Boundary element formulation for elastoplastic analysis of axisymmetric bodies

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The complete formulation of B.E.M. applied to the analysis of axisymmetric bodies acting in the plastic range is presented in this paper. The concept of derivative of a singular integral given by Mikhlin has been used in order to calculate the stresses in internal points.

Also a semianalytical approach is proposed to compute the matrix coefficients, presenting the way in which it can be done and the results obtained.

Key words: mathematical model, boundary element formulation

Introduction

The analysis of bodies of revolution has many practical applications in engineering, such as pressure vessels, certain pipes, rotating disks, and many different types of containers, including nuclear pressure vessels. Their importance justifies applying more efficient and accurate methods of analysis, such as boundary elements to them. This paper deals with the elastoplastic analysis of bodies of revolution subject to axisymmetric loads.

Several authors have discussed the application of elastoplasticity using boundary elements. The more complete formulations for three and two dimensional plane stress-plane strain are found elsewhere,[1-4] and the first application was due to Ricardella.[5] Cruse was the first to present a formulation for axisymmetric elasticity[6] based on the fundamental solution as presented by Massonet.[7] Other authors[8] have solved the axisymmetric potential problem using boundary elements. In a recent paper[9] the axisymmetric elastoplastic case was analysed but the complete formulation was not presented.

In addition, internal stresses were calculated using an approximate formulation based on a simple finite difference procedure. By contrast Telles and Brebbia[3] have calculated the stresses at internal points using the proper integral equation. This correct formulation is extended in the present paper to deal with axisymmetric bodies.

The paper starts by formulating the integral equations of elastoplasticity and the corresponding stress tensor components for axisymmetric cases in the way previously presented for three and two dimensions (plane stress-plane strain) in reference 1. This paper also applies the technique developed by Wrobel and Brebbia[8] to integrate the Legendre function around the singularity for potential axisymmetric problems.

The analytical expressions required to complete the formulation are presented in the Appendix.

General formulation

The Navier equation for the elastoplastic case can be written as:

$$\dot{u}_{j,ll} + \frac{1}{1-2\nu}\dot{u}_{l,lj} = 2\left(\dot{\epsilon}^p_{ij,i} + \frac{\nu}{1-2\nu}\dot{\epsilon}^p_{kk,j}\right) - \frac{\dot{b}_j}{G} \quad \text{in } \Omega \tag{1}$$

where $\dot{u}_j$ are the components of the displacement rates, G and ν are the shear modulus and Poisson's ratio of the material $\dot{b}_j$ are the body forces rate components and $\dot{\epsilon}^p$ indicates the plastic strain rates. The comma indicates derivatives.

Using the weighted residual method, the theory of distribution or similar techniques one can obtain a Somigliana type identity from equation (1), i.e.:

$$c_{ik}\dot{u}_i(x) = \int_\Omega u_{ik}^* \dot{b}_i(y)\,d\Omega - \int_\Gamma p_{ik}^* u_i(y)\,d\Gamma + \int_\Gamma u_{ik}^* \dot{p}_i\,d\Gamma + \int_\Omega \Sigma_{ijk}\dot{\epsilon}_{ij}^p\,d\Omega \quad (2)$$

where u^* and p^* indicate the fundamental solution for displacements and surface tractions. The c_{ik} components given by the Kronecker delta δ_{ij} for internal points are all zero for external points and they have different values for points on the surface. For smooth boundaries for instance they are equal to the Kronecker delta divided by 2, i.e. $\frac{1}{2}\delta_{ij}$.

The stresses are given by:

$$\dot{\sigma}_{ij} = \frac{2G\nu}{1-2\nu}\dot{u}_{l,l}\delta_{ij} + G(\dot{u}_{i,j} + \dot{u}_{j,i}) - 2G\dot{\epsilon}_{ij}^p - \frac{2G\nu}{1-2\nu}\dot{\epsilon}_{kk}^p\delta_{ij} \quad (3)$$

The tensor components Σ_{ijk} are as follows:

