Triomensional plasticity using BIEM

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This paper presents the application of BIE techniques to elastoplastic three-dimensional problems. Along with the general procedures the needed integrations are described in detail and so is the flow chart of the written program.

Introduction

It is well known that the Boundary Integral Equation Method (BIEM) can be based on a reciprocal relationship between two elastic states. In our case, we shall take an auxiliary one defined by the fields ($\sigma_{ij}^*, E_{ij}^*, u_j^*$) and the elastic part of the increment of deformation components in an elastoplastic system ($\dot{\sigma}_{ij}, \;^e_{ij}, \dot{u}_i$).
That is:

$$\int_D \sigma_{ij}^* \dot{\epsilon}_{ij}^e \, dV = \int_D \epsilon_{ij}^* \dot{\sigma}_{ij} \, dV \tag{1}$$

or:

$$\int_D F_i^* \dot{u}_i \, dV + \int_{\partial D} T_i^* \dot{u}_i \, dS$$

$$= \int_D \dot{F}_i u_i^* \, dV + \int_{\partial D} \dot{T}_i u_i^* \, dS + \int_D \sigma_{ik}^* \dot{\epsilon}_{ik}^p \, dS$$

Taking the (*) system as that corresponding to the Kelvin solution, that is:

$$F_i^* = \delta(x - x)\, e_j$$
$$T_i^* = T_{ji} e_j$$
$$u_i^* = u_{ji} e_j$$
$$\sigma_{ik}^* = \Sigma_{jik} e_j$$

we obtain:

$$\dot{u}_j(x) + \int_{\partial D} T_{ji} \dot{u}_i \, dS$$

$$= \int_D \dot{F}_i u_{ji} \, dV + \int_{\partial D} \dot{T}_i u_{ji} \, dS + \int_D \Sigma_{jik} \dot{\epsilon}_{ik}^p \, dV \tag{2}$$

$x \in D$

which is a representation formula for the incremental displacement at an interior point x.

By the usual limiting procedure, equation (2) can be transformed in:

$$c_{ij} \dot{u}_i(x) + \int T_{ji} \dot{u}_i \, dS \quad \epsilon_{ik}^p \, dV$$

$$= \int_{\partial D} u_{ji} \dot{t}_i \, dS + \int_D u_{ji} \dot{F}_i \, dV + \int_D \Sigma_{jik} \qquad x \in \partial D \tag{3}$$

The first domain integral on the right hand side can be expressed as a boundary one for several useful cases (self-weight, centrifugal forces, thermoelasticity, etc.) in the form:

$$\int_D u_{ji} \dot{F}_i \, dV = \int_{\partial D} \phi u_{ji} n_i \, dS + \int_{\partial D} \Omega_j \frac{\partial \phi}{\partial n} \, dS - \int_{\partial D} \phi \frac{\partial \Omega_j}{\partial n} \, dS$$

$$- k_0 \int_{\partial D} w_{ji} n_i \, dS \tag{4}$$

with:

$$F = \nabla \phi$$
$$k_0 = \nabla^2 \phi$$
$$\Omega_j = \frac{1 - 2\nu}{16\pi G(1 - \nu)} r_{,j}$$
$$\frac{\partial \Omega_j}{\partial n} = \frac{1 - 2\nu}{16\pi G(1 - \nu)} r_{,jk} n_k \tag{5}$$

Inside the body under study:

$$\dot{\sigma}_{ij} = \frac{2G\nu}{1 - 2\nu} \delta_{ij} \dot{\epsilon}_{kk} + 2G \dot{\epsilon}_{ij}$$

where:

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^p$$

Assuming:

$$\dot{\epsilon}_{kk}^p = 0$$

we obtain the following material law:

$$\dot{\sigma}_{ij} = \frac{2G\nu}{1-2\nu}\,\delta_{ij}\dot{u}_{m,m} + G(\dot{u}_{i,j} + \dot{u}_{j,i}) - 2G\dot{\mathfrak{E}}^{p}_{ij} \tag{6}$$

