\partial\phi_t^h$; and subject further to the constraint imposed by the conservation of momentum. Here, $\boldsymbol{n}$ is the outward unit normal to the deformed surface $\partial\phi_t(\mathcal{B})$ and q is positive when fluid is being supplied to the system.

2.3. Balance of energy

This section describes the balance of energy, or the 'first law of thermodynamics,' for a saturated soil–water mixture. Balance of energy is important in interpreting the so-called stored energy function that will be used extensively in the next sections.

Assume there exist internal energy functions e_{s} and e_{w} representing the internal energy function per unit solid mass and the internal energy function per unit fluid mass, respectively. Ignoring kinetic energy and non-mechanical power, and assuming balance of momentum and balance of mass hold, balance of energy for the solid phase reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\phi_t(\mathcal{U})}\rho_s(1-\varphi)e_s\,\mathrm{d}v=\int_{\phi_t(\mathcal{U})}\rho_s(1-\varphi)\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}v+\int_{\phi_t(\mathcal{U})}\boldsymbol{h}^{\mathrm{s}}\cdot\boldsymbol{v}\,\mathrm{d}v+\int_{\partial\phi_t(\mathcal{U})}\boldsymbol{\sigma}^{\mathrm{s}}:\boldsymbol{v}\otimes\boldsymbol{n}\,\mathrm{d}a \tag{2.21a}$$

with the following localization

$$\rho_s(1-\varphi)\dot{e}_s=\boldsymbol{\sigma}^{\mathrm{s}}:\boldsymbol{d}\,, \tag{2.21b}$$

where $(\boldsymbol{v}\otimes\boldsymbol{n})_{ij}=v_in_j$, and

$$\boldsymbol{d}=\mathrm{symm}(\boldsymbol{l})\,;\qquad \boldsymbol{l}=\mathrm{grad}\,\boldsymbol{v}\,. \tag{2.22}$$

One often refers to $\boldsymbol{d}$ as the rate of deformation tensor and $\boldsymbol{l}$ as the spatial velocity gradient.

Under the same sets of assumptions, balance of energy for the fluid phase can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\phi_t(\mathcal{U})}\rho_w\varphi e_w\,\mathrm{d}v=\int_{\phi_t(\mathcal{U})}\rho_w\varphi\boldsymbol{g}\cdot\boldsymbol{v}^{\mathrm{w}}\,\mathrm{d}v+\int_{\phi_t(\mathcal{U})}\boldsymbol{h}^{\mathrm{w}}\cdot\boldsymbol{v}^{\mathrm{w}}\,\mathrm{d}v+\int_{\partial\phi_t(\mathcal{U})}\boldsymbol{\sigma}^{\mathrm{w}}:\boldsymbol{v}^{\mathrm{w}}\otimes\boldsymbol{n}\,\mathrm{d}a \tag{2.23a}$$

with the localization

$$\rho_w\varphi\dot{e}_w=\boldsymbol{\sigma}^{\mathrm{w}}:\boldsymbol{d}^{\mathrm{w}}\,, \tag{2.23b}$$

where

$$\boldsymbol{d}^{\mathrm{w}}=\mathrm{symm}(\boldsymbol{l}^{\mathrm{w}})\,;\qquad \boldsymbol{l}^{\mathrm{w}}=\mathrm{grad}\,\boldsymbol{v}^{\mathrm{w}}\,. \tag{2.24}$$

The localized versions of balance of energy can be derived in the following fashion. Consider the left-hand side of (2.23a), for example. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\phi_t(\mathcal{U})}\rho_w\varphi e_w\,\mathrm{d}v&=\int_{\phi_t(\mathcal{U})}\left\{\frac{\partial(\rho_w\varphi e_w)}{\partial t}+\mathrm{div}[\rho_w\varphi e_w\boldsymbol{v}^{\mathrm{w}}]\right\}\mathrm{d}v\\&=\int_{\phi_t(\mathcal{U})}\left\{e_w\left[\frac{\partial(\rho_w\varphi)}{\partial t}+\mathrm{div}(\rho_w\varphi\boldsymbol{v}^{\mathrm{w}})\right]+\rho_w\varphi\left(\frac{\partial e_w}{\partial t}+\mathrm{grad}\,e_w\cdot\boldsymbol{v}^{\mathrm{w}}\right)\right\}\mathrm{d}v\\&\equiv\int_{\phi_t(\mathcal{U})}\rho_w\varphi\dot{e}_w\,\mathrm{d}v\,,\end{aligned}$$

by virtue of the localized version of balance of mass (2.15b). Also,

$$\int_{\partial\phi_t(\mathcal{U})}\boldsymbol{\sigma}^{\mathrm{w}}:\boldsymbol{v}^{\mathrm{w}}\otimes\boldsymbol{n}\,\mathrm{d}a=\int_{\phi_t(\mathcal{U})}[(\mathrm{div}\,\boldsymbol{\sigma}^{\mathrm{w}})\cdot\boldsymbol{v}^{\mathrm{w}}+\boldsymbol{\sigma}^{\mathrm{w}}:\boldsymbol{l}^{\mathrm{w}}]\,\mathrm{d}v\,.$$

Substituting in (2.23a), using the localized version of balance of linear momentum (2.3b), and noting that $\boldsymbol{\sigma}^{\mathrm{w}}:\boldsymbol{l}^{\mathrm{w}}=\boldsymbol{\sigma}^{\mathrm{w}}:\boldsymbol{d}^{\mathrm{w}}$ since $\boldsymbol{\sigma}^{\mathrm{w}}$ is symmetric, we obtain the localized version (2.23b).

Balance of energy for the soil–water mixture can be derived by summing the stress powers of (2.21b) and (2.23b). The result takes the form

$$\rho\dot{\bar{e}}=\boldsymbol{\sigma}^{\mathrm{s}}:\boldsymbol{d}+\boldsymbol{\sigma}^{\mathrm{w}}:\boldsymbol{d}^{\mathrm{w}}\,, \tag{2.25}$$

where $\dot{\bar{e}}$ is the rate of internal energy for the soil–water mixture obtained from the volume average

$$\dot{\bar{e}}=\frac{\rho_s(1-\varphi)\dot{e}_s+\rho_w\varphi\dot{e}_w}{\rho}\,. \tag{2.26}$$

It is often convenient to describe balance of energy in the material picture because the domain of integration of the function remains fixed. To this end, we make use of the following transformation. Let the right leg of the tensor $\tilde{\boldsymbol{P}}$ be pushed forward by the configuration ϕ_t. The result is the Kirchhoff total stress tensor $\tilde{\boldsymbol{\tau}}$, which differs from the Cauchy total stress tensor $\tilde{\boldsymbol{\sigma}}$ by the factor J, i.e.

$$\tilde{\boldsymbol{\tau}}=J\tilde{\boldsymbol{\sigma}}=\tilde{\boldsymbol{P}}\cdot\boldsymbol{F}^{\mathrm{t}}\,. \tag{2.27}$$

By the additive decomposition of $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{P}}$, we can also decompose $\tilde{\boldsymbol{\tau}}$ into a solid part and a fluid part in any of the following ways (cf. (2.6) and (2.7))

$$\tilde{\boldsymbol{\tau}} = \boldsymbol{\tau}^{s} + \boldsymbol{\tau}^{w} = \boldsymbol{\tau} + \frac{\boldsymbol{\tau}^{w}}{\varphi} . \tag{2.28}$$

We shall use the second decomposition, based on the effective stress $\boldsymbol{\tau}$, whenever possible. Further, we will assume a 'perfect' fluid (i.e. no capability to generate shear tractions) and rewrite the effective stress equation based on Kirchhoff stresses as follows

$$\tilde{\boldsymbol{\tau}} = \boldsymbol{\tau} - \theta \mathbf{1} , \tag{2.29}$$

where θ is the Kirchhoff pore water pressure (compression positive) and $\mathbf{1}$ is the second-order identity tensor.

If we pull back the left leg of $\tilde{\boldsymbol{P}}$ by the inverse motion $\phi_t^* =$ inverse (ϕ_t), then we obtain the symmetric second Piola–Kirchhoff total stress tensor $\tilde{\boldsymbol{S}}$

$$\tilde{\boldsymbol{S}} = \boldsymbol{F}^{-1} \cdot \tilde{\boldsymbol{P}} = \boldsymbol{F}^{-1} \cdot \tilde{\boldsymbol{\tau}} \cdot \boldsymbol{F}^{-t} = J\boldsymbol{F}^{-1} \cdot \tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{F}^{-t} . \tag{2.30}$$

Again, by the additive decomposition of the solid and fluid stresses, we have the following effective stress equation

$$\tilde{\boldsymbol{S}} = \boldsymbol{S} - \theta \boldsymbol{C}^{-1} , \tag{2.31}$$

where $\boldsymbol{S}$ is the second Piola–Kirchhoff effective stress tensor and $\boldsymbol{C}$ is the symmetric right Cauchy–Green tensor given explicitly by

$$\boldsymbol{C} = \boldsymbol{F}^{t} \cdot \boldsymbol{F} , \tag{2.32}$$

where it is again recalled that $\boldsymbol{F}$ is the deformation gradient computed from the solid phase motion $\phi(X, t)$.

Now, let $x = \phi(X, t)$, $E_s(X, t) = e_s(x, t)$ and $E_w(X, t) = e_w(x, t)$. If we multiply the localized balance of energy for the solid phase (2.21b) by J, and use the porosity expression (2.11), then we obtain the following expression for balance of energy for the solid phase in localized material form

$$\rho_s(1 - \varphi_0)\dot{E}_s = \boldsymbol{\tau}^{s} : \boldsymbol{d} \equiv \frac{1}{2} \boldsymbol{S}^{s} : \dot{\boldsymbol{C}} . \tag{2.33}$$

The similarity in form between (2.21b) and (2.33) is not surprising since the mass of the solid phase is conserved by its own motion. On the other hand, multiplying the localized balance of energy for the fluid phase by the same Jacobian gives

$$\rho_w[J - (1 - \varphi_0)]\dot{E}_w = \boldsymbol{\tau}^{w} : \boldsymbol{d}^{w} \equiv \frac{1}{2} \boldsymbol{S}^{w} : \dot{\boldsymbol{C}}^{w} , \tag{2.34}$$

with an equivalent interpretation given to the tensor $\boldsymbol{C}^{w}$. In contrast to (2.33), (2.34) has no physical meaning since the motion of the fluid phase is not associated with the Jacobian J. Balance of energy for the soil–water mixture in the material picture is now given by

$$J\rho \dot{\bar{E}} = \boldsymbol{\tau}^{s} : \boldsymbol{d} + \boldsymbol{\tau}^{w} : \boldsymbol{d}^{w} = \frac{1}{2} \boldsymbol{S}^{s} : \dot{\boldsymbol{C}} + \frac{1}{2} \boldsymbol{S}^{w} : \dot{\boldsymbol{C}}^{w} , \tag{2.35}$$

where $\dot{\bar{E}}$ is obtained from the volume average

$$\dot{\bar{E}} = \frac{\rho_s(1 - \varphi_0)\dot{E}_s + \rho_w[J - (1 - \varphi_0)]\dot{E}_w}{J\rho} \equiv \dot{\bar{e}} . \tag{2.36}$$

The quantity $J\rho\dot{\bar{E}}$ is the mechanical power generated per unit reference volume of the soil–water mixture.

So far we have presented the results for balance of energy in terms of partial stresses. However, in geotechnical engineering practice, partial stresses are almost never used, and effective stresses are very highly favored. The following proposition unifies the concepts presented and demonstrates that effective stresses can be used for interpreting energy balance laws even more 'effectively' than partial stresses.

PROPOSITION 1. Assuming incompressible solid grains and fluids and that balance of mass holds for a saturated soil–water mixture, then

$$J\rho\dot{\bar{E}} = \boldsymbol{\tau} : \boldsymbol{d} \,, \tag{2.37}$$

i.e. the sum of the mechanical powers of the partial stresses is equal to the mechanical power of the effective stresses with respect to the deformation of the solid matrix computed from its own motion.

To prove (2.37) we need the following lemma

LEMMA. The spatial gradient of the Jacobian is zero.