$$\begin{aligned}
\Sigma_{rrr} &= 2G\left[\frac{1-\nu}{1-2\nu}\frac{\partial U_{rr}}{\partial r} + \frac{\nu}{1-2\nu}\left(\frac{1}{r}U_{rr} + \frac{\partial U_{zr}}{\partial z}\right)\right] \\
\Sigma_{zrr} &= \Sigma_{rzr} = G\left(\frac{\partial U_{rr}}{\partial z} + \frac{\partial U_{zr}}{\partial r}\right) \\
\Sigma_{zzr} &= 2G\left[\frac{1-\nu}{1-2\nu}\frac{\partial U_{zr}}{\partial z} + \frac{\nu}{1-2\nu}\left(\frac{1}{r}U_{rr} + \frac{\partial U_{rr}}{\partial r}\right)\right] \\
\Sigma_{\theta\theta r} &= 2G\left[\frac{1-\nu}{1-2\nu}\frac{U_{rr}}{r} + \frac{\nu}{1-2\nu}\left(\frac{\partial U_{rr}}{\partial r} + \frac{\partial U_{zr}}{\partial z}\right)\right] \\
\Sigma_{\theta rr} &= \Sigma_{r\theta r} = \Sigma_{\theta zr} = \Sigma_{z\theta r} = 0 \qquad (4) \\
\Sigma_{rrz} &= 2G\left[\frac{1-\nu}{1-2\nu}\frac{\partial U_{rz}}{\partial r} + \frac{\nu}{1-2\nu}\left(\frac{1}{r}U_{rz} + \frac{\partial U_{zz}}{\partial z}\right)\right] \\
\Sigma_{rzz} &= \Sigma_{zrz} = G\left(\frac{\partial U_{rz}}{\partial z} + \frac{\partial U_{zz}}{\partial r}\right) \\
\Sigma_{zzz} &= 2G\left[\frac{1-\nu}{1-2\nu}\frac{\partial U_{zz}}{\partial z} + \frac{\nu}{1-2\nu}\left(\frac{1}{r}U_{rz} + \frac{\partial U_{rz}}{\partial r}\right)\right] \\
\Sigma_{\theta\theta z} &= 2G\left[\frac{1-\nu}{1-2\nu}\frac{U_{rz}}{r} + \frac{\nu}{1-2\nu}\left(\frac{\partial U_{rz}}{\partial r} + \frac{\partial U_{zz}}{\partial z}\right)\right] \\
\Sigma_{\theta rz} &= \Sigma_{r\theta z} = \Sigma_{\theta zz} = \Sigma_{z\theta z} = 0
\end{aligned}$$

Finding the derivatives of Somigliana's equation (2) for internal points we have:

$$\delta_{ki}\dot{u}_{i,m} = \int u_{ik,m}^* \dot{p}_i\,d\Gamma - \int p_{ik,m}^* \dot{u}_i\,d\Gamma + \frac{\partial}{\partial x_m}\int_\Omega u_{ik}^* \dot{b}_i\,d\Omega + \frac{\partial}{\partial x_m}\int_\Omega \Sigma_{ijk}\dot{\epsilon}_{ij}^p\,d\Omega \quad (5)$$

where the derivatives due to the singularity have to be taken into consideration in the way proposed by Mikhlin.[10]

As indicated by Telles and Brebbia[1] the above integrals can be taken to the limit which for axisymmetric cases will give:

$$\delta_{ki}\dot{u}_{i,m} = \int_\Gamma u_{ik,m}^* \dot{p}_i\,d\Gamma - \int_\Gamma p_{ik,m}^* \dot{u}_i\,d\Gamma + \int_\Omega u_{ik,m}^* \dot{b}_i\,d\Omega + \int_\Omega \Sigma_{ijk,m}\dot{\epsilon}_{ij}^p\,d\Omega - \dot{\epsilon}_{ij}^p \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} \Sigma_{ijk}\hat{r}_{,m}\,d\Gamma$$

where $\hat{r}$ represents the distance between the field and singular points.

Taking this into consideration expression (3) for stresses now becomes:

$$\begin{aligned}
\dot{\sigma}_{km} = {} & \int_\Gamma D_{kmi}\dot{p}_i\,d\Gamma - \int S_{kmi}\dot{u}_i\,d\Gamma + \int_\Omega D_{kmi}\dot{b}_i\,d\Omega + \int_\Omega \Sigma_{kmij}\dot{\epsilon}_{ij}^p\,d\Omega - \dot{\epsilon}_{ij}^p \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} \\
& \times\left[\frac{2G\nu}{1-2\nu}\Sigma_{ijl}r_{,l}\delta_{km} + G(\Sigma_{ijk}r_{,m} + \Sigma_{ijm}r_{,k})\right]d\Gamma + 2G\dot{\epsilon}_{km}^p + \frac{2G\nu}{1-2\nu}\dot{\epsilon}_{ll}^p\delta_{km} \qquad (6)
\end{aligned}$$

In the limit:

$$\begin{aligned}
\dot{\sigma}_{km} = {} & \int_\Gamma D_{kmi}\dot{p}_i\,d\Gamma - \int_\Gamma S_{kmi}\dot{u}_i\,d\Gamma + \int_\Omega D_{kmi}\dot{b}_i\,d\Omega + \int_\Omega \Sigma_{kmij}\dot{\epsilon}_{ij}^p\,d\Omega - \frac{G}{4(1-\nu)} \\
& \times [2\dot{\epsilon}_{km}^p + \delta_{km}(\dot{\epsilon}_{ll}^p - \dot{\epsilon}_\theta^p)] \qquad (7)
\end{aligned}$$