After some algebra, the introduction of (2) inside (6) produces:

$$\dot{\sigma}_{ij} = \int_{\partial D} D_{kij}\dot{T}_k\,\mathrm{d}S - \int_{\partial D} S_{kij}\dot{u}_k\,\mathrm{d}S + \int_{D} D_{kij}\dot{F}_k\,\mathrm{d}V$$

$$+ \int_{D} \Sigma_{ijkl}\ {}^{p}_{kl}\,\mathrm{d}V - \frac{7-5\nu}{15(1-\nu)}\,2G\dot{\mathfrak{E}}^{p}_{ij} \tag{7}$$

In this expression, it has been necessary to use the principal value of the plastic singular integral developed by Bui and Mikhlin,[1] that is:

$$\frac{\partial}{\partial xh}\int_{D} \Sigma_{ml\nu}\mathfrak{E}^{p}_{l\nu}\,\mathrm{d}V = \int_{D} \frac{\partial\sigma_{ml\nu}}{\partial xh}\,\dot{\mathfrak{E}}^{p}_{l\nu}\,\mathrm{d}V + \frac{8-10\nu}{15(1-\nu)}\,\dot{\mathfrak{E}}^{p}_{mk} \tag{8}$$

The discretization of the above written equations can easily be obtained when we assume a constant value of the functions on boundary elements as well as inside cells in which the whole domain is divided.
The final results are:

$$\dot{u}_j(k) + \sum_{L=1}^{\text{N.Ele.}} \mathfrak{T}_{ji}\dot{u}_i(L)$$

$$= \sum_{L=1}^{\text{N.Ele.}} \mathfrak{U}_{ji}\dot{T}_i(L) + \sum_{L=1}^{\text{N.Ele.}} I_{\partial D} + \sum_{M=1}^{\text{N.Cel.}} E_{jik}\mathfrak{E}^{p}_{ik}(M)$$

$$K, M \in D, L \in \partial D$$

$$\sigma_{ij}(m) = \sum_{L=1}^{\text{N.Ele.}} \mathfrak{D}_{kij}\dot{T}_k(L) - \sum_{L=1}^{\text{N.Ele.}} \mathfrak{F}_{kij}\dot{u}_k(L)$$

$$+ \sum_{M=1}^{\text{N.Cel.}} E_{jikl}\dot{\mathfrak{E}}^{p}_{kl}(M) + \sum_{N=1}^{\text{N.Ele.}} I'_{\partial D}$$

$$- \frac{7-5\nu}{15(1-\nu)}\,2G\dot{\mathfrak{E}}^{p}_{ij}(M) \quad K, L \in \partial D, M \in D$$

$$\frac{1}{2}\dot{u}_i(k) + \sum_{L=1}^{\text{N.Ele.}} \mathfrak{T}_{ji}\dot{u}_i(L)$$

$$= \sum_{L=1}^{\text{N.Ele.}} \mathfrak{U}_{ji}\dot{t}_i(L) + \Sigma\dot{I}_{\partial D} + \sum_{M=1}^{\text{N.Cel.}} E_{jik}\mathfrak{E}^{p}_{ik}(M) \tag{9}$$

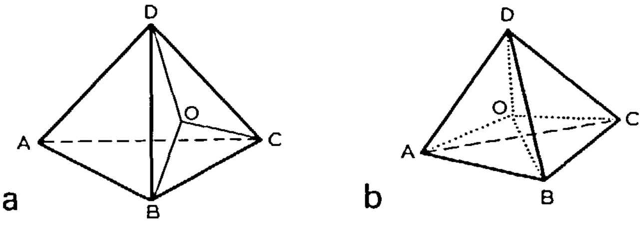


Figure 1 Integration from points contained in cells where integral is being carried. a, boundary equation (3) ABCD = ABOD U ABOC U ACOD. b, internal stress, equation (7) ABCD = OABC U OBCD U OACD U OADB