PROOF. One can show from matrix algebra that if $\boldsymbol{a}(z)$ is a matrix function of z, then

$$\frac{\mathrm{d}}{\mathrm{d}z} \det \boldsymbol{a}(z) = \frac{\mathrm{d}a_{ij}}{\mathrm{d}z} \mathrm{COF}(a_{ij}) \,,$$

where $\mathrm{COF}(a_{ij})$ is the (i, j)th cofactor of the element a_{ij} of the matrix $\boldsymbol{a}$. Therefore,

$$\frac{\partial J}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\frac{\partial \phi_j}{\partial X_K} \right) \mathrm{COF}(F_{jK}) \equiv 0 \,.$$

COROLLARY. An important corollary to this result, in view of (2.5) and (2.11), is that

$$\operatorname{grad} J = \operatorname{grad} \rho = \operatorname{grad} \varphi = \boldsymbol{0} \,. \tag{2.38}$$

PROOF OF (2.37). By definition,

$$J\rho\dot{\bar{E}} = \boldsymbol{\tau}^{\mathrm{s}} : \boldsymbol{d} + \boldsymbol{\tau}^{\mathrm{w}} : \boldsymbol{d}^{\mathrm{w}} = \boldsymbol{\tau}^{\mathrm{s}} : \boldsymbol{d} - \varphi\theta \boldsymbol{1} : \boldsymbol{d}^{\mathrm{w}} = \boldsymbol{\tau}^{\mathrm{s}} : \boldsymbol{d} - \varphi\theta \operatorname{div} \boldsymbol{v}^{\mathrm{w}} \,.$$

Now, expanding the volume conservation equation (2.19) and using (2.38), we obtain

$$\operatorname{div} \boldsymbol{v}^{\mathrm{w}} = \left(1 - \frac{1}{\varphi}\right) \operatorname{div} \boldsymbol{v} \,.$$

Inserting in the previous equation, we have

$$\begin{aligned} J\rho\dot{\bar{E}} &= \boldsymbol{\tau}^{\mathrm{s}} : \boldsymbol{d} + \varphi\theta\left(\frac{1}{\varphi} - 1\right) \operatorname{div} \boldsymbol{v} = \boldsymbol{\tau}^{\mathrm{s}} : \boldsymbol{d} - \left(\frac{1}{\varphi} - 1\right)\boldsymbol{\tau}^{\mathrm{w}} : \boldsymbol{d} \\ &= \left[\boldsymbol{\tau}^{\mathrm{s}} - \left(\frac{1}{\varphi} - 1\right)\boldsymbol{\tau}^{\mathrm{w}}\right] : \boldsymbol{d} \equiv \boldsymbol{\tau} : \boldsymbol{d} \,. \quad \square \end{aligned}$$

REMARK 1. Proposition 1 states that the total mechanical power in the soil–water mixture is absorbed by the energy rate $\boldsymbol{\tau} : \boldsymbol{d}$, and that the tensor $\boldsymbol{\tau}^{\mathrm{w}}/\varphi$ in (2.28) performs no work. This might appear like a paradox, but it must be recalled that $\boldsymbol{\tau}^{\mathrm{w}}/\varphi$ is a tensor of fluid forces per unit fluid area, and since the fluid is assumed to be incompressible and has no shear strength, then it cannot store volumetric nor deviatoric energy, i.e. it has no mechanical power. This example establishes an absolutely clear distinction between the partial fluid stress tensor $\boldsymbol{\tau}^{\mathrm{w}}$ and the pore pressure tensor $\boldsymbol{\tau}^{\mathrm{w}}/\varphi$.

3. Variational equations, constitutive theories and algorithms

This section establishes the weak form of the non-linear consolidation theory and outlines how the constitutive models for the solid and fluid phases may be incorporated in the formulation. The idea is to use the undeformed configuration whenever possible since the undeformed domain is fixed throughout the entire solution process.

3.1. *Weak form of the boundary-value problem*

Following the standard arguments of variational principles, we define the following spaces. Let the space of configurations be

$$\mathscr{C}_\phi = \{\phi : \mathscr{B} \to R^{n_{sd}} \mid \phi_i \in H^1, \phi = \phi_d \text{ on } \partial\mathscr{B}^d\}$$

and the space of variations be

$$\mathscr{V}_\phi = \{\boldsymbol{\eta} : \mathscr{B} \to R^{n_{sd}} \mid \eta_i \in H^1, \boldsymbol{\eta} = \mathbf{0} \text{ on } \partial\mathscr{B}^d\},$$

where H^1 is the usual Sobolev space of functions of degree one. Further, let $G: \mathscr{C}_\phi \times \mathscr{V}_\phi \to R$ be given by

$$G(\phi, \Pi, \boldsymbol{\eta}) = \int_{\mathscr{B}} (\text{GRAD}\, \boldsymbol{\eta} : \tilde{\boldsymbol{P}} - \rho_0 \boldsymbol{\eta} \cdot \boldsymbol{G})\, \mathrm{d}V - \int_{\partial\mathscr{B}^t} \boldsymbol{\eta} \cdot \boldsymbol{t}\, \mathrm{d}A \tag{3.1}$$

One can easily show that balance of linear momentum is given by the condition $G(\phi, \Pi, \boldsymbol{\eta}) = 0$, which is equivalent to (2.12) if $\tilde{\boldsymbol{P}}$ and $\boldsymbol{\eta}$ are assumed to be C^1.

Next, we define the space of potentials as

$$\mathscr{C}_\theta = \{\Pi : \phi_t(\mathscr{B}) \to R \mid \Pi \in H^1, \Pi = \Pi_\theta \text{ on } \partial\phi_t^\theta\}$$

and the corresponding space of variations as

$$\mathscr{V}_\theta = \{\psi : \phi_t(\mathscr{B}) \to R \mid \psi \in H^1, \psi = 0 \text{ on } \partial\phi_t^\theta\}.$$

Further, let $H: \mathscr{C}_\theta \times \mathscr{V}_\theta \to R$ be given by

$$H(\phi, \Pi, \psi) = \int_{\phi_t(\mathscr{B})} (\psi \,\text{div}\, \boldsymbol{v} - \text{grad}\, \psi \cdot \tilde{\boldsymbol{v}})\, \mathrm{d}v - \int_{\partial\phi_t^h(\mathscr{B})} \psi q \, \mathrm{d}a. \tag{3.2}$$

Again, one can show that balance of mass is given by the condition $H(\phi, \Pi, \psi) = 0$, which is equivalent to (2.20) if $\psi, \boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ are assumed to be C^1.

The weak form of the boundary-value problem is as follows. Find $\phi \in \mathscr{C}_\phi$ and $\Pi \in \mathscr{C}_\theta$ such that

$$G(\phi, \Pi, \boldsymbol{\eta}) = H(\phi, \Pi, \psi) = 0 \tag{3.3}$$

for all $\boldsymbol{\eta} \in \mathscr{V}_\phi$ and $\psi \in \mathscr{V}_\theta$.

Condition (3.3) emanates directly from the strong form of the boundary-value problem. However, the functions G and H possess an awkward structure that is not directly amenable to standard matrix manipulations. Hereafter, we will restructure these functions, particularly $H(\phi, \Pi, \psi)$, in such a way that the integration is done with respect to the common undeformed reference configuration $\mathscr{B}$.

Consider first the function $G(\phi, \Pi, \boldsymbol{\eta})$. Using the results described in the previous section, we can rewrite G in the following form

$$G(\phi, \Pi, \boldsymbol{\eta}) = \int_{\mathscr{B}} (\text{grad}\, \boldsymbol{\eta} : \boldsymbol{\tau} - \theta \,\text{div}\, \boldsymbol{\eta} - \rho_0 \boldsymbol{\eta} \cdot \boldsymbol{G})\, \mathrm{d}V - \int_{\partial\mathscr{B}^t} \boldsymbol{\eta} \cdot \boldsymbol{t}\, \mathrm{d}A. \tag{3.4}$$

Next, consider the function $H(\phi, \Pi, \psi)$. This is an integral function reckoned from the deformed configuration. The domain of integration can be reckoned quite easily from the undeformed configuration by introducing the Jacobian J. Since J will play a central role in the volume equations, we will describe one of its more important properties in the following proposition.

PROPOSITION 2. The time derivative of the Jacobian is

$$\dot{J} = J \,\text{div}\, \boldsymbol{v}. \tag{3.5}$$

PROOF. This is a standard result, see [31]. □

Now, let $\tilde{\boldsymbol{V}} \cdot \boldsymbol{N} = -Q$ be the prescribed volumetric rate of flow per unit undeformed area across the boundary $\partial\mathscr{B}^h$. Here, $\tilde{V} = JF^{-1} \cdot \tilde{v}$ is the Piola transform of the Darcy velocity $\tilde{v}$, and Q is positive when pointing inward relative to the undeformed surface $\partial\mathscr{B}$ with outward unit normal N. Inserting identity (3.5) in (3.2) results in the variational equation for balance of volume, now reckoned with respect to the undeformed configuration $\mathscr{B}$

$$H(\phi, \Pi, \psi) = \int_{\mathscr{B}} (\psi \dot{J} - \mathrm{grad}\, \psi \cdot J\tilde{v})\, \mathrm{d}V - \int_{\partial\mathscr{B}^h} \psi Q\, \mathrm{d}A\,. \tag{3.6}$$

A usual boundary condition is $q = Q = 0$, i.e. no fluid is supplied to the system, as would be the case with impermeable walls.

3.2. *Reduced dissipation inequality*

Let $\mathscr{D}$ denote the local dissipation function per unit reference volume of the soil matrix associated with the material point $X \in \mathscr{B}$. Further, let Ψ denote the stored energy function, or free energy, per unit reference volume of the soil matrix. Ignoring non-mechanical power and kinetic energy production, the second law states that

$$\mathscr{D} = \boldsymbol{\tau} : \boldsymbol{d} - \frac{\mathrm{d}\Psi}{\mathrm{d}t} = \frac{1}{2}\boldsymbol{S} : \dot{\boldsymbol{C}} - \frac{\mathrm{d}\Psi}{\mathrm{d}t} \geq 0\,. \tag{3.7}$$

Clearly, $\mathrm{d}\Psi/\mathrm{d}t = J\rho\dot{E}$ and $\mathscr{D} = 0$ for an elastic material (cf. (2.37)). Furthermore, for isothermal and elastic processes, Ψ depends only on X and $\boldsymbol{C}$ if it is to satisfy the axiom of material frame indifference [30, 31]. Equally well, we could say that for isothermal, elastic processes Ψ is a function only of X and the left Cauchy–Green tensor

$$\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^{\mathrm{t}}\,, \tag{3.8}$$

provided that $\boldsymbol{b}$ satisfies an objective transformation.

Now, for a more general elastoplastic process, one can employ the following multiplicative decomposition of the deformation gradient [15, 16]

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}^{\mathrm{u}}} \cdot \frac{\partial \boldsymbol{x}^{\mathrm{u}}}{\partial \boldsymbol{X}} \equiv \boldsymbol{F}^{\mathrm{e}} \cdot \boldsymbol{F}^{\mathrm{p}}\,, \quad \forall X \in \mathscr{B}; \quad t \geq 0\,, \tag{3.9}$$

where $\boldsymbol{x}^{\mathrm{u}}$ are the coordinates of the unloaded configuration ϕ_t^{u}, $\boldsymbol{F}^{\mathrm{e}} = \partial\boldsymbol{\phi}/\partial\boldsymbol{x}^{\mathrm{u}}$, and $\boldsymbol{F}^{\mathrm{p}} = \partial\boldsymbol{x}^{\mathrm{u}}/\partial\boldsymbol{X}$, see Fig. 2. Note that this product decomposition emanates from the chain rule. From a micromechanical standpoint, $\boldsymbol{F}^{\mathrm{p}}$ is an internal variable related to the amount of slipping, crushing, yielding, and (for

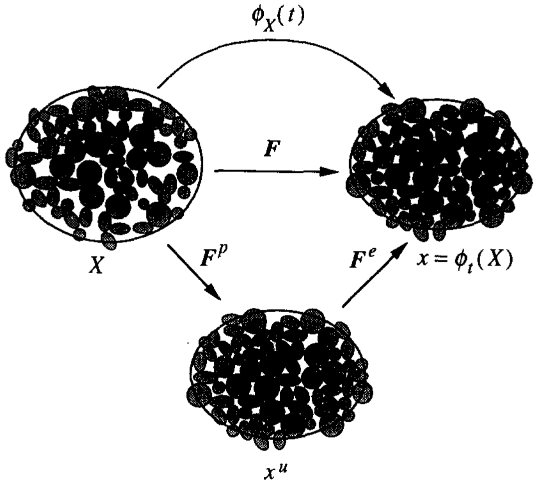

Fig. 2. Illustration of multiplicative decomposition of deformation gradient based on overall solid phase motion: product decomposition relies on the assumption that an unloaded, or intermediate, configuration x^{u} exists.

plate-like particles) plastic bending of the granules comprising the soil assembly [29]. Conversely, $(\boldsymbol{F}^{e})^{-1}$ defines the stress-free, unloaded configuration of the material point X. The product decomposition (3.9) then represents the overall kinematics of deformation of the macroscopic material point X and may be interpreted as the volume average of the responses derived from the aforementioned micromechanical processes [26–28].