Tensors components such as D_{kmi} and S_{kmi} have the same expressions as for elasticity.[6] The Σ_{kmij} components are given by:

$$\begin{aligned}
\Sigma_{zzrr} = {} & \frac{4G^2}{(1-2\nu)^2}\left\{(1-\nu)^2\frac{\partial^2 U_{rz}}{\partial r\,\partial Z} + \nu(1-\nu)\right. \\
& \times\left[\frac{1}{r}\frac{\partial U_{rz}}{\partial Z} + \frac{\partial^2 U_{zz}}{\partial z\,\partial Z} + \frac{\partial^2 U_{rr}}{\partial r\,\partial R} + \frac{1}{R}\frac{\partial U_{rr}}{\partial r}\right] \\
& \left. + \nu^2\left[\frac{1}{Rr}U_{rr} + \frac{1}{R}\frac{\partial U_{zr}}{\partial z} + \frac{1}{r}\frac{\partial U_{rr}}{\partial R} + \frac{\partial^2 U_{zr}}{\partial z\,\partial R}\right]\right\} \qquad (8)
\end{aligned}$$

$$\begin{aligned}
\Sigma_{\theta\theta rz} = \Sigma_{\theta\theta zr} = {} & \frac{2G^2}{1-2\nu}\left\{(1-\nu)\left[\frac{\partial U_{rr}}{\partial z} + \frac{\partial U_{zr}}{\partial r}\right]\frac{1}{R}\right. \\
& \left. + \nu\left[\frac{\partial}{\partial R}\left(\frac{\partial U_{rr}}{\partial z} + \frac{\partial U_{zr}}{\partial r}\right) + \frac{\partial}{\partial Z}\left(\frac{\partial U_{zr}}{\partial z} + \frac{\partial U_{zz}}{\partial r}\right)\right]\right\} \\
\Sigma_{rrzz} = {} & \frac{4G^2}{(1-2\nu)^2}\left\{(1-\nu)^2\frac{\partial^2 U_{zr}}{\partial z\,\partial R} + \nu(1-\nu)\right. \\
& \times\left[\frac{1}{r}\frac{\partial U_{rr}}{\partial R} + \frac{1}{R}\frac{\partial U_{zr}}{\partial z} + \frac{\partial^2 U_{rr}}{\partial r\,\partial R} + \frac{\partial^2 U_{zz}}{\partial z\,\partial Z}\right] \\
& \left. + \nu^2\left[\frac{1}{Rr}U_{rr} + \frac{1}{R}\frac{\partial U_{rr}}{\partial r} + \frac{1}{r}\frac{\partial U_{rz}}{\partial Z} + \frac{\partial^2 U_{rz}}{\partial r\,\partial Z}\right]\right\}
\end{aligned}$$

$$\Sigma_{rzzz} = \Sigma_{zrzz} = \frac{2G^2}{1-2\nu}\left\{(1-\nu)\left[\frac{\partial^2 U_{zr}}{\partial z\,\partial Z} + \frac{\partial^2 U_{zz}}{\partial z\,\partial R}\right] + \nu\left[\frac{1}{r}\left(\frac{\partial U_{rr}}{\partial Z} + \frac{\partial U_{rz}}{\partial R}\right) + \frac{\partial^2 U_{rr}}{\partial r\,\partial Z} + \frac{\partial^2 U_{rz}}{\partial r\,\partial R}\right]\right\}$$

$$\Sigma_{zzzz} = \frac{4G^2}{(1-2\nu)^2}\left\{(1-\nu)^2\frac{\partial^2 U_{zz}}{\partial z\,\partial Z} + \nu(1-\nu)\times\left[\left(\frac{1}{r}\frac{\partial U_{rz}}{\partial Z} + \frac{1}{R}\frac{\partial U_{zr}}{\partial z}\right) + \frac{\partial^2 U_{rz}}{\partial r\,\partial Z} + \frac{\partial^2 U_{zr}}{\partial z\,\partial R}\right] + \nu^2\left[\frac{1}{Rr}U_{rr} + \frac{1}{R}\frac{\partial U_{rr}}{\partial r} + \frac{1}{r}\frac{\partial U_{rr}}{\partial R} + \frac{\partial^2 U_{rr}}{\partial r\,\partial R}\right]\right\}$$