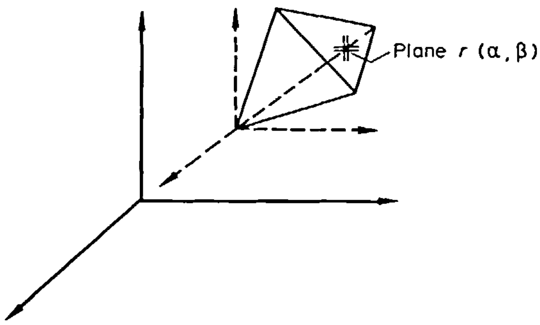


Figure 2 Tetrahaedron, D_i. $I_1 = \int_{D_i}\Sigma_{kij}\,\mathrm{d}V$

where there is an obvious correspondence among the capitals $\mathfrak{T}, \mathfrak{U}, \mathfrak{E}, \mathfrak{D}, \mathfrak{F}$ and the integrals with kernels T, U, Σ, D, S and where N.Ele. and N.Cel. means 'number of elements' and 'number of cells', respectively and:

$$\sum_{L=1}^{\text{N.Ele.}} I_{\partial D} \qquad \sum_{L=1}^{\text{N.Ele.}} I_{\partial D}$$

represent the discretization of the (4) expression.

Computation of the coefficients

None of the boundary integrals presents any computational difficulty and the procedures used in elastic solutions can be utilized without difficulty. This is why we shall concentrate on the analysis of the volume integrals corresponding to plastic deformation. To compute those coefficients the domain D is divided into tetrahaedrical cells in which the value of $\mathfrak{E}^{p}$ is assumed to be constant.

There are several cases, among which we shall distinguish two groups: when the integration is carried from points contained in a cell or when it refers to a point far from the cell.

The first category can be reduced to a simpler problem of whether the point is on the cell boundary or inside it (*Figure 1*). Taking the point as a vertex the cell can be subdivided into three (*Figure 1a*) or four (*Figure 1b*) subdomains in which the common feature is that the integral has to be carried from a vertex of a tetrahaedron.

The second category is generally computed by numerical methods, while the first is done by semianalytical methods.

Let us discuss this later problem. The integration from boundary points is:

$$I_1 = \int_{D_i} \Sigma_{kij}\,\mathrm{d}V \tag{10}$$

where:

$$\Sigma_{kij} = -\frac{C_1}{r^2}f_{kij}$$

$$f_{kij} = (\delta_{ki}r_{,j} + \delta_{kj}r_{,j} - \delta_{1j}r_{,k}) + 3C_3\,r_{,i}r_{,j}r_{,k}$$

$$C_1 = \frac{1-2\nu}{8\pi(1-\nu)} \qquad C_3 = \frac{\nu}{1-2\nu} \tag{11}$$

so that, integrating in D_i (*Figure 2*),

$$I_1 = -C_1\int_{\alpha}\int_{\beta}\int^{r(\alpha,\beta)} \frac{1}{r^2}f_{kij}\,\mathrm{d}V$$

$$= -C_1 \int_\alpha \int_\beta \int^{r(\alpha,\beta)} f_{kij} \cos\beta \, d\alpha \, d\beta \qquad (12)$$

where $r(\alpha, \beta)$ represents the equation of the face opposite the vertex from which the integration is done.

Changing to a system of natural coordinates (*Figure 3*) contained in that face:

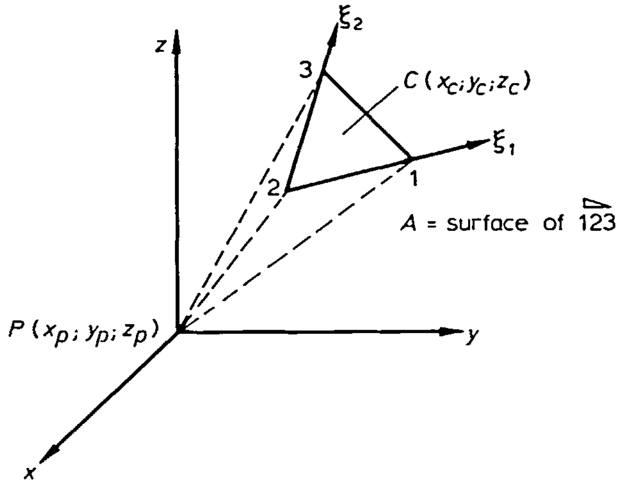


Figure 3

be utilized without difficulty. This is why we shall concentrate on the analysis of the volume integrals correspondong to plastic deformation. To compute these coefficients the domain D is divided into tetrahaedrical cells in which the value of $\mathfrak{C}^p$ is assumed to be constant.

There are several cases, among which we shall distinguish two groups: when the integration is carried from points contained in a cell or when it refers to a point far from the cell.

The first category can be reduced to a simpler problem of whether the point is on the cell boundary or inside it we have:

$$d\alpha \, d\beta = \frac{2A}{r^2 \cos\beta} d\xi_1 \, d\xi_2$$

and so:

$$I_1 = -2AC_1 \int_\alpha \int_\beta \frac{1}{r} f_{kij} \, d\xi_1 \, d\xi_2$$

$$= -2AC_1 \sum_{i=1}^{N} f(\xi_1^i \xi_2^i) \, W_i \qquad (13)$$

where W_i are the appropriate weights for a numerical Hammer quadrature.

The other case is the integral needed for the equation (7) of internal points. In this case:

$$I_2 = \int_{\mathfrak{D}} \Sigma_{ijkl} \, dV$$

where:

$$\Sigma_{ijkl} = \frac{C}{r^3} g_{ijkl} \qquad C = \frac{G}{4\pi(1-\nu)}$$

$$\begin{aligned} g_{ijkl} = {} & 3(1-2\nu)(\delta_{ij} r_{,k} r_{,l} + \delta_{kl} r_{,i} r_{,j}) \\ & + 3(\delta_{li} r_{,j} r_{,k} + \delta_{jk} r_{,l} r_{,i} + \delta_{ik} r_{,l} r_{,j} + \delta_{jl} r_{,i} r_{,k}) \\ & - 15 r_{,i} r_{,j} r_{,l} + (1-2\nu)(\delta_{ik}\delta_{lj} + \delta_{jk}\delta_{li}) \\ & - (1-4\nu)\,\delta_{ij}\delta_{kl} \end{aligned} \qquad (14)$$

or:

$$I_{\mathfrak{D}i} = \int_{\mathfrak{D}_i} \frac{C}{r^3} g \, d\theta = \int_\alpha \int_\beta \int_{\mathfrak{C}}^{r} \frac{C}{r^3} g r^2 \cos\beta \, d\alpha \, d\beta \, dr$$

$$= C \int_\alpha \int_\beta \ln r g \cos\beta \, d\alpha \, d\beta - C \ln$$

$$- C \ln \xi \int_\alpha \int_\beta g \cos\beta \, d\alpha \, d\beta \qquad (15)$$

Summing up the contributions of the four tetraedra confluenting in this vertex, the second integral will be defined inside a sphere of radius ξ.

As:

$$\int_\alpha \int_\beta r_{,i} r_{,j} \cos\beta \, d\alpha \, d\beta = \delta_{ij} \frac{4}{3}\pi$$

we obtain:

$$\int_\alpha \int_\beta g \cos\beta \, d\alpha \, d\beta = 0 \qquad (16)$$

and the integral will be:

$$I_2 = C \int_\alpha \int_\beta \ln r g \cos\beta \, d\alpha \, d\beta$$

$$= 2CA \int_\alpha \int_\beta \frac{\ln r}{r^2} g \, d\xi_1 \, d\xi_2 \qquad (17)$$

which is solved numerically in the same way as I_1.