An elastoplastic process requires a yield function, a hardening rule, and the imposition of the consistency condition. Let $\mathscr{F}$ be the yield function. Under the setting provided by (3.9), the stored energy function takes the functional form

$$\Psi = \Psi(X, \boldsymbol{b}^{e}, \xi), \qquad \boldsymbol{b}^{e} = \boldsymbol{F}^{e} \cdot \boldsymbol{F}^{e^{t}}, \tag{3.10}$$

where $\boldsymbol{b}^{e}$ is the left elastic Cauchy–Green tensor and ξ is an internal plastic variable defined such that $\chi := \partial\Psi/\partial\xi$ characterizes the hardening response of the soil.

In order for (3.7) to hold for all admissible processes, and to satisfy the postulate of maximum plastic dissipation, the results of [25] may be used to arrive at the following constitutive equations

$$\boldsymbol{\tau} = 2\frac{\partial\Psi}{\partial\boldsymbol{b}^{e}} \cdot \boldsymbol{b}^{e}, \qquad -\frac{1}{2}\mathscr{L}_{v}\boldsymbol{b}^{e} = \dot{\gamma}\frac{\partial\mathscr{F}}{\partial\boldsymbol{\tau}} \cdot \boldsymbol{b}^{e}; \qquad \dot{\xi} = \dot{\gamma}\frac{\partial\mathscr{F}}{\partial\chi}, \tag{3.11}$$

where $\mathscr{L}_{v}\boldsymbol{b}^{e}$ is the Lie derivative of $\boldsymbol{b}^{e}$. The consistency condition is given by the set of equations $\dot{\gamma} \geq 0$, $\mathscr{F} \leq 0$ and $\dot{\gamma}\mathscr{F} = 0$, where $\mathscr{F}$ again is the yield function.

REMARK 2. Since all the mechanical power is contained within the tensor $\boldsymbol{\tau}$, constitutive equations for soils are best developed in terms of effective stresses. In practice, effective stress-based constitutive models, such as the Cam-clay model and its enhancements, are developed without regard for the presence or absence of water in the voids of the soil. Models of this type typically replicate the phenomenological responses of dry soils, or those of water-saturated soils deforming under fully drained conditions. The use of $\boldsymbol{\tau}$ also allows the reckoning of the stored energy function Ψ from a fixed, undeformed reference volume of the soil matrix. This is a mathematical convenience over formulations based on partial stresses since it completely avoids the need to track the material motion of the fluid in modeling the soil response.

REMARK 3. A central assumption underlying (3.11), in addition to the assumption of material frame indifference, is the restriction of the theory to isotropy. For example, if $\mathscr{F} = \mathscr{F}(\boldsymbol{\tau}, \chi) = 0$ is the yield function, then $\mathscr{F}$ must be an isotropic function of $\boldsymbol{\tau}$. This implies that $\boldsymbol{\tau}, \boldsymbol{b}^{e}$ and $\partial\Psi/\partial\boldsymbol{b}^{e}$ all commute. Issues pertaining to finite strain elastoplasticity of one-phase bodies are elaborated extensively in [25] within the context of the multiplicative plasticity theory. Since this theory blends well with the proposed non-linear consolidation theory, we will adopt it in this paper and describe its main features in Section 4. Note that the following mathematical formulation is valid for any yield function of the form $\mathscr{F} = \mathscr{F}(\boldsymbol{\tau}, \chi) = 0$ describing the constitutive behavior of the soil skeleton.

3.3. *Multiplicative plasticity model for soil skeleton*

Let the elastic left Cauchy–Green tensor $\boldsymbol{b}^{e}$ be decomposed spectrally into

$$\boldsymbol{b}^{e} = \sum_{A=1}^{3} (\lambda_{A}^{e})^{2}\boldsymbol{m}^{(A)}; \qquad \boldsymbol{m}^{(A)} = \boldsymbol{n}^{(A)} \otimes \boldsymbol{n}^{(A)}, \tag{3.12}$$

where λ_{A}^{e} is the elastic principal stretch corresponding to the principal direction $\boldsymbol{n}^{(A)}$, and A is an index which takes on the values 1, 2 and 3. We recall that $\boldsymbol{b}^{e}$ is a measure of elastic deformation of the solid matrix, or soil skeleton. By restriction to isotropy, the Kirchhoff effective stress tensor $\boldsymbol{\tau}$ also can be decomposed spectrally in the form

$$\boldsymbol{\tau} = \sum_{A=1}^{3} \beta_{A}\boldsymbol{m}^{(A)}, \tag{3.13}$$

where β_A are the principal Kirchhoff effective stresses, for $A = 1, 2, 3$. Note that isotropy implies that the principal directions of $\boldsymbol{\tau}$ coincide with those of $\boldsymbol{b}^e$.

Frame indifference and isotropy also imply that the free energy function is a symmetric function of the elastic principal stretches. Equivalently, therefore, we have

$$\Psi(X, \boldsymbol{b}^e) = \tilde{\Psi}(X, \epsilon_1^e, \epsilon_2^e, \epsilon_3^e); \qquad \epsilon_A^e = \ln(\lambda_A^e), \quad A = 1, 2, 3. \tag{3.14}$$

The ϵ_A^e's are called principal elastic logarithmic stretches. Thus, the elastic constitutive equation $(3.11)_1$ boils down to scalar relationships between the principal Kirchhoff effective stresses β_A and the principal elastic logarithmic stretches ϵ_A^e through the function $\tilde{\Psi}$. The relationship takes the form

$$\beta_A = \frac{\partial \tilde{\Psi}}{\partial \epsilon_A^e}, \quad A = 1, 2, 3. \tag{3.15}$$

Note that (3.15) is valid for any form of stored energy function $\tilde{\Psi}$.

In the elastoplastic regime, there lies an additional task of enforcing the consistency condition, $\mathscr{F}(\boldsymbol{\tau}, \chi) = 0$, for the soil skeleton. The approach is best done by incrementation, starting from the converged configuration $\phi_{t_n}(\mathscr{B})$ and moving on to the unknown configuration $\phi_{t_{n+1}}(\mathscr{B})$. We again use the results of [25] to describe the following two-step procedure: In the first step, plastic flow is frozen and an elastic step is taken ignoring the constraints imposed by the yield criterion. This leads to a trial elastic state and the sets of equations

$$\dot{\boldsymbol{f}} = \boldsymbol{l} \cdot \boldsymbol{f}; \qquad \dot{\boldsymbol{b}}^e = 2\,\mathrm{symm}(\boldsymbol{l} \cdot \boldsymbol{b}^e); \qquad \dot{\xi} = 0, \tag{3.16}$$

where $\boldsymbol{f} = \partial \boldsymbol{\phi} / \partial \boldsymbol{x}_n$ is the deformation gradient evaluated relative to the configuration $\phi_{t_n}(\mathscr{B})$. In the second step, the trial state is held fixed and plastic relaxation is introduced. The algorithm is given explicitly by

$$\dot{\boldsymbol{f}} = \boldsymbol{0}; \qquad \dot{\boldsymbol{b}}^e = -2\dot{\gamma} \frac{\partial \mathscr{F}}{\partial \boldsymbol{\tau}} \cdot \boldsymbol{b}^e; \qquad \dot{\xi} = \dot{\gamma} \frac{\partial \mathscr{F}}{\partial \chi}, \tag{3.17}$$

subject to $\dot{\gamma} \geqslant 0$, $\mathscr{F} \leqslant 0$, and $\dot{\gamma}\mathscr{F} = 0$. In (3.16) and (3.17), $\boldsymbol{l}$ is the spatial velocity gradient of the solid phase and takes a form identical to that given by $(2.22)_2$.

The incremental counterpart of the evolution equations (3.16) and (3.17) is obtained from the so-called product formula algorithm. For (3.16), the incremental counterpart of the trial elastic left Cauchy–Green tensor is obtained by freezing plastic flow, and is computed from

$$\boldsymbol{b}^{e\,tr} = \boldsymbol{f} \cdot \boldsymbol{b}_n^e \cdot \boldsymbol{f}^t; \qquad \xi = \xi_n, \tag{3.18}$$

where $\boldsymbol{b}_n^e$ and ξ_n are the respective values of $\boldsymbol{b}^e$ and ξ at configuration ϕ_{t_n}. Next, the tensor $\boldsymbol{b}^{e\,tr}$ is decomposed spectrally in the form

$$\boldsymbol{b}^{e\,tr} = \sum_{A=1}^{3} (\lambda_A^{e\,tr})^2 \boldsymbol{m}^{tr(A)}; \qquad \boldsymbol{m}^{tr(A)} = \boldsymbol{n}^{tr(A)} \otimes \boldsymbol{n}^{tr(A)}. \tag{3.19}$$

An exponential approximation may then be introduced into the plastic flow equation $(3.11)_2$ via

$$\boldsymbol{b}^e = \exp\left(-2\,\Delta\gamma \frac{\partial \mathscr{F}}{\partial \boldsymbol{\tau}}\right) \cdot \boldsymbol{b}^{e\,tr}; \qquad \xi = \xi_n + \Delta\gamma \frac{\partial \mathscr{F}}{\partial \chi}, \tag{3.20}$$

where $\Delta\gamma$ is an incremental consistency parameter that satisfies the condition $\Delta\gamma \geqslant 0$, $\mathscr{F} \leqslant 0$ and $\Delta\gamma\, \mathscr{F} = 0$.

Now, by invoking isotropy we conclude that there exists a function $\tilde{\mathscr{F}} = \tilde{\mathscr{F}}(\beta_1, \beta_2, \beta_3, \xi)$ such that

$$\frac{\partial \mathscr{F}}{\partial \boldsymbol{\tau}} = \sum_{A=1}^{3} \frac{\partial \tilde{\mathscr{F}}}{\partial \beta_A} \boldsymbol{m}^{(A)}. \tag{3.21}$$

Inserting (3.21) in (3.20) and inverting gives

$$b^{\mathrm{e\,tr}} = \sum_{A=1}^{3} \left[(\lambda_A^{\mathrm{e}})^2 \exp\left(2\,\Delta\gamma \frac{\partial \tilde{\mathscr{F}}}{\partial \beta_A}\right) \right] m^{(A)} . \tag{3.22}$$

Comparing (3.19) and (3.22) then leads to the following useful result (see [25])

$$m^{(A)} = m^{\mathrm{tr}(A)} ; \qquad (\lambda_A^{\mathrm{e}})^2 = \exp\left(-2\,\Delta\gamma \frac{\partial \tilde{\mathscr{F}}}{\partial \beta_A}\right) (\lambda_A^{\mathrm{e\,tr}})^2 . \tag{3.23}$$

Eq. (3.23) states that the principal directions $n^{(A)}$ coincide with the trial principal directions $n^{\mathrm{tr}(A)}$, and that the plastic relaxation equation $(3.23)_2$ takes place along the fixed axis defined by the trial elastic state. Furthermore, an additive form of the plastic relaxation equation is obtained if one takes the natural logarithm of both sides of $(3.23)_2$. The result reads

$$\epsilon_A^{\mathrm{e}} = \epsilon_A^{\mathrm{e\,tr}} - \Delta\gamma \frac{\partial \tilde{\mathscr{F}}}{\partial \beta_A} . \tag{3.24}$$

Eq. (3.24) consists of an elastic logarithmic principal stretch predictor $\epsilon_A^{\mathrm{e\,tr}}$ and a plastic relaxation corrector in the direction of the linear map $\partial \tilde{\mathscr{F}} / \partial \beta_A$, and thus represents a linear return mapping algorithm in the space defined by the principal logarithmic stretches.

Finally, a linear return mapping algorithm similar to that presented in [43], but now taking place in the Kirchhoff effective stress space, is recovered if one assumes a constant elasticity operator α_{AB} from the equation

$$\beta_A = \frac{\partial \tilde{\Psi}}{\partial \epsilon_A^{\mathrm{e}}} = \sum_{B=1}^{3} \alpha_{AB} \epsilon_B^{\mathrm{e}} , \quad A = 1, 2, 3 . \tag{3.25}$$

The result reads

$$\beta_A = \beta_A^{\mathrm{tr}} - \Delta\gamma \sum_{B=1}^{3} \alpha_{AB} \frac{\partial \tilde{\mathscr{F}}}{\partial \beta_B} , \quad A = 1, 2, 3 . \tag{3.26}$$

Thus, $\boldsymbol{\tau}$ can be defined completely from (3.13) and (3.26), and may be inserted directly into the variational equation (3.4).