$$\Sigma_{\theta\theta rr} = \frac{4G^2}{(1-2\nu)^2}\left\{(1-\nu)^2\frac{1}{R}\frac{\partial U_{rr}}{\partial r} + \nu(1-\nu)\times\left[\frac{1}{Rr}U_{rr} + \frac{1}{R}\frac{\partial U_{zr}}{\partial z} + \frac{\partial^2 U_{rr}}{\partial r\,\partial R} + \frac{\partial^2 U_{rz}}{\partial r\,\partial Z}\right] + \nu^2\left[\frac{1}{r}\frac{\partial U_{rr}}{\partial R} + \frac{\partial^2 U_{zr}}{\partial z\,\partial R} + \frac{1}{r}\frac{\partial U_{rz}}{\partial Z} + \frac{\partial U_{zz}}{\partial z\,\partial Z}\right]\right\}$$

$$\Sigma_{rrrz} = \Sigma_{rrzr} = \frac{2G^2}{1-2\nu}\left\{(1-\nu)\frac{\partial}{\partial R}\left(\frac{\partial U_{rr}}{\partial z} + \frac{\partial U_{zr}}{\partial r}\right) + \nu\left[\frac{1}{R}\left(\frac{\partial U_{rr}}{\partial z} + \frac{\partial U_{zr}}{\partial r}\right) + \frac{\partial}{\partial Z}\left(\frac{\partial U_{rz}}{\partial z} + \frac{\partial U_{zz}}{\partial r}\right)\right]\right\}$$

$$\Sigma_{rzrz} = \Sigma_{zrrz} = \Sigma_{rzzr} = \Sigma_{zrzr} = G^2\left[\frac{\partial}{\partial Z}\left(\frac{\partial U_{rr}}{\partial z}\frac{\partial U_{zr}}{\partial r}\right) + \frac{\partial}{\partial R}\left(\frac{\partial U_{rz}}{\partial z} + \frac{\partial U_{zz}}{\partial r}\right)\right]$$

$$\Sigma_{zzrz} = \Sigma_{zzzr} = \frac{2G^2}{1-2\nu}\left\{(1-\nu)\frac{\partial}{\partial Z}\left(\frac{\partial U_{rz}}{\partial z} + \frac{\partial U_{zz}}{\partial r}\right) + \nu\left[\frac{1}{R}\left(\frac{\partial U_{rr}}{\partial z} + \frac{\partial U_{zr}}{\partial r}\right) + \frac{\partial}{\partial R}\left(\frac{\partial U_{rr}}{\partial z} + \frac{\partial U_{zr}}{\partial r}\right)\right]\right.$$

$$\Sigma_{rrrr} = \frac{4G^2}{(1-2\nu)^2}\left\{(1-\nu)^2\frac{\partial^2 U_{rr}}{\partial r\,\partial R} + \nu(1-\nu)\times\left[\frac{1}{r}\frac{\partial U_{rr}}{\partial R} + \frac{\partial^2 U_{zr}}{\partial z\,\partial R} + \frac{1}{R}\frac{\partial U_{rr}}{\partial r} + \frac{\partial^2 U_{rz}}{\partial r\,\partial Z}\right] + \nu^2\left[\frac{U_{rr}}{rR} + \frac{1}{R}\frac{\partial U_{zr}}{\partial z} + \frac{1}{r}\frac{\partial U_{rz}}{\partial Z} + \frac{\partial^2 U_{zz}}{\partial z\,\partial Z}\right]\right\}$$

$$\Sigma_{rzrr} = \Sigma_{zrrr} = \frac{2G^2}{1-2\nu}\left\{(1-\nu)\left[\frac{\partial^2 U_{rr}}{\partial r\,\partial Z} + \frac{\partial^2 U_{rz}}{\partial r\,\partial R}\right] + \nu\left[\frac{1}{r}\left(\frac{\partial U_{rz}}{\partial R} + \frac{\partial U_{rr}}{\partial Z}\right) + \frac{\partial^2 U_{zr}}{\partial z\,\partial Z} + \frac{\partial^2 U_{zz}}{\partial z\,\partial R}\right]\right\}$$

$$\Sigma_{\theta\theta zz} = \frac{4G^2}{(1-2\nu)^2}\left\{(1-\nu)^2\frac{1}{R}\frac{\partial U_{zr}}{\partial z} + \nu(1-\nu)\times\left[\frac{1}{rR}U_{rr} + \frac{1}{R}\frac{\partial U_{rr}}{\partial r} + \frac{\partial^2 U_{zr}}{\partial z\,\partial R} + \frac{\partial^2 U_{zz}}{\partial z\,\partial Z}\right] + \nu^2\left[\frac{1}{r}\frac{\partial U_{rr}}{\partial R} + \frac{\partial^2 U_{rr}}{\partial r\,\partial R} + \frac{1}{r}\frac{\partial U_{rz}}{\partial Z} - \frac{\partial^2 U_{rz}}{\partial r\,\partial Z}\right]\right\}$$