In the previously called 'second category', that is, when the integration point is far from the cell, the computation is done numerically from the beginning. That is:

$$I_1' = \int_{\mathfrak{D}} \Sigma_{kij} \, dV \simeq -C_1 \int_{\mathfrak{D}} \frac{1}{r^2} f_{kij} 6V \, d\xi_1 \, d\xi_2 \, d\xi_3 \qquad (18)$$

and:

$$I_2' = \int_{\mathfrak{D}} \Sigma_{ijkl} \, dV \simeq C \int_{\mathfrak{D}} \frac{1}{r^3} g 6V \, d\xi_1 \, d\xi_2 \, d\xi_3 \qquad (19)$$

Iterative procedure

The establishment of equations (9) for as many points as elements on the boundary leads to a system:

$$H \cdot \dot{u} = G \cdot \dot{t} + \dot{I} + D\mathfrak{C}^p \qquad (20)$$

and for internal cells:

$$\underline{\dot{\sigma}} = G'\underline{t} - H' \cdot \underline{\dot{u}} + \dot{I}' + (D' + C')\,\mathfrak{C}^p \qquad (21)$$

Reordering (20) and (21) according to proper data and unknowns of the problem will produce:

$$A\dot{\underline{x}} = \dot{F} + D\mathfrak{C}^p \tag{22}$$

$$\underline{\dot{\sigma}} = \dot{A}\dot{x} + \underline{\dot{F}}' + D^* \cdot \mathfrak{C}^p \tag{23}$$

or:

$$\dot{x} = A^{-1}\dot{F} + A^{-1}D\mathfrak{C}^p \tag{24}$$

$$\dot{\sigma} = A'A^{-1}\underline{\dot{F}} + A'A^{-1}D\mathfrak{C}^p + \dot{F}' + D^*\mathfrak{C}^p \tag{25}$$

In compact form:

$$\dot{x} = \dot{M} + k\mathfrak{C}^p$$

$$\dot{\sigma} = \dot{N} + B\mathfrak{C}^p \tag{26}$$

As (26) represent incremental values, it is possible to write, for a step I:

$$x_i = \sum_0^{i-1} \dot{x} + \dot{x}_i = \sum_0^{i-1} \dot{M}_k + k\sum_0^{i-1} \mathfrak{C}_k^p + \dot{M}_i + k\mathfrak{C}_i^p$$

$$= \lambda_i M + k \sum_0^{i-1} \mathfrak{C}_k^p + \mathfrak{C}_i^p$$

$$\sigma_i = \sum_0^{i-1} \dot{\sigma}_k + \dot{\sigma}_i = \sum_0^{i-1} \dot{N}_k + B\sum_0^{i-1} \mathfrak{C}_k^p + \dot{N}_i + B\mathfrak{C}_i^p$$

$$= \lambda_i N + B \sum_0^{i-1} \mathfrak{C}_k^p + \mathfrak{C}_i^p \tag{27}$$

or:

$$x_i = \lambda_i M + k(\mathfrak{C}^p + \Delta\mathfrak{C}^p) \qquad \sigma_i = \lambda_i N + B(\mathfrak{C}^p + \Delta\mathfrak{C}^p) \tag{28}$$

where M and N are elastic boundary and domain solutions and λ_i the corresponding load factor. The iterative procedure is clearly shown in the flow chart contained in the next paragraph, where:

σ_e is the equivalent or effective stress (Von Mises criterion).

$$\sigma^e = \frac{1}{2}\,[(\sigma x - \sigma y)^2 + (\sigma x - \sigma z)^2 + (\sigma y - \sigma z)^2$$

$$+ 6(\tau x^2 y + \tau x^2 z + \tau y^2 z)]^{1/2}$$

$\sigma_e^{\max}$ is the maximum value of σ_e in the whole domain:

$$E'_{ij} = E^e_{ij} + \Delta E^p_{ij}$$

$$e'_{ij} = E'_{ij} - \frac{1}{3}E'_{kk}\delta_{ij}$$

E_{et} is the modified equivalent deformation:

$$_{et} = \frac{2}{3}e'_{ij}e'_{ij}$$

$$\Delta E_p = \frac{3GE_{et} - \sigma_{e_{i-1}}}{3G + H'_{i-1}}$$

$$\Delta E^p_{ij} = \frac{\Delta E_p}{E_{et}}\,e'_{ij}$$

H' is the tangent to the curve equivalent stress versus equivalent strain.