3.4. Specialization of the multiplicative plasticity theory to undrained loading

Assuming that the solid grains and fluids in a saturated soil–water mixture are both incompressible, then an undrained deformation is obtained from any one of the following equivalent conditions: (i) ϕ_t is volume preserving, (ii) $J(X, t) = 1$, and (iii) $\operatorname{div} \boldsymbol{v} = 0$. The equivalence of (ii) and (iii) can be established from (3.5) and from the condition that $J(X, 0) = 1$. Undrained conditions also imply that the fluid and solid phases follow the same motion so that $\boldsymbol{v} = \boldsymbol{v}^{\mathrm{w}}$ and $\tilde{\boldsymbol{v}} = \boldsymbol{0}$. Undrained conditions prevail in cases where the soil is loaded at a fast enough rate that fluid flow is inhibited.

Let us incorporate the condition of undrained deformation within the multiplicative plasticity theory (see [44]). Let $J^{\mathrm{e}} = \det(\boldsymbol{F}^{\mathrm{e}})$ and $J^{\mathrm{p}} = \det(\boldsymbol{F}^{\mathrm{p}})$. Taking the determinant of (3.9) results in the following product relation for undrained loading

$$J = J^{\mathrm{e}} J^{\mathrm{p}} \equiv 1 , \qquad \forall X \in \mathscr{B} ; \quad t \geq 0 . \tag{3.27}$$

Imposing (3.27) at time station t_n and at any time instant $t \in [t_n, t_{n+1}]$, we obtain

$$\frac{J^{\mathrm{e}}(X, t)}{J_n^{\mathrm{e}}} = \frac{J_n^{\mathrm{p}}}{J^{\mathrm{p}}(X, t)} , \tag{3.28}$$

where J_n^{e} and J_n^{p} are the respective values of $J^{\mathrm{e}}(X)$ and $J^{\mathrm{p}}(X)$ at time $t = t_n$.

To obtain a more explicit expression for $J^{\mathrm{e}}(X, t)$ and $J^{\mathrm{p}}(X, t)$, let us use the fact that $J^{\mathrm{e}} = (\det \boldsymbol{b}^{\mathrm{e}})^{1/2}$ from $(3.10)_2$; the time-derivative of J^{e}, using $(3.16)_2$ and $(3.17)_2$, is then given by

$$\dot{J}^{e} = \frac{1}{2}\frac{\dot{\boldsymbol{b}}^{e}}{J^{e}} : \mathrm{COF}(\boldsymbol{b}^{e}) = \frac{1}{2} J^{e} \boldsymbol{b}^{e^{-1}} : \dot{\boldsymbol{b}}^{e}$$
$$= J^{e} \boldsymbol{b}^{e^{-1}} : \left[\mathrm{symm}(\boldsymbol{l} \cdot \boldsymbol{b}^{e}) - \dot{\gamma} \frac{\partial \mathcal{F}}{\partial \boldsymbol{\tau}} \cdot \boldsymbol{b}^{e} \right]$$
$$\equiv - J^{e} \dot{\gamma} \operatorname{tr}\left(\frac{\partial \mathcal{F}}{\partial \boldsymbol{\tau}} \right) \tag{3.29}$$

since $\boldsymbol{b}^{e^{-1}} : \mathrm{symm}(\boldsymbol{l} \cdot \boldsymbol{b}^{e}) = \operatorname{div} \boldsymbol{v} \equiv 0$ for undrained loading. Integrating (3.29) using the backward difference scheme for the stresses yields

$$\frac{J^{e}(X, t)}{J^{e}_{n}} = \frac{J^{p}_{n}}{J^{p}(X, t)} = \exp\left[-\Delta\gamma \operatorname{tr}\left(\frac{\partial \mathcal{F}}{\partial \boldsymbol{\tau}} \right) \right], \tag{3.30}$$

where $\Delta\gamma = \dot{\gamma}(t - t_n)$. Since the logarithmic part was integrated exactly while the stresses were integrated only approximately, the accuracy of the approximation is carried over in (3.30), i.e. the integration is first-order accurate.

Now, from the spectral decomposition of $\boldsymbol{b}^{e}$, we have

$$J^{e}(X, t) = \det(\boldsymbol{F}^{e}) = \sqrt{\det(\boldsymbol{b}^{e})} = \lambda_1^{e} \lambda_2^{e} \lambda_3^{e}, \tag{3.31}$$

where the λ_A^{e}'s are the principal elastic stretches. Hence, the incompressibility constraint for the undrained problem becomes

$$\lambda_1^{e} \lambda_2^{e} \lambda_3^{e} = (\lambda_1^{e} \lambda_2^{e} \lambda_3^{e})_n \exp\left[-\Delta\gamma \operatorname{tr}\left(\frac{\partial \mathcal{F}}{\partial \boldsymbol{\tau}} \right) \right]. \tag{3.32}$$

Taking the natural logarithm of both sides gives

$$\operatorname{tr}(\boldsymbol{\epsilon}^{e}) = \operatorname{tr}(\boldsymbol{\epsilon}^{e}_{n}) - \Delta\gamma \operatorname{tr}\left(\frac{\partial \mathcal{F}}{\partial \boldsymbol{\tau}} \right), \tag{3.33}$$

where $\boldsymbol{\epsilon}^{e}$ and $\boldsymbol{\epsilon}^{e}_{n}$ are the diagonal tensors of logarithmic principal stretches evaluated at time instants t and t_n, respectively.

A similar expression to (3.33) can be obtained by taking the trace of (3.24)

$$\operatorname{tr}(\boldsymbol{\epsilon}^{e}) = \operatorname{tr}(\boldsymbol{\epsilon}^{e\,\mathrm{tr}}) - \Delta\gamma \operatorname{tr}\left(\frac{\partial \mathcal{F}}{\partial \boldsymbol{\tau}} \right). \tag{3.34}$$

Now, subtracting (3.33) from (3.34) results in

$$\operatorname{tr}(\boldsymbol{\epsilon}^{e\,\mathrm{tr}}) - \operatorname{tr}(\boldsymbol{\epsilon}^{e}_{n}) = \operatorname{tr}(\Delta\boldsymbol{\epsilon}^{e\,\mathrm{tr}}) = 0. \tag{3.35}$$

In other words, the incompressibility constraint is satisfied by the condition that the sum of the incremental trial principal elastic logarithmic stretches vanishes for each load increment. In the following proposition, we shall show that condition (3.35) actually results in *exact* conservation of volume for all $t \geqslant t_n$.

PROPOSITION 3. The following statements are equivalent: (*i*) $\operatorname{tr}(\Delta\boldsymbol{\epsilon}^{e\,\mathrm{tr}}) = \operatorname{tr}(\boldsymbol{\epsilon}^{e\,\mathrm{tr}}) - \operatorname{tr}(\boldsymbol{\epsilon}^{e}_{n}) = 0$; (*ii*) $\dot{J}^{e\,\mathrm{tr}} = 0$, *where* $J^{e\,\mathrm{tr}} = \sqrt{\det \boldsymbol{b}^{e\,\mathrm{tr}}}$ *and* (*iii*) $\operatorname{div} \boldsymbol{v} = 0$.

PROOF. To prove the equivalence of (i) and (ii), let us rewrite (i) as

$$\lambda_1^{e\,\mathrm{tr}} \lambda_2^{e\,\mathrm{tr}} \lambda_3^{e\,\mathrm{tr}} = (\lambda_1^{e} \lambda_2^{e} \lambda_3^{e})_n = \text{constant}.$$

Taking the time derivative gives

$$\frac{\dot{\lambda}_1^{e\,\mathrm{tr}}}{\lambda_1^{e\,\mathrm{tr}}} + \frac{\dot{\lambda}_2^{e\,\mathrm{tr}}}{\lambda_2^{e\,\mathrm{tr}}} + \frac{\dot{\lambda}_3^{e\,\mathrm{tr}}}{\lambda_3^{e\,\mathrm{tr}}} \equiv \frac{1}{2} (\boldsymbol{b}^{e\,\mathrm{tr}})^{-1} : \dot{\boldsymbol{b}}^{e\,\mathrm{tr}} = 0.$$

Multiplying the result by $J^{\mathrm{e\,tr}}$ gives

$$\frac{1}{2} J^{\mathrm{e\,tr}} \boldsymbol{b}^{\mathrm{e\,tr}-1} : \dot{\boldsymbol{b}}^{\mathrm{e\,tr}} = \frac{1}{2} \frac{1}{J^{\mathrm{e\,tr}}} \dot{\boldsymbol{b}}^{\mathrm{e\,tr}} : \mathrm{COF}(\boldsymbol{b}^{\mathrm{e\,tr}}) \equiv \dot{J}^{\mathrm{e\,tr}} = 0\,.$$

To prove the equivalence of (ii) and (iii), we argue that since $\dot{J}^{\mathrm{e\,tr}} = 0$, then the time derivative of $\det \boldsymbol{b}^{\mathrm{e\,tr}}$ also must vanish. From (3.18), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} (\det \boldsymbol{b}^{\mathrm{e\,tr}}) = 2 \det \boldsymbol{f} \det \boldsymbol{b}_n^{\mathrm{e}} \frac{\mathrm{d}}{\mathrm{d}t} (\det \boldsymbol{f}) = 0\,.$$

Since neither $\det \boldsymbol{f}$ nor $\det \boldsymbol{b}_n^{\mathrm{e}}$ is zero, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} (\det \boldsymbol{f}) = \dot{\boldsymbol{f}} : \mathrm{COF}(\boldsymbol{f}) = \frac{J}{J_n} \operatorname{div} \boldsymbol{v} = 0\,.$$

Since $J \neq 0$, we have $\operatorname{div} \boldsymbol{v} = 0$, resulting in exact conservation of volume. □

A corollary to Proposition 3 is that the backward difference algorithm (3.30) is equivalent to the product formula algorithm (3.20), and that even though they are both first-order accurate the numerical errors cancel in (3.35) in such a way that the total volume of the solid matrix is conserved exactly.

The weak form of the boundary value problem for undrained loading can now be stated as follows (cf. (3.3)). Let $\mathscr{C}_{\mathrm{vol}} = \{\boldsymbol{\phi} \in \mathscr{C}_{\phi} \mid J(\boldsymbol{\phi}) = 1\}$ be the space of all volume-preserving configurations, then find $\boldsymbol{\phi} \in \mathscr{C}_{\mathrm{vol}}$ such that for all $\boldsymbol{\eta} \in \mathscr{V}_{\phi}$,

$$G(\boldsymbol{\phi}, \Pi, \boldsymbol{\eta}) = 0\,, \tag{3.36}$$

where $G(\boldsymbol{\phi}, \Pi, \boldsymbol{\eta})$ is given by (3.4). Now, if $\theta = 0$, then $G = G(\boldsymbol{\phi}, \boldsymbol{\eta})$, and we obtain a well-posed boundary value problem of finding the motion $\boldsymbol{\phi}$ of a one-phase continuum, subject to (3.36) and to the prescribed boundary conditions. The results of [45] can be used to show that setting $\theta \neq 0$ does not destroy the well-posedness of the resulting boundary value problem.

The solution of (3.36) may be obtained by setting (cf. (3.6))

$$H(\boldsymbol{\phi}, \Pi, \psi) \equiv H^*(\psi) = \int_{\mathscr{B}} \psi \dot{J} \,\mathrm{d}V = 0\,, \tag{3.37}$$

in which case, the pore pressure variable θ plays the role of a Lagrange multiplier. Alternately, we can borrow the idea of [46, 47] and replace the pore pressure variable θ by the following constitutive law

$$\theta = \theta_n - \frac{\lambda_{\mathrm{w}}}{\varphi_0} \operatorname{tr}(\Delta \boldsymbol{\epsilon}^{\mathrm{e\,tr}})\,, \tag{3.38}$$

where λ_{w} is the bulk modulus of the fluid phase, φ_0 is the reference porosity of the soil, and θ_n is the Kirchhoff pore water pressure at time station t_n. Note in (3.38) that $J = 1$ implies that the Kirchhoff pore water pressure θ becomes identical to the Cauchy pore water pressure, or simply, the 'pore water pressure,' and that the porosity φ is conserved at φ_0 from (2.11). Now, as $\lambda_{\mathrm{w}} \to \infty$, it is clear from the energy equation (2.35) that $J\rho\dot{E}$ can be conserved if and only if $\operatorname{tr}(\Delta \boldsymbol{\epsilon}^{\mathrm{e\,tr}}) \to 0$ (which also implies $\operatorname{div} \boldsymbol{v} \to 0$). The constitutive equation (3.38) is identical in form to that used in [48, 49] for undrained analysis of saturated soil media at infinitesimal strains.

REMARK 4. Formula (3.38) allows the pore water pressure to be backfigured naturally for large but finite values of λ_{w}. The presence of the porosity φ_0 in the denominator makes (3.38) physically meaningful if one uses the actual bulk modulus of the slightly compressible fluid; otherwise, $\lambda_{\mathrm{w}}/\varphi_0$ simply becomes a penalty parameter. Note that θ in (3.38) has the physical meaning of being the 'pore water pressure,' and is not merely an artificial parameter as, for example, the pressure terms used in [46, 47] to impose the incompressibility constraint.