$$\Sigma_{rr\theta\theta} = \frac{4G^2}{(1-2\nu)^2}\left\{(1-\nu)^2\frac{1}{r}\frac{\partial U_{rr}}{\partial R} + \nu(1-\nu)\times\left[\frac{\partial^2 U_{rr}}{\partial r\,\partial R} + \frac{\partial^2 U_{zr}}{\partial z\,\partial R} + \frac{U_{rr}}{Rr} + \frac{\partial^2 U_{zz}}{\partial z\,\partial Z}\right] + \nu^2\left[\frac{1}{R}\frac{\partial U_{rr}}{\partial r} + \frac{1}{R}\frac{\partial U_{zr}}{\partial z} + \frac{1}{r}\frac{\partial U_{rz}}{\partial Z} + \frac{\partial U_{rz}}{\partial r\,\partial Z}\right]\right.$$

$$\Sigma_{rz\theta\theta} = \Sigma_{zr\theta\theta} = \frac{2G^2}{1-2\nu}\left\{(1-\nu)\left[\frac{1}{r}\frac{\partial U_{rr}}{\partial Z} + \frac{1}{r}\frac{\partial U_{rz}}{\partial R}\right] + \nu\left[\frac{\partial^2 U_{rr}}{\partial r\,\partial Z} + \frac{\partial^2 U_{zr}}{\partial z\,\partial Z} + \frac{\partial^2 U_{rz}}{\partial r\,\partial R}\frac{\partial^2 U_{zz}}{\partial z\,\partial R}\right]\right\}$$

$$\Sigma_{zz\theta\theta} = \frac{4G^2}{(1-2\nu)^2}\left\{(1-\nu)^2\frac{1}{r}\frac{\partial U_{rz}}{\partial Z} + \nu(1-\nu)\times\left[\frac{\partial^2 U_{rz}}{\partial r\,\partial Z} + \frac{\partial^2 U_{zz}}{\partial z\,\partial Z} + \frac{U_{rr}}{Rr} + \frac{1}{r}\frac{\partial U_{rr}}{\partial R}\right] + \nu^2\left[\frac{1}{R}\frac{\partial U_{rr}}{\partial r} + \frac{1}{R}\frac{\partial U_{zr}}{\partial z} + \frac{\partial^2 U_{rr}}{\partial r\,\partial R} + \frac{\partial^2 U_{zr}}{\partial z\,\partial R}\right]\right\}$$

$$\Sigma_{\theta\theta\theta\theta} = \frac{4G^2}{(1-2\nu)^2}\left\{(1-\nu)^2\frac{U_{rr}}{Rr} + \nu(1-\nu)\times\left[\frac{1}{R}\frac{\partial U_{rr}}{\partial r} + \frac{1}{R}\frac{\partial U_{zr}}{\partial z} + \frac{1}{r}\frac{\partial U_{rr}}{\partial R} + \frac{1}{r}\frac{\partial U_{rz}}{\partial Z}\right] + \nu^2\left[\frac{\partial^2 U_{rr}}{\partial r\,\partial R} + \frac{\partial^2 U_{zr}}{\partial z\,\partial R} + \frac{1}{r}\frac{\partial U_{rz}}{\partial z} + \frac{\partial^2 U_{rz}}{\partial r\,\partial Z}\right]\right\}$$

The axisymmetric elastoplastic boundary problem does not imply any additional compilation over the elastic case and in both cases the integrals are treated as shown in Cruse *et al.*[6]

Discretization

Let us assume that the boundary is discretized into N constant elements and the domain into M internal cells. This produces, as shown in reference 11, the following system of equations:

$$\begin{aligned} \dot{\sigma} &= G'\dot{P} - H'\dot{U} + \dot{B}' + Q'\dot{\epsilon}^P \\ H\dot{U} &= G\dot{P} + \dot{B} + Q\dot{\epsilon}^P \end{aligned} \tag{9}$$

where G' and H' are matrices of dimensions $4M \times 2N$, Q' is $4M \times 4M$, H and G are $2N \times 2N$, Q is $2N \times 4M$ and the others are vectors of dimension $2N$, except for the $\dot{\sigma}^P$ vector which is $4M$.

For elements without the singularity one can use the standard Gaussian quadrature formula.[12] When the element to be integrated presents a singularity these formulae are not efficient and a large number of integration points are needed to obtain an accurate result. For this case the semianalytical integration can be used.[8] Another possibility is to develop more accurate integration formulae for the axisymmetric case, such as the logarithmic presented in reference 12 or the special ones of reference 13.

In the present paper the integrals are calculated analytically near the singularity where the singular functions are of Legendre's type and have a simple asymptotic expansion over the rest of the element.