Program characteristics

The numerical process, presented above, has been developed in a FORTRAN V program and implemented in a UNIVAC 1108. The mean properties of this program are:

(1) It uses a constant interpolation function for the displacements, stresses and plastic strain fields.
(2) It uses flat triangular elements in order to discretize the boundary of the domain.
(3) It introduces tetraedral cells with flat faces to discretize the plastic part of the domain.
(4) It considers homogeneous and isotropic materials only.
(5) It is possible to take volume loads into account.
(6) It is possible to take concentrated loads such as those corresponding to prestressing cables into account.
(7) Materials have been considered for which the incompressibility of plastic strains (metals, clays) is accepted.
(8) The strain-hardening model that has been used is the isotropic one.

The main parts of the flow chart of the programe are described below:

Input data
Print data

↓

Computation of integral coefficients for elastic case
Generation of equation system for elastic case

↓

Boundary conditions

↓

Computation of elastic solution
$X = M$

↓

Computation of A, I, D and K matrices

↓

Computation of A', N and U matrices

↓

Vector displacements in internal cells
for elastic solution

↓

Print elastic results

↓

Computation of matrix D and
$B = D^* + A'K$

↓

Read σ_0, W and M

↓

Iterative process
Computation of plastic strains
in internal cells

↓

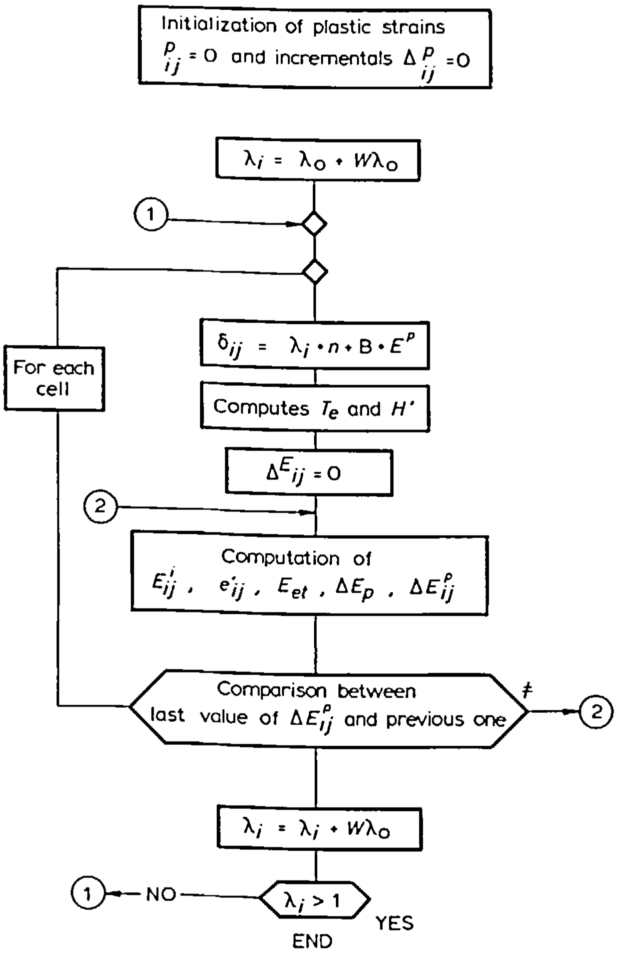
Print results u, t, σ

↓

End

The iterative process is described below with more detail.

Iterative process



Conclusions

This paper presents the application of B.I.E.M. to tri-dimensional plasticity and its implementation on the computer. This theme is being developed at present for more complicated cases at the Polytechnical University in Madrid and the University of Southampton, by the present authors and other members of these groups.