3.5. *Constitutive law for fluid flow*

We now turn to the fluid flow problem and describe a constitutive law similar to that developed for the solid phase. Assuming laminar flow, we can use the generalized Darcy's law to obtain the following linear constitutive equation

$$\tilde{\boldsymbol{v}} = -\boldsymbol{k} \cdot \operatorname{grad} \Pi \,, \tag{3.39}$$

where $\boldsymbol{k}$ is the second-order permeability tensor and Π is the same fluid potential used in (2.20). The negative sign in (3.39) implies that the fluid always flows in the direction of decreasing potential. The permeability tensor $\boldsymbol{k}$ may be assumed to be symmetric and positive-definite in the majority of cases.

For incompressible flow the potential Π may be decomposed into a pressure part Π^{θ} and an elevation part Π^{e}. Let the elevation part of the potential be measured in the direction of the gravity acceleration vector $\boldsymbol{G}$; then the decomposition of Π takes the form

$$\Pi = \Pi^{\theta} + \Pi^{e} = \frac{\theta}{J g \rho_w} + \Pi^{e} \,, \tag{3.40}$$

where g is the gravity acceleration constant. Taking the spatial gradient of (3.40) and using (2.38), wc obtain

$$\operatorname{grad} \Pi = \frac{\operatorname{grad} \theta}{J g \rho_w} + \frac{\boldsymbol{G}}{g} \,. \tag{3.41}$$

Thus, the variational equation (3.6) for the volume conservation may be written as

$$H(\phi, \theta, \psi) = \int_{\mathcal{B}} \psi \dot{J} \, \mathrm{d}V + \int_{\mathcal{B}} \operatorname{grad} \psi \cdot \boldsymbol{k} \cdot \left(\frac{\operatorname{grad} \theta}{g \rho_w} + J \frac{\boldsymbol{G}}{g} \right) \mathrm{d}V - \int_{\partial \mathcal{B}^h} \psi Q \, \mathrm{d}A \,, \tag{3.42}$$

where the Kirchhoff pore pressure variable θ now replaces the potential Π. The condition $H(\phi, \theta, \psi) = 0$ is the variational form of the volume constraint imposed by fluid flow.

4. Linearization

The objectives of this section are two-fold: (i) to provide a link between the linear and non-linear theories of consolidation, and (ii) to develop *exact* expressions for the first derivatives of the functions $G(\phi, \theta, \boldsymbol{\eta})$ and $H(\phi, \theta, \psi)$ for use in Newton and Newton-type iterations. More specifically, we want the linearization of the non-linear two-field linear momentum and mass conservation equations at some configuration ϕ^0 and pressure θ^0, which corresponds to some infinitesimal variations $\delta \boldsymbol{u}$ and $\delta\theta$. We develop this idea in the following sections using both the material and spatial descriptions.

4.1. *Preliminaries*

Some useful formulas are summarized below. The first of these formulas, the Piola transformation first introduced in Section 3.1 (see (3.6)), is of great importance and is used extensively throughout the remainder of this paper. We will define it formally in this section as follows

DEFINITION. Let y be a vector field in $R^{n_{sd}}$ and let the motion ϕ be regular in $\mathcal{B}$. Then, the Piola transform of y is

$$\boldsymbol{Y} = J \boldsymbol{F}^{-1} \cdot \boldsymbol{y} \,. \quad \square \tag{4.1}$$

The Piola identity is next given by the following theorem.

THEOREM. Let $\boldsymbol{Y}$ *be the Piola transform of* $\boldsymbol{y}$. *Then the following equation holds*

$$\mathrm{DIV}\, \boldsymbol{Y} = J \,\mathrm{div}\, \boldsymbol{y} \,. \tag{4.2}$$

The proof is given in [31]. This theorem may be extended to cases where $\boldsymbol{Y}$ and $\boldsymbol{y}$ are vectors derived from tensors of order greater than or equal to two by fixing all but one of the tensor's legs (for example, fixing one leg of the Cauchy stress tensor $\boldsymbol{\sigma}$ produces a vector of Cauchy stresses). □

We complete this section with the following propositions concerning the linearization of some basic terms.

PROPOSITION 4. Let $\delta\boldsymbol{u}$ be the variation of the displacement field; then the linearizations of $\boldsymbol{F}$ and $\boldsymbol{F}^{-1}$ at configuration ϕ^0 are given, respectively, by

$$\mathscr{L}\boldsymbol{F} = \boldsymbol{F}^0 + \mathrm{grad}\, \delta\boldsymbol{u} \cdot \boldsymbol{F}^0 \,; \tag{4.3a}$$

$$\mathscr{L}\boldsymbol{F}^{-1} = \boldsymbol{F}^{0^{-1}} - \boldsymbol{F}^{0^{-1}} \cdot \mathrm{grad}\, \delta\boldsymbol{u} \,. \tag{4.3b}$$

PROOF. We use the notion of directional derivative of a function to obtain the variation

$$\delta\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \boldsymbol{F}(\boldsymbol{\phi}^0 + \epsilon\, \delta\boldsymbol{u}) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \frac{\partial(\boldsymbol{\phi}^0 + \epsilon\, \delta\boldsymbol{u})}{\partial \boldsymbol{X}} = \frac{\partial(\delta\boldsymbol{u})}{\partial \boldsymbol{X}} = \mathrm{grad}\, \delta\boldsymbol{u} \cdot \boldsymbol{F}^0 \,,$$

where $\boldsymbol{\phi}^0$ are the spatial coordinates of the point whose motion is ϕ^0. The inverse relationship can be derived from the identity $\boldsymbol{F} \cdot \boldsymbol{F}^{-1} = \mathbf{1}$. Taking the variation using the chain rules gives

$$\delta(\boldsymbol{F} \cdot \boldsymbol{F}^{-1}) = \delta\boldsymbol{F} \cdot \boldsymbol{F}^{-1} + \boldsymbol{F} \cdot \delta\boldsymbol{F}^{-1} = \mathbf{0} \,,$$

which gives $\delta\boldsymbol{F}^{-1}$ from the previously determined $\delta\boldsymbol{F}$. □

PROPOSITION 5. The linearizations of the Jacobian and the rate of the Jacobian at configuration ϕ^0 are given, respectively, by

$$\mathscr{L}J = J^0 + J^0[\mathrm{div}(\delta\boldsymbol{u}) \,; \tag{4.4a}$$

$$\mathscr{L}\dot{J} = \dot{J}^0 + J^0\, \mathrm{div}(\delta\boldsymbol{v}) - \mathrm{grad}\, \boldsymbol{v}^0 : \mathrm{grad}^{\mathrm{t}}(\delta\boldsymbol{u}) + \mathrm{div}(\delta\boldsymbol{u})\, \mathrm{div}\, \boldsymbol{v}^0] \,, \tag{4.4b}$$

where $\delta\boldsymbol{v}$ is the variation of the velocity field $\boldsymbol{v}$.

PROOF. The first of (4.4) can be demonstrated from the identity

$$J(\boldsymbol{\phi}^0 + \epsilon\, \delta\boldsymbol{u}) = \det \boldsymbol{F}(\boldsymbol{\phi}^0 + \epsilon\, \delta\boldsymbol{u}) \,.$$

Hence, the variation of the Jacobian is

$$\delta J = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} J(\boldsymbol{\phi}^0 + \epsilon\, \delta\boldsymbol{u}) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \left[\frac{\partial(\boldsymbol{\phi}^0 + \epsilon\, \delta\boldsymbol{u})}{\partial \boldsymbol{X}}\right] : \mathrm{COF}(\boldsymbol{F}^0) = J^0\, \mathrm{div}(\delta\boldsymbol{u}) \,.$$

See Proposition 2 for a similar result.

For the variation of the Jacobian rate, we have, by the chain rule,

$$\delta\dot{J} = \delta(J^0\, \mathrm{div}\, \boldsymbol{v}^0) = J^0\, \delta(\mathrm{div}\, \boldsymbol{v}^0) + \delta J\, \mathrm{div}\, \boldsymbol{v}^0 \,.$$

Now, since

$$\begin{aligned}
\delta(\mathrm{div}\, \boldsymbol{v}^0) &= \delta(\mathrm{GRAD}\, \boldsymbol{v}^0 : \boldsymbol{F}^{0^{-\mathrm{t}}}) = \delta(\mathrm{GRAD}\, \boldsymbol{v}^0) : \boldsymbol{F}^{0^{-\mathrm{t}}} + \mathrm{GRAD}\, \boldsymbol{v}^0 : \delta\boldsymbol{F}^{-\mathrm{t}} \\
&= \mathrm{GRAD}\, \delta\boldsymbol{v} : \boldsymbol{F}^{0^{-\mathrm{t}}} - \mathrm{GRAD}\, \boldsymbol{v}^0 : (\boldsymbol{F}^{0^{-1}} \cdot \mathrm{grad}\, \delta\boldsymbol{u})^{\mathrm{t}} \\
&= \mathrm{div}(\delta\boldsymbol{v}) - \mathrm{grad}\, \boldsymbol{v}^0 : \mathrm{grad}^{\mathrm{t}}(\delta\boldsymbol{u}) \,,
\end{aligned}$$

we obtain the desired result. □

COROLLARY. The linearization of the reference saturated mass density $\rho_0 = J\rho$ at the configuration ϕ^0 is

$$\mathscr{L}\rho_0 = \rho_0^0 + \rho_w J^0 \operatorname{div}(\delta \boldsymbol{u}) . \tag{4.5}$$

PROOF. The proof follows from (2.5), (2.11) and (4.4a). Note that ρ_0 is *not* constant since, as pointed out previously, the total mass of the soil–water mixture in $\mathscr{B}$ is not necessarily conserved in $\phi_t(\mathscr{B})$. The variation of ρ_0 reflects the amount of fluid that enters into or escapes from the soil matrix due to the variation of the Jacobian. □

4.2. Linearization of the field equations

We now apply the results of the previous section to the equations of linear momentum and mass conservation. This will provide the link between the linear and non-linear theories of consolidation. As in the previous sections, we assume a two-field mixed formulation involving the finite deformation ϕ^0 and the Kirchhoff pore pressure θ^0. Then, we write the linearizations of the field equations consistent with the imposed infinitesimal variations $\delta\boldsymbol{u}$ and $\delta\theta$. In the following propositions, we assume a condition of dead loading, which implies that the gravity acceleration vectors $\boldsymbol{G}$ and $\boldsymbol{g}$ are unique functions of X (otherwise, an extra term must be added to the linearization to represent their variations).

PROPOSITION 6. Let $\boldsymbol{E} = \operatorname{DIV} \tilde{\boldsymbol{P}} + \rho_0 \boldsymbol{G}$ be the linear momentum equation. Then, for dead loading the linearization of $\boldsymbol{E}$ at (ϕ^0, θ^0) is

$$\begin{aligned}\mathscr{L}\boldsymbol{E} = \boldsymbol{E}^0 &+ \operatorname{DIV}(\boldsymbol{A}^0 : \operatorname{GRAD} \delta\boldsymbol{u}) + \operatorname{DIV}(\theta^0 \boldsymbol{F}^{0^{-t}} \cdot \operatorname{GRAD}^t \delta\boldsymbol{u} \cdot \boldsymbol{F}^{0^{-t}}) \\ &- \operatorname{DIV}(\delta\theta\, \boldsymbol{F}^{0^{-t}}) + \rho_w \operatorname{DIV}(J^0 \boldsymbol{F}^{0^{-1}} \cdot \delta\boldsymbol{u}) \boldsymbol{G}^0 ,\end{aligned} \tag{4.6}$$

where $\boldsymbol{E}^0 = \operatorname{DIV} \tilde{\boldsymbol{P}}^0 + \rho_0^0 \boldsymbol{G}^0$ and $\boldsymbol{A}^0 = \partial \boldsymbol{P} / \partial \boldsymbol{F}$ is the first tangential elasticity tensor for the solid matrix evaluated at the configuration ϕ^0.