Consider a boundary element as shown in *Figure 1* and let us assume that the integration will be carried out over a length as indicated in the figure.

Similarly, for domain integrations the two situations indicated in *Figures 2* and *3* can be present. *Figure 2* represents the integration of terms in the matrix Q (i.e. from a point on the boundary) and *Figure 3* the integration of terms in Q' (i.e. for a singularity in the domain). The dark area in the figures corresponds to the area where analytical integration is used. The analytical integration refers only to the variable S (see *Figure 4*) as the integration with respect to θ can be accurately done numerically. Hence for *Figure 4* we have:

$$\int_V f\,\mathrm{d}v = 2\pi \int_A rsf\,\mathrm{d}A = 2\pi \int_{\theta_1}^{\theta_2} \mathrm{d}\theta \lim_{\epsilon \to 0} \int_{\epsilon}^{S(\theta)} rsf\,\mathrm{d}s$$

$$\times \int_V f\,\mathrm{d}v = 2\pi \int_{\theta_1}^{\theta_2} \mathrm{d}\theta \cdot I \tag{10}$$

$$I = \lim_{\epsilon \to 0} \int_{\epsilon}^{S(\theta)} rsf\,\mathrm{d}s \tag{11}$$

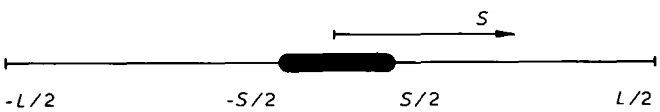

Figure 1

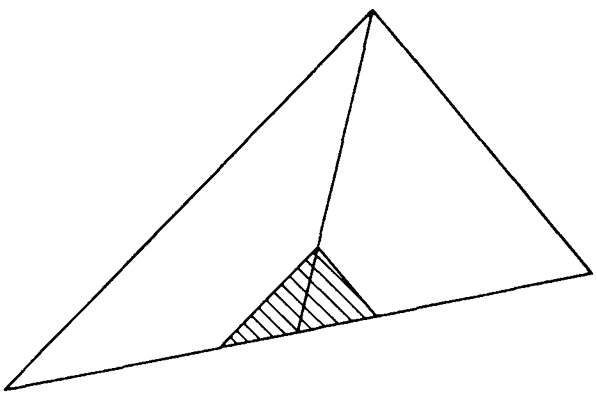

Figure 2

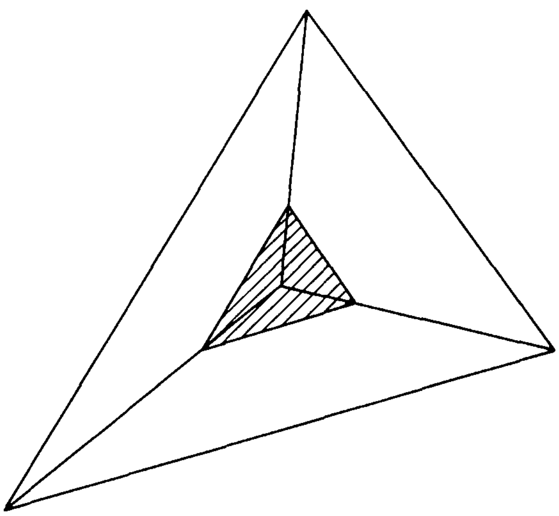

Figure 3

We need to consider how to calculate the I integral for each different case. The f functions to be integrated correspond to the first and second derivatives of the displacements and are the ones that appear in the tensors Σ_{ijk} and Σ_{ijkl}. In what follows we present the particular case of the integrals in the Q matrix. The same development applies for those integrals in Q'.

Matrix Q

For the calculation of the terms in this matrix it is necessary to consider terms in the first derivatives of the displacements in the Ω domain, see *Figure 6*. These integrals take the following expression:

$$\int_V \frac{U_{rr}}{r}\,\mathrm{d}v = \frac{-1}{8\pi G(1-\nu)\sqrt{R}} \int_{\theta_1}^{\theta_2} \times \left[\frac{3-4\nu}{2} I_1 + (6 - 8\nu + t_z^2) I_2\right] \mathrm{d}\theta \tag{12}$$

$$\int_V \frac{\partial U_{zr}}{\partial r}\,\mathrm{d}v = \frac{1}{8\pi G(1-\nu)\sqrt{R}} \int_{\theta_1}^{\theta_2} \times t_z \left[t_z^2 R I_4 - t_r^2 I_6 + \frac{t_r R}{4} I_7 + \frac{(4t_z^2 - 1)\,t_r}{4} I_2 - \frac{3 + t_z^2}{4} I_5 - \tfrac{3}{8} I_3\right] \mathrm{d}\theta$$

$$\int_V \frac{\partial U_{zr}}{\partial z}\,\mathrm{d}v = \frac{1}{8\pi G(1-\nu)\sqrt{R}} \int_{\theta_1}^{\theta_2} \times \left[(1 - 2t_z^2)\, t_r I_6 + \tfrac{1}{2}(1 + t_z^2) I_2 + \tfrac{1}{4} I_1\right] \mathrm{d}\theta$$