Notations

	domain
∂	boundary of domain
x	interior point x
y	boundary point y
$E_{ij} = E^e_{ij} + E^p_{ij}$	total strain
E^e_{ij}	elastic strain
E^p_{ij}	plastic strain
$u_i(x)$	component i of displacement of point x
$t_i(x)$	component i of stress vector in point x
X_k	volume loads
$\sigma_{ij}(x)$	stress tensor in an internal point
δ_{ij}	Kronecker's delta
$(^*) = \Delta$	increment of a variable
f, j	derivative of function to component j

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Appendix

Tensors

Here we present the Kelvin solution and the associated tensors, which have been used in previous equations.
If we define:

x: Point where the concentrated load is applied

y: Field point

r: $|x - y|$

we will have:

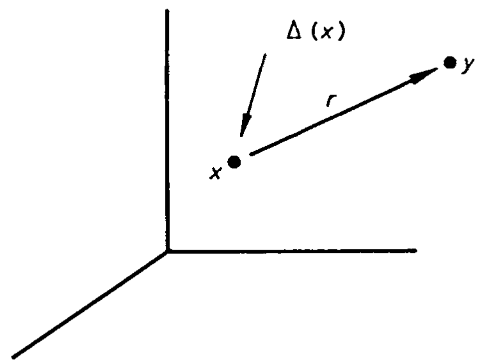

Figure A1

$$x^*_i(y) = F^*_i = \Delta(x)\, e_j$$
$$t^*_i(y) = T_{ji}(x, y)\, e_j$$
$$u^*_i(y) = u_{ji}(x, y)\, e_j$$
$$\sigma^*_{ik}(y) = \sigma_{jik}(x, y)\, e_j$$

where:

$$u_{ji} = \frac{1}{16\pi G(1-\nu)\, r}\,[(3 - 4\nu)\,\delta_{ij} + r_{,i} r_{,l}]$$

$$T_{ji} = -\frac{1}{8\pi G(1-\nu)\, r^2}\;\frac{\partial r}{\partial n}\,((1 - 2\nu)\,\delta_{ij} + 3 r_{,i} r_{,j})$$
$$+ (1 - 2\nu)(r_{,i} n_j - n_i r_{,i})]$$

$$\sigma_{kij} = \Sigma_{kij} = -D_{kij} = -\frac{1}{8\pi(1-\nu)\,r^2} \times [(1-2\nu)(\delta_{ik}r_{,j} + \delta_{jk}r_{,i} - \delta_{ij}r_{,k}) + 3r_{,i}r_{,j}r_k]$$

$$S_{kij} = \frac{G/\pi}{4(1-\nu)\,r^3} \quad 3\frac{\partial r}{\partial n}[(1-2\nu)\,\delta_{ij}r_{,k} + \nu(\delta_{jk}r_{,i} + \delta_{ij}r_{,j}) - 5r_{,i}r_{,j}r_k] + 3\nu(n_ir_{,j}r_{,k} + n_jr_{,i}r_{,k}) + (1-2\nu)(3n_kr_{,i}r_{,j} + \delta_{ik}n_j + \delta_{jk}n_i) - (1-4\nu)\,\delta_{ij}n_k$$

$$\sigma_{jikl} = \Sigma_{ijkl} = \frac{G}{4(1-\nu)\,\pi r^3}\{3(1-2\nu)(\delta_{ij}r_{,k}r_{,l} + \delta_{kl}r_{,i}r_{,j}) + 3\nu(\delta_{li}r_{,j}r_{,k} + \delta_{jk}r_{,l}r_{,i} + \delta_{jl}r_{,k}) - 15r_{,i}r_{,j}r_{,k}r_{,l} + (1-2\nu)(\delta_{ik}\cdot\delta_{lj} + \delta_{jk}\delta_{li} - (1-4\nu)\,\delta_{ij}\delta_{kl}$$

where:

$$r_{,i} = \frac{y_i - x_i}{r}$$

$$r_{,ij} = -\frac{1}{r}(-\delta_{ij} + r_{,i}r_{,j})$$