PROOF. Rewrite $\boldsymbol{E}$ as

$$\boldsymbol{E} = \operatorname{DIV}(\boldsymbol{P} - \theta \boldsymbol{F}^{-t}) + J\rho \boldsymbol{G} .$$

Taking the variations yields

$$\delta\boldsymbol{E} = \operatorname{DIV}(\boldsymbol{A} : \delta\boldsymbol{F} - \theta\, \delta\boldsymbol{F}^{-t} - \delta\theta\, \boldsymbol{F}^{-t}) + \delta(J\rho)\boldsymbol{G} .$$

Substituting (4.3), (4.4) and (4.5) gives the desired result. □

REMARK 5. Eq. (4.6) is a linearization in the material description. The two-point tensor $\boldsymbol{A}$ has the structure $A_{iAjB} = \partial P_{iA} / \partial F_{jB}$, and may be replaced by the equivalent expression

$$\boldsymbol{A} = 2\boldsymbol{F} \cdot \boldsymbol{D} \cdot \boldsymbol{F}^t + \boldsymbol{S} \oplus \boldsymbol{1} , \quad \text{or} \quad A_{iAjB} = 2F_{iC} F_{jD} D_{CADB} + S_{AB} \delta_{ij} , \tag{4.7}$$

where $\boldsymbol{D} = \partial \boldsymbol{S} / \partial \boldsymbol{C}$ (i.e. $D_{CADB} = \partial S_{CA} / \partial C_{DB}$) is the second tangential elasticity tensor of order four. By the symmetry of the second Piola–Kirchhoff effective stress tensor $\boldsymbol{S}$ and of the right Cauchy–Green tensor $\boldsymbol{C}$, and by the axiom of material frame indifference, the tensor $\boldsymbol{D}$ possesses both the major and minor symmetries.

REMARK 6. Note that the variation of the Piola transform $\boldsymbol{U}$ is *not* equal to the Piola transform of the variation of $\boldsymbol{u}$, i.e. $\delta\boldsymbol{U} \equiv \delta(J\boldsymbol{F}^{-1} \cdot \delta\boldsymbol{u}) \neq J\boldsymbol{F}^{-1} \cdot \delta\boldsymbol{u}$. Hence, the argument of the DIV-operator on the last term of (4.6) may not be replaced by $\delta\boldsymbol{U}$.

The spatial counterpart of (4.6) may be derived directly from the Piola transformation. For example, let the spatial tangential elasticity tensors $\boldsymbol{a}$ and $\boldsymbol{d}$ be defined from the push-forwards on each large index of $\boldsymbol{A}$ and $\boldsymbol{D}$ as

$$a_{iajb} = F_{aA} F_{bB} A_{iA_jB} ; \tag{4.8a}$$

$$d_{ijkl} = 2F_{iA}F_{jB}F_{kC}F_{lD}D_{ABCD} \ . \tag{4.8b}$$

Then, the linearization of $\boldsymbol{E}$ in the spatial picture takes the form

$$\mathscr{L}\boldsymbol{E} = \boldsymbol{E}^0 + \mathrm{div}(\boldsymbol{a}^0 : \mathrm{grad}\,\delta\boldsymbol{u}) + \mathrm{div}(\theta^0\,\mathrm{grad}^{\mathrm{t}}\,\delta\boldsymbol{u}) - \mathrm{grad}(\delta\theta) + J^0\rho_{\mathrm{w}}\,\mathrm{div}(\delta\boldsymbol{u})\boldsymbol{g}^0 \ , \tag{4.9}$$

where $\boldsymbol{g}^0 \equiv \boldsymbol{G}^0$.

An equivalent form to (4.9), using the spatial tangential elasticity tensor $\boldsymbol{d}$, is

$$\mathscr{L}\boldsymbol{E} = \boldsymbol{E}^0 + \mathrm{div}[(\boldsymbol{d}^0 + \boldsymbol{\tau}^0 \oplus \boldsymbol{1}) : \mathrm{grad}\,\delta\boldsymbol{u}] + \mathrm{div}(\theta^0\,\mathrm{grad}^{\mathrm{t}}\,\delta\boldsymbol{u}) - \mathrm{grad}(\delta\theta) + J^0\rho_{\mathrm{w}}\,\mathrm{div}(\delta\boldsymbol{u})\boldsymbol{g}^0 \ , \tag{4.10}$$

where $(\boldsymbol{\tau}^0 \oplus \boldsymbol{1})_{ijkl} = \tau^0_{jl}\delta_{ik}$ represents the 'initial stress' contribution to the spatial stiffness. The equivalence of (4.6) and (4.9) may be established directly by noting that $\boldsymbol{A}^0 : \mathrm{GRAD}\,\delta\boldsymbol{u}$ is the Piola transform of $J^{0^{-1}}\boldsymbol{a}^0 : \mathrm{grad}\,\delta\boldsymbol{u}$, and so on; and from the fact that $\mathrm{grad}\,J = \boldsymbol{0}$ (see (2.38)).

PROPOSITION 7. Let $\boldsymbol{K} = \boldsymbol{F}^{-1} \cdot \boldsymbol{k} \cdot \boldsymbol{F}^{-\mathrm{t}}$ *be the pull-back permeability tensor, and let* $\boldsymbol{U}, \boldsymbol{V}$ *and* $\tilde{\boldsymbol{V}}$ *be the Piola transforms of* $\boldsymbol{u}, \boldsymbol{v}$ *and* $\tilde{\boldsymbol{v}}$*, respectively. Further, let* $M = \mathrm{DIV}\,\boldsymbol{V} + \mathrm{DIV}\,\tilde{\boldsymbol{V}}$ *be the volume conservation equation for a saturated soil–water mixture with incompressible solid grains and fluid. Then, the linearization of* M *at* (ϕ^0, θ^0) *is*

$$\mathscr{L}M = M^0 + \mathrm{DIV}\,\delta\boldsymbol{V} + \mathrm{DIV}\,\delta\tilde{\boldsymbol{V}} \ , \tag{4.11}$$

where

$$\delta\boldsymbol{V} = \mathrm{DIV}(J^0\boldsymbol{F}^{0^{-1}} \cdot \delta\boldsymbol{u})\boldsymbol{F}^{0^{-1}} \cdot \boldsymbol{v}^0 - J^0\boldsymbol{F}^{0^{-1}} \cdot \mathrm{GRAD}(\delta\boldsymbol{u}) \cdot \boldsymbol{F}^{0^{-1}} \cdot \boldsymbol{v}^0 + J^0\boldsymbol{F}^{0^{-1}} \cdot \delta\boldsymbol{v} \ ; \tag{4.12a}$$

$$\begin{aligned} -\delta\tilde{\boldsymbol{V}} = {} & \boldsymbol{K}^0 \cdot \left\{ \frac{\mathrm{GRAD}\,\delta\theta}{g\rho_{\mathrm{w}}} + [\mathrm{DIV}(J^0\boldsymbol{F}^{0^{-1}} \cdot \delta\boldsymbol{u})\boldsymbol{F}^{0^{\mathrm{t}}} + J^0\,\mathrm{GRAD}^{\mathrm{t}}(\delta\boldsymbol{u})] \cdot \frac{\boldsymbol{G}}{g} \right\} \\ & + \delta\boldsymbol{K} \cdot \left(\frac{\mathrm{GRAD}\,\theta^0}{g\rho_{\mathrm{w}}} + J^0\boldsymbol{F}^{0^{\mathrm{t}}} \cdot \frac{\boldsymbol{G}}{g} \right) ; \end{aligned} \tag{4.12b}$$

$$-\delta\boldsymbol{K} = 2\boldsymbol{F}^{0^{-1}} \cdot \mathrm{symm}(\mathrm{GRAD}\,\delta\boldsymbol{u} \cdot \boldsymbol{K}^0 \cdot \boldsymbol{F}^{0^{\mathrm{t}}}) \cdot \boldsymbol{F}^{0^{-\mathrm{t}}} \ . \tag{4.12c}$$

PROOF. The expression for M can be derived by multiplying (2.20) by J and using the Piola identity (4.2). Eq. (4.11) then follows from the variation of M. Note that $\delta(\mathrm{DIV}\,\boldsymbol{V}) = \mathrm{DIV}\,\delta\boldsymbol{V} \equiv \delta\dot{J}$ by (3.5) and (4.2). The constitutive equation in terms of the Piola transform $\tilde{\boldsymbol{V}}$ is

$$\tilde{\boldsymbol{V}} = J\boldsymbol{F}^{-1} \cdot \tilde{\boldsymbol{v}} = -\boldsymbol{K} \cdot \left(\frac{\mathrm{GRAD}\,\theta}{g\rho_{\mathrm{w}}} + J\boldsymbol{F}^{\mathrm{t}} \cdot \frac{\boldsymbol{G}}{g} \right) ,$$

where $\boldsymbol{K}$ is the tensor obtained from the pull-backs on each small index of the spatial permeability tensor $\boldsymbol{k}$. The variations $\delta\boldsymbol{V}, \delta\tilde{\boldsymbol{V}}$ and $\delta\boldsymbol{K}$ then follow from the chain rule. □

The linearization of M in the spatial picture may be written as follows

$$\mathscr{L}M = M^0 + \delta\dot{J} + \mathrm{div}[\delta(J\tilde{\boldsymbol{v}})] - \mathrm{grad}(J^0\tilde{\boldsymbol{v}}^0) : \mathrm{grad}^{\mathrm{t}}(\delta\boldsymbol{u}) \ , \tag{4.13}$$

where

$$\delta\dot{J} = J^0[\mathrm{div}(\delta\boldsymbol{v}) - \mathrm{grad}\,\boldsymbol{v}^0 : \mathrm{grad}^{\mathrm{t}}(\delta\boldsymbol{u}) + \mathrm{div}(\delta\boldsymbol{u})\,\mathrm{div}\,\boldsymbol{v}^0] \ ; \tag{4.14a}$$

$$J^0\tilde{\boldsymbol{v}}^0 = -\boldsymbol{k} \cdot \left(\frac{\mathrm{grad}\,\theta^0}{g\rho_{\mathrm{w}}} + J^0\frac{\boldsymbol{G}}{g} \right) ; \tag{4.14b}$$

$$\delta(J\tilde{\boldsymbol{v}}) = -\boldsymbol{k} \cdot \left[\frac{\mathrm{grad}(\delta\theta) - \mathrm{grad}\,\theta^0 \cdot \mathrm{grad}\,\delta\boldsymbol{u}}{g\rho_{\mathrm{w}}} + J^0\,\mathrm{div}(\delta\boldsymbol{u})\frac{\boldsymbol{G}}{g} \right] , \tag{4.14c}$$

REMARK 7. Both the material and spatial forms for $\mathscr{L}M$ contain the variation of the solid velocity vector $\boldsymbol{v} \equiv \dot{\boldsymbol{u}}$ due to the presence of the rate of the Jacobian, $\dot{J}$, which is mathematically awkward. This variation may be eliminated altogether by a semi-discretization of the volume conservation equation in time prior to linearization, via finite differencing, for example. This idea will be pursued in Section 4.3 within the context of the variational equation for the volume constraint.

The linearization of the equilibrium equation takes a specially simple form when applied to undrained loading. The following proposition summarizes the results when a constitutive law of the form (3.38) is substituted in lieu of the pore pressure variable θ:

PROPOSITION 8. Undrained Loading: Let $\boldsymbol{E} = \mathrm{DIV}\,\tilde{\boldsymbol{P}} + \rho_0 \boldsymbol{G}$ *be the linear momentum equation, where* $\tilde{\boldsymbol{P}} = \boldsymbol{P} - \theta \boldsymbol{F}^{-t}$ *and* $\theta = \theta_n - (\lambda_w/\varphi_0)\,\mathrm{tr}(\Delta\boldsymbol{\epsilon}^{e\,tr})$, *with* $\lambda_w \gg 0$. *Then, under a condition of dead loading, and in the limit as* $\lambda_w \to \infty$, *the linearization of* $\boldsymbol{E}$ *at* ϕ^0 *is*

$$\mathscr{L}\boldsymbol{E} = \boldsymbol{E}^0 + \mathrm{DIV}(\boldsymbol{A}^* : \mathrm{GRAD}\,\delta\boldsymbol{u})\,, \tag{4.15a}$$

where

$$\boldsymbol{A}^* = \boldsymbol{A}^0 + (\lambda_w/\varphi_0)\boldsymbol{F}^{0^{-t}} \otimes \boldsymbol{F}^{0^{-t}} + \theta^0 \boldsymbol{F}^{0^{-t}} \ominus \boldsymbol{F}^{0^{-1}}\,, \tag{4.15b}$$

with $(\boldsymbol{a} \otimes \boldsymbol{b})_{ijkl} = a_{ij} b_{kl}$ *and* $(\boldsymbol{a} \ominus \boldsymbol{b})_{ijkl} = a_{il} b_{jk}$ *for any tensors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *of order two.*