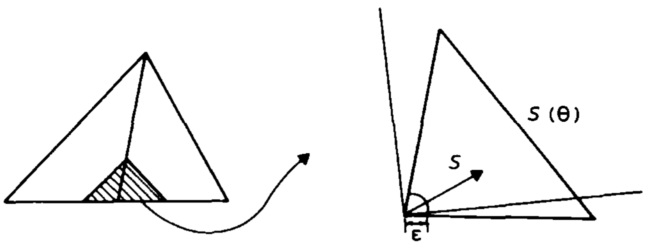

Figure 4

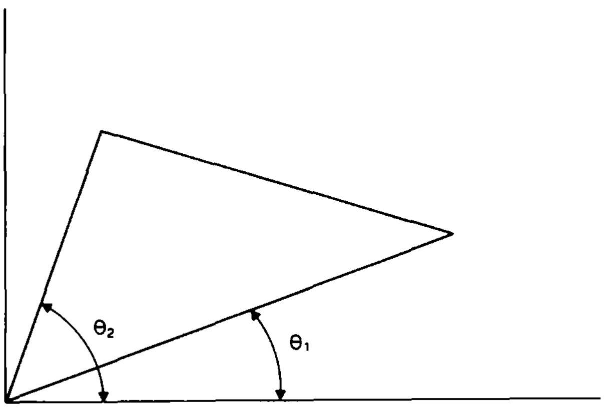

Figure 5

$$\int_V \frac{\partial U_{rz}}{\partial r}\,dv = \frac{1}{8\pi G(1-\nu)\sqrt{R}}\int_{\theta_1}^{\theta_2}$$

$$\times\, t_z\left[Rt_z^2 I_4 - t_r^2 I_6 + (t_z^2 - \tfrac{3}{4})\,t_r I_2 - \frac{t_r R}{4} I_7 + \frac{3+t_z^2}{4} I_5 + \tfrac{3}{8} I_3\right] d\theta$$

$$\int_V \frac{U_{rz}}{r}\,dv = \frac{-1}{8\pi G(1-\nu)\sqrt{R}}\int_{\theta_1}^{\theta_2}$$

$$\times\, t_z[\tfrac{1}{4} I_3 + R I_4 + 1/2 I_5 - I_6]\,d\theta$$

$$\int_V \frac{\partial U_{rr}}{\partial r}\,dv = \frac{1}{8\pi G(1-\nu)\sqrt{R}}\int_{\theta_1}^{\theta_2} d\theta$$

$$\left\{t_r\left(t_z^2 - \frac{3-4\nu}{2}\right) I_6 + t_r R\left(t_z^2 - \frac{3-4\nu}{2}\right)\right.$$
$$\left.\times I_4 + [(3-4\nu) + (3-2\nu)\,t_z^2 - t_z^4]\,I_2 + \tfrac{1}{4} I_1\right\}$$

$$\int_V \frac{\partial U_{rr}}{\partial z}\,dv = \frac{-1}{8\pi G(1-\nu)\,R}\int_{\theta_1}^{\theta_2} t_z(3 + 2t_r^2 - 4\nu)\,I_6\,d\theta$$

$$\int_V \frac{\partial U_{rz}}{\partial z}\,dv = \frac{1}{8\pi G(1-\nu)\sqrt{R}}\int_{\theta_1}^{\theta_2}$$

$$\times\, [t_r(1 - 2t_z^2)\,I_6 - \tfrac{1}{2}(1 + t_z^2) I_2 - \tfrac{1}{4} I_1]\,d\theta$$

$$\int_V \frac{\partial U_{zz}}{\partial r}\,dv = \frac{1}{8\pi G(1-\nu)\sqrt{R}}\int_{\theta_1}^{\theta_2}$$

$$\times\, [t_z^2(t_z^2 - 2\nu)\,I_2 - t_r(3 - 4\nu + 2t_z^2)/2$$
$$\times\, (I_6 - R I_4) + \tfrac{1}{4} I_1]\,d\theta$$

$$\int_V \frac{\partial U_{zz}}{\partial z}\,dv = \frac{1}{8\pi G(1-\nu)\,R}\int_{\theta_1}^{\theta_2} t_z[2t_r^2 - (3-4\nu)]\,I_6\,d\theta \qquad (12)$$