PROOF. We shall prove (4.15) using the spatial form for $\mathscr{L}\boldsymbol{E}$ given by (4.10). First, observe that

$$\mathrm{grad}^t\,\delta\boldsymbol{u} = \boldsymbol{1} \ominus \boldsymbol{1} : \mathrm{grad}\,\delta\boldsymbol{u}\,,$$

where $(\boldsymbol{1} \ominus \boldsymbol{1})_{ijkl} = \delta_{il}\,\delta_{jk}$. Next, note that the variation of θ is

$$\delta\theta = \delta\left(\theta_n - \frac{\lambda_w}{\varphi_0}\,\mathrm{tr}\,\Delta\boldsymbol{\epsilon}^{e\,tr}\right) = -\frac{\lambda_w}{\varphi_0}\,\mathrm{tr}\,\delta\boldsymbol{\epsilon}^{e\,tr} = -\frac{\lambda_w}{2\varphi_0}\boldsymbol{b}^{e\,tr-1} : \delta\boldsymbol{b}^{e\,tr}\,,$$

while the variation of $\boldsymbol{b}^{e\,tr}$ is

$$\delta\boldsymbol{b}^{e\,tr} = \delta(\boldsymbol{f} \cdot \boldsymbol{b}^e_n \cdot \boldsymbol{f}^t) = \delta\boldsymbol{f} \cdot \boldsymbol{b}^e_n \cdot \boldsymbol{f}^t + \boldsymbol{f} \cdot \boldsymbol{b}^e_n \cdot \delta\boldsymbol{f}^t = 2\,\mathrm{symm}(\mathrm{grad}\,\delta\boldsymbol{u} \cdot \boldsymbol{b}^{e\,tr})\,,$$

since $\delta\boldsymbol{f} = \partial(\delta\boldsymbol{u})/\partial\boldsymbol{x}_n = \mathrm{grad}\,\delta\boldsymbol{u} \cdot \boldsymbol{f}$ from the chain rule. Therefore

$$\delta\theta = -\frac{\lambda_w}{\varphi_0}\boldsymbol{b}^{e\,tr-1} : \mathrm{symm}(\mathrm{grad}\,\delta\boldsymbol{u} \cdot \boldsymbol{b}^{e\,tr}) = -\frac{\lambda_w}{\varphi_0}\,\mathrm{div}\,\delta\boldsymbol{u}\,,$$

and

$$\begin{aligned}-\mathrm{grad}(\delta\theta) &= \frac{\lambda_w}{\varphi_0}\,\mathrm{grad}(\mathrm{div}\,\delta\boldsymbol{u}) = \mathrm{div}\left[\frac{\lambda_w}{\varphi_0}\,\mathrm{div}(\delta\boldsymbol{u})\boldsymbol{1}\right] \\ &= \mathrm{div}\left[\frac{\lambda_w}{\varphi_0}\boldsymbol{1} \otimes \boldsymbol{1} : \mathrm{grad}\,\delta\boldsymbol{u}\right],\end{aligned}$$

where $(\boldsymbol{1} \otimes \boldsymbol{1})_{ijkl} = \delta_{ij}\,\delta_{kl}$ (contrast the orders of indices generated by the operators $\otimes$, $\ominus$ and $\oplus$). Hence, the linearization of $\boldsymbol{E}$ in the spatial picture takes the form

$$\mathscr{L}\boldsymbol{E} = \boldsymbol{E}^0 + \mathrm{div}[(\boldsymbol{d}^0_1 + \boldsymbol{d}^0_2) : \mathrm{grad}\,\delta\boldsymbol{u}]\,, \tag{4.16a}$$

where

$$\boldsymbol{d}^0_1 = \boldsymbol{d}^0 + \frac{\lambda_w}{\varphi_0}\boldsymbol{1} \otimes \boldsymbol{1} \quad \text{and} \quad \boldsymbol{d}^0_2 = \boldsymbol{\tau}^0 \oplus \boldsymbol{1} + \theta^0 \boldsymbol{1} \ominus \boldsymbol{1}\,. \tag{4.16b}$$

Eq. (4.15) then follows from pull-backs on the second and fourth legs of the spatial elasticity tensors $\boldsymbol{d}^0_1$ and $\boldsymbol{d}^0_2$. An alternative approach involves a direct linearization of $\tilde{\boldsymbol{P}}$, which yields $\boldsymbol{A}^* = \partial\tilde{\boldsymbol{P}}/\partial\boldsymbol{F}$ of the form identical to (4.15b). Note that as $\lambda_w \to \infty$, the incompressibility condition $\mathrm{div}\,\delta\boldsymbol{u} = 0$ is recovered from (4.16a). This condition also causes the variation of the reference mass density to drop from the linearized terms. □

A crucial step in the linearization of the linear momentum balance equation is the evaluation of the tangential elasticity tensor for the solid matrix. We have introduced four of them in this section: the tensors $\boldsymbol{A}, \boldsymbol{D}, \boldsymbol{a}$ and $\boldsymbol{d}$. Each of these tensors can be derived directly from the other. Let us focus our attention on the spatial tensor $\boldsymbol{d}$ and describe a procedure, based on the results presented in [24, 25], for evaluating it.

Let the second Piola–Kirchhoff effective stress tensor $\boldsymbol{S}$ be obtained from the pull backs of the Kirchhoff effective stress tensor $\boldsymbol{\tau}$ as defined by the constitutive equation (3.13)

$$\boldsymbol{S} = \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-t} = \sum_{A=1}^{3} \beta_A \boldsymbol{M}^{(A)}; \qquad \boldsymbol{M}^{(A)} = \boldsymbol{F}^{-1} \cdot \boldsymbol{m}^{(A)} \cdot \boldsymbol{F}^{-t}. \tag{4.17}$$

We recall that the β_A's are the principal Kirchhoff effective stresses and the $\boldsymbol{m}^{(A)}$'s are dyads formed by juxtaposing the principal directions of the elastic stretches, as given explicitly by $(3.12)_2$. Using the chain rule, we obtain the following expression for the tensor $\boldsymbol{D}$

$$\boldsymbol{D} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}} = \frac{1}{2} \sum_{A=1}^{3} \sum_{B=1}^{3} \frac{\partial \beta_A}{\partial \epsilon_B} \boldsymbol{M}^{(A)} \otimes \boldsymbol{M}^{(B)} + \sum_{A=1}^{3} \beta_A \frac{\partial \boldsymbol{M}^{(A)}}{\partial \boldsymbol{C}}, \tag{4.18}$$

where use is made of the identity $\partial \epsilon_B / \partial \boldsymbol{C} = \boldsymbol{M}^{(B)}/2$ (see [25]). A push-forward on all large indices of $\boldsymbol{D}$ then gives the following expression for the spatial tensor $\boldsymbol{d}$

$$\boldsymbol{d} = \sum_{A=1}^{3} \sum_{B=1}^{3} \frac{\partial \beta_A}{\partial \epsilon_B} \boldsymbol{m}^{(A)} \otimes \boldsymbol{m}^{(B)} + 2 \sum_{A=1}^{3} \beta_A \boldsymbol{d}^{(A)}, \tag{4.19}$$

where $\boldsymbol{d}^{(A)}$ is a completely defined fourth-order tensor with a form given in [24] for the general case of $\boldsymbol{b}^{\mathrm{e}}$ having *distinct* eigenvalues $(\lambda_1^{\mathrm{e}})^2$, $(\lambda_2^{\mathrm{e}})^2$ and $(\lambda_3^{\mathrm{e}})^2$.

REMARK 8. The tensors $\boldsymbol{D}$ and $\boldsymbol{d}$ given by (4.18) and (4.19), respectively, are algorithmic moduli tensors obtained from the linearization of the corresponding algorithmic stresses. The first component contains the partial derivative $\partial \beta_A / \partial \epsilon_B$, which can be obtained from a consistent linearization of the return mapping algorithm in the principal stress axes, and is thus a function of the specific plasticity model chosen to replicate the behavior of the soil matrix. The second term depends solely on geometric non-linearities and is valid for any form of the stored energy function. See [25] for specific details pertaining to the implementation of this algorithm.

4.3. Linearization of the variational equations

We can also apply the results of the previous section to the linear momentum and mass conservation equations in variational form. Because of the simplicity of the linearization in the spatial picture, as well as its amenability to finite element implementation, we will linearize the integral functions $G(\phi, \theta, \boldsymbol{\eta})$ and $H(\phi, \theta, \psi)$ using the spatial description for the integrands evaluated over the same fixed, undeformed domain $\mathscr{B}$. A summary of the results is given in the following propositions.

PROPOSITION 9. Let $G(\phi, \theta, \boldsymbol{\eta})$ be the linear momentum balance equation of the form given by the variational equation (3.4). Assuming a condition of dead loading, the linearization of G at (ϕ^0, θ^0) is

$$\begin{aligned} \mathscr{L}G = G^0 &+ \int_{\mathscr{B}} \operatorname{grad} \boldsymbol{\eta} : (\boldsymbol{d}^0 + \boldsymbol{\tau}^0 \oplus \boldsymbol{1}) : \operatorname{grad} \delta \boldsymbol{u} \, \mathrm{d}V \\ &- \int_{\mathscr{B}} (\delta\theta \operatorname{div} \boldsymbol{\eta} - \theta^0 \operatorname{grad}^{\mathrm{t}} \boldsymbol{\eta} : \operatorname{grad} \delta \boldsymbol{u}) \, \mathrm{d}V \\ &- \int_{\mathscr{B}} \rho_{\mathrm{w}} J^0 \operatorname{div}(\delta \boldsymbol{u}) \boldsymbol{\eta} \cdot \boldsymbol{g} \, \mathrm{d}V - \int_{\partial\mathscr{B}} \boldsymbol{\eta} \cdot \delta \boldsymbol{t} \, \mathrm{d}A, \end{aligned} \tag{4.20}$$

where $G^0 = G(\phi^0, \theta^0, \boldsymbol{\eta})$ and $\delta \boldsymbol{u}$, $\delta\theta$ and $\delta \boldsymbol{t}$ are the variations of the displacement vector, Kirchhoff pore water pressure, and traction vector, respectively.

PROOF. The variation

$$\delta \int_{\mathscr{B}} \operatorname{grad} \boldsymbol{\eta} : \boldsymbol{\tau} \, dV = \delta \int_{\mathscr{B}} \operatorname{GRAD} \boldsymbol{\eta} : \boldsymbol{P} \, dV = \delta \int_{\mathscr{B}} \operatorname{GRAD} \boldsymbol{\eta} : \boldsymbol{F} \cdot \boldsymbol{S} \, dV$$

$$= \int_{\mathscr{B}} \operatorname{GRAD} \boldsymbol{\eta} : (\boldsymbol{F} \cdot \delta \boldsymbol{S} + \delta \boldsymbol{F} \cdot \boldsymbol{S}) \, dV$$

produces the first integral term on the right-hand side of (4.20), upon substitution of the identities $\delta \boldsymbol{S} = \boldsymbol{D} : \delta \boldsymbol{C} = \boldsymbol{D} : \boldsymbol{F}^{t} \oplus \boldsymbol{F}^{t} : \operatorname{grad} \delta \boldsymbol{u}$ and $\delta \boldsymbol{F} = \operatorname{grad} \boldsymbol{u} \cdot \boldsymbol{F}$. The variation

$$\delta \int_{\mathscr{B}} \theta \operatorname{div} \boldsymbol{\eta} \, dV = \int_{\mathscr{B}} [\delta\theta \operatorname{div} \boldsymbol{\eta} + \theta \, \delta(\operatorname{div} \boldsymbol{\eta})] \, dV$$

produces the second integral term upon substitution of the identity $\delta(\operatorname{div} \boldsymbol{\eta}) = \operatorname{div} \delta\boldsymbol{\eta} - \operatorname{grad}^{t} \boldsymbol{\eta} : \operatorname{grad} \delta \boldsymbol{u}$ from Proof of Proposition 5 (note that $\delta \boldsymbol{\eta} \equiv \boldsymbol{0}$). The third integral term emanates from the linearization of ρ_0 based on (4.5), while the last integral term is produced from a straight-forward linearization of the traction vector $\boldsymbol{t}$. □

We next consider the linearization of $H(\phi, \theta, \psi)$. As pointed out in the previous section the presence of the velocity term $\boldsymbol{v}$ makes the linearization mathematically awkward, and so we will eliminate it at the outset through a semi-discretization of the variational equation in time.

Let us rewrite H in the following form

$$H(\phi, \theta, \psi) = \int_{\mathscr{B}} \psi \dot{J} \, dV - \int_{\mathscr{B}} \operatorname{grad} \psi \cdot J\tilde{\boldsymbol{v}} \, dV - \int_{\mathscr{B}} \psi Q \, dA \, , \tag{4.21a}$$

where

$$J\tilde{\boldsymbol{v}} = -\boldsymbol{k} \cdot \left(\frac{\operatorname{grad} \theta}{g \rho_{w}} + J \frac{\boldsymbol{G}}{g} \right) . \tag{4.21b}$$

Now, consider the following time-integrated variational equation [4, 6]

$$\begin{aligned} H_{\Delta t}(\phi, \theta, \psi) = & \int_{\mathscr{B}} \frac{\psi}{\Delta t} \left(J_{n+1} - \sum_{m=1}^{k} \alpha_m J_{n+1-m} \right) dV \\ & - \beta_0 \int_{\mathscr{B}} [\beta(\operatorname{grad} \psi \cdot J\tilde{\boldsymbol{v}})_{n+1} + (1-\beta)(\operatorname{grad} \psi \cdot J\tilde{\boldsymbol{v}})_n] \, dV \\ & - \beta_0 \int_{\partial \mathscr{B}} \psi [\beta Q_{n+1} + (1-\beta) Q_n] \, dA \, , \end{aligned} \tag{4.22}$$

where $\Delta t = t_{n+1} - t_n$ and β, β_0 and the α_m's are time-integration parameters. The well-known trapezoidal rule is recovered from (4.22) if $k = 1, \beta_0 = 1, \alpha_1 = 1$, and $\beta \in [0, 1]$. The stability and accuracy characteristics of this method are well-documented [4, 50] and are known to be functions solely of the trapezoidal integration parameter β. If $\beta = 1$ and $k \geqslant 1$, then we recover the family of unconditionally stable, k-step backward differentiation formula (BDF) methods. The accuracy of these methods depends on the order k, as well as on the values of β_0 and $\alpha_1, \alpha_2, \ldots, \alpha_k$, which in turn are functions of Δt [51, 52]. See [4] for further results on the performance of these families of time-stepping algorithms for consolidation problems.

Our objective is to linearize (4.22) for *fixed* Δt. This assumption is crucial since allowing Δt to vary will produce the convected terms that we want to avoid. The result of the linearization with fixed Δt is summarized in the following proposition.

PROPOSITION 10. Let $H_{\Delta t}$ be the time-integrated volume conservation equation of the form (4.22). For a fixed Δt the linearization of $H_{\Delta t}$ at the configuration (ϕ^0, θ^0) is

$$
\begin{aligned}
\mathscr{L}H_{\Delta t} = H^0_{\Delta t} &+ \int_{\mathscr{B}} \frac{\psi}{\Delta t} J^0 \operatorname{div} \delta \boldsymbol{u} \, \mathrm{d}V + \beta\beta_0 \int_{\mathscr{B}} \operatorname{grad} \psi \cdot \frac{\boldsymbol{k}}{g\rho_{\mathrm{w}}} \cdot \operatorname{grad} \delta\theta \, \mathrm{d}V \\
&- 2\beta\beta_0 \int_{\mathscr{B}} \operatorname{grad} \psi \cdot \operatorname{symm}\left(\frac{\boldsymbol{k}}{g\rho_{\mathrm{w}}} \cdot \operatorname{grad}^{\mathrm{t}} \delta \boldsymbol{u}\right) \cdot \operatorname{grad} \theta^0 \, \mathrm{d}V \\
&- \beta\beta_0 \int_{\mathscr{B}} \operatorname{grad} \psi \cdot [\operatorname{grad} \delta \boldsymbol{u} - (\operatorname{div} \delta \boldsymbol{u})\mathbf{1}] \cdot \boldsymbol{k} \cdot \frac{\boldsymbol{G}}{g} J^0 \, \mathrm{d}V \\
&- \beta\beta_0 \int_{\partial\mathscr{B}} \psi \, \delta Q \, \mathrm{d}A \, ,
\end{aligned}
\tag{4.23}
$$

where $H^0_{\Delta t} = H_{\Delta t}(\phi^0, \theta^0, \psi)$ and δQ is the variation of the fluid flux Q.

PROOF. The first integral term on the right-hand side of (4.23) results from the linearization of J. The second, third and fourth integral terms result from the linearization

$$
-\beta\beta_0 \delta \int_{\mathscr{B}} \operatorname{grad} \psi \cdot J\tilde{\boldsymbol{v}} \, \mathrm{d}V = -\beta\beta_0 \int_{\mathscr{B}} [\delta(\operatorname{grad} \psi) \cdot J\tilde{\boldsymbol{v}} + \operatorname{grad} \psi \cdot \delta(J\tilde{\boldsymbol{v}})] \, \mathrm{d}V \, ,
$$

in which the identity $\delta(\operatorname{grad} \psi) = -\operatorname{grad} \psi \cdot \operatorname{grad} \delta \boldsymbol{u}$ is used, and where $\delta(J\tilde{\boldsymbol{v}})$ is given by (4.14c). The last term results from the linearization of Q. Note that the undrained condition

$$
\delta H_{\Delta t} = \int_{\mathscr{B}} \psi J^0 \operatorname{div} \delta \boldsymbol{u} \, \mathrm{d}V \equiv \int_{\mathscr{B}} \psi \, \delta J \, \mathrm{d}V = 0
$$

is recovered from (4.23) in the limit as $\Delta t \to 0$ (cf. (3.37)). □

REMARK 9. It is sometimes assumed that the permeability of the soil skeleton varies with the soil's porosity φ, or, equivalently, with the Jacobian J, i.e. $\boldsymbol{k} = \boldsymbol{k}(J)$. If this constitutive theory exists, then $\boldsymbol{k}$ will cease to be constant, and an additional linearization term associated with the variation $\delta \boldsymbol{k} = (\partial \boldsymbol{k}/\partial J)\, \delta J \equiv (\partial \boldsymbol{k}/\partial J) J \operatorname{div} \delta \boldsymbol{u}$ must be added to the terms in (4.23).

We conclude this section with the following proposition for undrained loading when a constitutive law of the form (3.38) is substituted in lieu of the pore pressure variable θ into the linearized equation (4.20).

PROPOSITION 11. Undrained Loading: Let $G(\phi, \theta, \boldsymbol{\eta})$ be the linear momentum balance equation of the form given by the variational equation (3.4), with $\theta = \theta_n - (\lambda_{\mathrm{w}}/\varphi_0) \operatorname{tr}(\Delta \boldsymbol{\epsilon}^{\mathrm{e\,tr}})$ and $\lambda_{\mathrm{w}} \gg 0$. Then, under a condition of dead loading, and in the limit as $\lambda_{\mathrm{w}} \to \infty$, the linearization of G at ϕ^0 is

$$
\mathscr{L}G = G^0 + \int_{\mathscr{B}} \operatorname{grad} \boldsymbol{\eta} : (\boldsymbol{d}^0_1 + \boldsymbol{d}^0_2) : \operatorname{grad} \delta \boldsymbol{u} \, \mathrm{d}V - \int_{\partial\mathscr{B}} \boldsymbol{\eta} \cdot \delta \boldsymbol{t} \, \mathrm{d}A \, ,
\tag{4.24}
$$

where $G^0 = G(\phi^0, \theta(\phi^0), \boldsymbol{\eta}) \equiv G(\phi^0, \boldsymbol{\eta})$, $\boldsymbol{d}^0_1 = \boldsymbol{d}^0 + (\lambda_{\mathrm{w}}/\varphi_0)\mathbf{1} \otimes \mathbf{1}$, and $\boldsymbol{d}^0_2 = \boldsymbol{\tau}^0 \oplus \mathbf{1} + \theta^0 \mathbf{1} \ominus \mathbf{1}$.

PROOF. The proof follows from Proposition 8. □

4.4. Linear consolidation theory

If the linearization of the previous section is performed about a stress-free, undeformed state, and only the first-order displacement terms are considered, then the small-strain consolidation theory results. Furthermore, if the elasticity tensor is assumed to be a fixed function of $\boldsymbol{x}$ and does not depend on the imposed load, then we obtain the linear consolidation theory of Biot [1, 2]. Under this simplified setting, (4.10) becomes

$$
\operatorname{div}(\boldsymbol{c} : \operatorname{grad} \boldsymbol{u} - \theta_{\mathrm{e}} \mathbf{1}) = \mathbf{0} \, ,
\tag{4.25}
$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is a vector field of infinitesimal solid displacements of $\mathscr{B}$ reckoned from a self-equilibrating condition of geostatic stresses [29], θ_e is the *excess* pore water pressure [4], and $\boldsymbol{c} = \boldsymbol{c}(\boldsymbol{x})$ is a given fourth-order, time-independent tensor field of elasticities in $\mathscr{B}$, which possesses both the major and minor symmetries. The term 'grad $\boldsymbol{u}$' may be replaced by the infinitesimal strain tensor '$\boldsymbol{\epsilon}$', where $\boldsymbol{\epsilon} = \text{symm}(\text{grad}\, \boldsymbol{u})$, due to the minor symmetry of $\boldsymbol{c}$ with respect to its third and fourth indices.

The volume conservation equation simplifies in a similar fashion. Here, (4.13) becomes

$$\text{div}\, \dot{\boldsymbol{u}} - \text{div}\left(\frac{\boldsymbol{k}}{g\rho_w} \cdot \text{grad}\, \theta_e\right) = 0\,, \tag{4.26}$$

where $\dot{\boldsymbol{u}}$ is the solid velocity field. Eqs. (4.25) and (4.26) are equivalent to those developed in [1, 2] and used in [3–13, 53, 54].

The functions $G(\phi, \theta_e, \boldsymbol{\eta})$ and $H(\phi, \theta_e, \psi)$ simplify accordingly. The function G takes the bilinear form

$$G(\phi, \theta_e, \boldsymbol{\eta}) = \int_{\mathscr{B}} (\text{grad}\, \boldsymbol{\eta} : \boldsymbol{c} : \text{grad}\, \boldsymbol{u} - \theta_e\, \text{div}\, \boldsymbol{\eta})\, dV - \int_{\partial\mathscr{B}^t} \boldsymbol{\eta} \cdot \boldsymbol{t}\, dA\,, \tag{4.27}$$

while the function H takes the form

$$H(\phi, \theta_e, \psi) = \int_{\mathscr{B}} \left(\psi\, \text{div}\, \dot{\boldsymbol{u}} + \text{grad}\, \psi \cdot \frac{\boldsymbol{k}}{g\rho_w} \cdot \text{grad}\, \theta_e\right) dV - \int_{\partial\mathscr{B}^h} \psi q\, dA\,. \tag{4.28}$$

Finally, a further simplification can be obtained from the assumption of undrained deformation. Assuming a constitutive equation for the excess pore pressure θ_e of the form (cf. (3.38))

$$\theta_e = -\frac{\lambda_w}{\varphi}\, \text{div}\, \boldsymbol{u}\,, \tag{4.29}$$

then (4.25) becomes

$$\text{div}(\tilde{\boldsymbol{c}} : \text{grad}\, \boldsymbol{u}) = \boldsymbol{0}\,, \tag{4.30}$$

where $\tilde{\boldsymbol{c}} = \boldsymbol{c} + (\lambda_w/\varphi)\mathbf{1} \otimes \mathbf{1}$ is the so-called *total* elasticity tensor for the soil–water mixture. The corresponding variational equation to (4.30) is

$$G(\phi, \theta_e, \boldsymbol{\eta}) = \int_{\mathscr{B}} \text{grad}\, \boldsymbol{\eta} : \tilde{\boldsymbol{c}} : \text{grad}\, \boldsymbol{u}\, dV - \int_{\mathscr{B}^t} \boldsymbol{\eta} \cdot \boldsymbol{t}\, dA\,, \tag{4.31}$$

in which $G = 0$ represents the linear momentum balance law.

5. Summary and conclusions

A mathematical model for finite strain elastoplastic consolidation of fully saturated soil media has been presented. The algorithmic treatment of finite strain elastoplasticity for the solid phase is based on multiplicative decomposition and is coupled with the algorithm for fluid flow via the Kirchhoff pore water pressure. To the authors' knowledge, the mathematical treatment of the formulation is unique and has not yet been explored, let alone reported in the literature, within the context of coupled solid displacement and diffusion problems. The mathematical formulation is valid for any plasticity model describing the constitutive behavior of the soil skeleton.

An interesting by-product of the analysis is the conclusion that the effective stress concept of Terzaghi is mathematically consistent even at finite strain, and that if there is a physically meaningful stress measure that must be used for finite strain formulation as well as for constitutive modeling, it is the effective stress. This conclusion will have its impact in the field of geotechnical engineering where the effective stress concept is the universal assumption used in almost all analyses. A further implication of this conclusion is that the infinitesimal plasticity models developed in soil mechanics may be used just as well for finite deformation consolidation analyses provided they have been developed based on effective stresses.

The incremental form and the corresponding linearization of the linear momentum and mass conservation equations are by no means trivial. Higher-order terms which could be crucial for optimal convergence of the iteration are present both in the linearization of the momentum balance equation (e.g. the $\boldsymbol{d}^A$-term in (4.19)) as well as that of the volume conservation equation (e.g. the first, third, and fourth integral terms in (4.23)). Exclusion of these terms in numerical integration algorithms based on direct incrementation will have a further consequence of loss of accuracy of the solution. Results of numerical simulations employing the proposed mathematical model will be reported in a future publication.

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