The expression for each of the I integrals is given in the Appendix. The integrals presented in (12) can then be used to compute the elements of Q. Similarly one can find the integrals necessary for Q', see Doblaré.[14]

Conclusions

Following the work by Telles and Brebbia[11] and Cruse *et al.*,[6] the complete boundary element formulation for the elastoplastic axisymmetric case is given in this paper. To this end the different tensor components corresponding to elastoplasticity have been calculated together with the principal value term for the plastic deformations. The boundary and domain integrals needed for the constant boundary element case are also given following the semi-analytical approach used by Telles and Brebbia[11] and Wrobel and Brebbia.[8]

References

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Appendix

$$I_1 = \int_\epsilon^S \frac{s}{\sqrt{r}} \ln\frac{s^2}{64Rr}\,ds = \frac{2}{3t_r^2}\left[(b - 2R)\sqrt{R+b}\right.$$
$$\times \ln\left|\frac{S^2}{64R(R+b)}\right| - 4/3(R+b)^{3/2} + 4R\sqrt{R+b}$$
$$\left. - 4R^{3/2}\ln\left|\frac{\sqrt{R+b}+\sqrt{R}}{\sqrt{R+b}-\sqrt{R}}\right| - 4R^{3/2}(\ln 2t_r + \tfrac{2}{3})\right]$$

$$I_2 = \int_\epsilon^S \frac{s}{\sqrt{r}}\,ds = \frac{2}{t_r^2}\,[(b - 2R)\sqrt{R+b} + 2R^{3/2}]$$

$$I_3 = \int_\epsilon^S \frac{s^2}{r^{3/2}} \ln\frac{s^2}{64Rr}\,ds = \frac{2}{t_r^2}\left[\frac{(R+b)^2 - 6R(R+b) - 3R^2}{3\sqrt{R+b}}\right.$$
$$\times \ln\frac{S^2}{64R(R+b)} - 2/9(R+b)^{3/2}$$
$$+ 8/3R\sqrt{R+b} + 2R^2\frac{1}{\sqrt{R+b}}$$
$$\left. - \frac{16}{3}\ln\left|\frac{\sqrt{R+b}+\sqrt{R}}{\sqrt{R+B}-\sqrt{R}}\right| - 8/3R^{3/2}(2\ln 2t_r + 5)\right]$$

$$I_4 = \int_\epsilon^S \frac{1}{\sqrt{r}}\,ds = \frac{2}{t_r}(\sqrt{R+b} - \sqrt{R})$$

$$I_5 = \int_\epsilon^S \frac{s^2}{r^{3/2}}\,ds = \frac{2}{t_r^3}\left[\frac{(R-b)^2 - 6R(R+b) - 3R^2}{3\sqrt{R+b}} + \frac{8R^{3/2}}{9}\right]$$

$$I_6 = \int_{\epsilon}^{S} \sqrt{r}\,\mathrm{d}s = \frac{2}{3t_r}\,[(R+b)^{3/2} - R^{3/2}]$$

$$I_7 = \int_{\epsilon}^{S} \frac{s}{r^{3/2}}\,\mathrm{d}s = \frac{2}{t_r^2}\left[\frac{2R+b}{\sqrt{R+b}} - 2\sqrt{R}\right]$$

With $b = t_r S$.

For the case $t_r = 0$, one has:

$$I_1 = \int_{\epsilon}^{S} \frac{s}{\sqrt{r}} \ln \frac{s^2}{64Rr}\,\mathrm{d}s = \frac{S^2}{2\sqrt{R}}\left(\ln \frac{S^2}{64R^2} - 1\right)$$

$$I_2 = \int_{\epsilon}^{S} \frac{s}{\sqrt{r}}\,\mathrm{d}s = \frac{S^2}{2\sqrt{R}}$$

$$I_3 = \int_{\epsilon}^{S} \frac{s^2}{r^{3/2}} \ln \frac{s^2}{64Rr}\,\mathrm{d}s = \frac{S^3}{3R^{3/2}}\left(\ln \frac{S^2}{64R^2} - \frac{2}{3}\right)$$

$$I_4 = \int_{\epsilon}^{S} \frac{1}{r}\,\mathrm{d}s = \frac{S}{\sqrt{R}}$$

$$I_5 = \int_{\epsilon}^{S} \frac{s^2}{r^{3/2}}\,\mathrm{d}s = \frac{S^3}{3R^{3/2}}$$

$$I_6 = \int_{\epsilon}^{S} \sqrt{r}\,\mathrm{d}s = S\sqrt{R}$$

$$I_7 = \int_{\epsilon}^{S} \frac{s}{r^{3/2}}\,\mathrm{d}s = \frac{S^2}{2R^{3/2}}$$