CONSERVATION LAWS FOR LIQUID BRIDGES

I. Da-Riva

Laboratorio de Aerodinámica, Universidad Politécnica, Madrid, Spain

ABSTRACT

Conservation laws for an inviscid liquid bridge set into motion by conservative forces are given in integral form. These laws provide useful information on the overall motion of the bridge in the presence of unexpected or uncontrolled disturbances and could, in addition, be monitored in a computational solution of the problem as an accuracy check.
Many of the resulting conservation laws are familiar to fluiddynamicists. Nevertheless, a systematic approach providing an exhaustive list of these laws reveals the existence of new conserved properties hardly deducible in the classical way. Although the present analysis concerns the case of axial, and constant, gravity it can be applied, with minor refinements, when the gravity field varies with time in both direction and intensity.

Keywords: Microgravity, Liquid bridge, Water waves, Symmetry groups, Conserved densities, Conservation laws.

1. INTRODUCTION

The idea of looking for groups of continuous transformations with respect to which the equations of some physical phenomena are invariant was developed by Birkhoff (Ref. 1). The mathematical algorithm for seeking the broad group of these transformations (symmetry groups) for ordinary differential equations is due to Lie, see the historical account in the preface to Ref. 2. In recent times a series of results of general character and applications to particular problems of mechanics and physics have been given by Ovsiannikov (Ref. 3), Bluman and Cole (Ref. 2), and Benjamin and Olver (Ref. 4). In particular, the problem of water waves in an otherwise undisturbed ocean is considered in the last reference.

Conservation laws written in integral form are linked to every one-parameter group of symmetries according to a theorem due to Noether which can be applied to variational problems associated to Euler-Lagrange systems (Ref. 3). For the present purposes a version of this theorem for Hamiltonian systems (Ref. 5) looks more adventageous.

2. THE LIQUID BRIDGE

The bridge is a liquid column, more or less cylindrical, held by surface tension forces between two solid supports not too far apart from each other.

The liquid is assumed to be inviscid, and set into motion from rest by conservative forces. The length of the bridge is larger than the characteristic axial length of the surface wave under consideration or the axial distance travelled by the wave during the observation time.

Gravity action either axial or non-axial, steady or non-steady can be taken into account.

2.1 The hydrodynamic problem

The hydrodynamic problem is defined as follows. See Figure 1 for the nomenclature and coordinate frame.

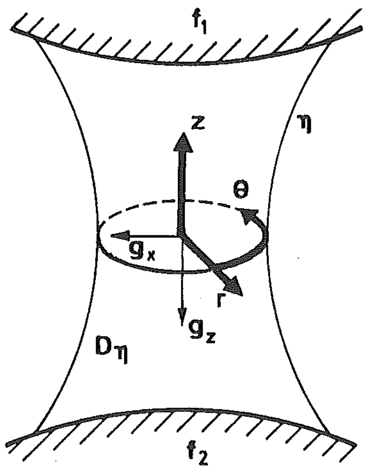

Figure 1. Geometry, coordinate system and fluid velocity components for the liquid bridge

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1) Differential equation for the velocity potential in cylindrical coordinates

$$\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial\theta^2} = 0 \quad \text{in } D_\eta \qquad (1)$$

2) Kinematic boundary condition at the free surface, $r = \eta(z,\theta,t)$,

$$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial r} - \frac{\partial\phi}{\partial z}\frac{\partial\eta}{\partial z} - \frac{1}{\eta^2}\frac{\partial\phi}{\partial\theta}\frac{\partial\eta}{\partial\theta} \quad \text{on } \eta. \qquad (2)$$

3) Dynamic boundary condition expressed as the Bernoulli integral on the free surface,

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\{(\frac{\partial\phi}{\partial r})^2 + (\frac{\partial\phi}{\partial z})^2 + \frac{1}{\eta^2}(\frac{\partial\phi}{\partial\theta})^2\} -$$
$$- g_z z - g_x x + \frac{\sigma}{\rho}K = F(t) \quad \text{on } \eta, \qquad (3)$$

where σ is the surface tension, ρ the density and K the local mean curvature of the free surface defined as

$$K = \frac{1}{\eta}\{\frac{1+\eta_z^2}{R} - (\frac{\eta\eta_z}{R})_z - (\frac{\eta_\theta}{\eta R})_\theta\} , \qquad (4)$$

with $\eta R = \sqrt{\eta^2(1+\eta_z^2) + \eta_\theta^2}$.

$F(t)$ is the integral of Bernoulli equation, which is usually eliminated by redefinition of $\partial\phi/\partial t$.

4) Kinematic boundary condition at the solid end supports, $f_i(z,\theta,t,r) = 0$, i=1,2,..., if relevant,

$$\frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial r}\frac{\partial\phi}{\partial r} + \frac{\partial f_i}{\partial z}\frac{\partial\phi}{\partial z} + \frac{1}{r^2}\frac{\partial f_i}{\partial\theta}\frac{\partial\phi}{\partial\theta} = 0 . \qquad (5)$$

In addition, contact angle conditions at the intersections of the free surface with the solid supports must be stated.

3. INFINITESIMAL ONE-PARAMETER TRANSFORMATION

Let an infinitesimal one-parameter transformation be defined as:

$$z^* = z + \alpha^1(z,\theta,t,r,\phi)\varepsilon$$
$$\theta^* = \theta + \alpha^2(z,\theta,t,r,\phi)\varepsilon$$
$$t^* = t + \tau(z,\theta,t,r,\phi)\varepsilon$$
$$r^* = r + \beta(z,\theta,t,r,\phi)\varepsilon$$
$$\phi^* = \phi + \gamma(z,\theta,t,r,\phi)\varepsilon ,$$

where α^1,α^2, ... are smooth functions defined in R^5 and ε a small parameter.

Let consider a smooth function $F(z^*,\theta^*,t^*,r^*,\phi^*)$, defined in R^5, which varies along the path curve (or orbit) of an arbitrary initial point z,θ,t,r,ϕ. For small values of ε,

$$\varepsilon\delta F = F(z^*,\theta^*,t^*,r^*,\phi^*) - F(z,\theta,t,r,\phi) =$$
$$= v\{F(z,\theta,t,r,\phi)\}\varepsilon ,$$

where the vector field

$$v = \alpha^1(z,\theta,t,r,\phi)\frac{\partial}{\partial z} + \alpha^2(z,\theta,t,r,\phi)\frac{\partial}{\partial\theta} +$$
$$+ \tau(z,\theta,t,r,\phi)\frac{\partial}{\partial t} + \beta(z,\theta,t,r,\phi)\frac{\partial}{\partial r} +$$
$$+ \gamma(z,\theta,t,r,\phi)\frac{\partial}{\partial\phi} , \qquad (6)$$

is the infinitesimal generator of the one-parameter group of symmetries (tangent vector field). These symmetries are obtained by integration of the system of ordinary differential equations

$$\alpha^1 = \frac{dz}{d\varepsilon} , \ \alpha^2 = \frac{d\theta}{d\varepsilon} , \ \tau = \frac{dt}{d\varepsilon} , \ \beta = \frac{dr}{d\varepsilon} , \ \gamma = \frac{d\phi}{d\varepsilon} , \qquad (7)$$

with $\varepsilon = 0$ taken as the identity element of the group.

The action of the group on a function $\mathfrak{f}$ defined as

$$\phi = \mathfrak{f}(z,\theta,t,r) ,$$

will be

$$\delta\mathfrak{f} = \delta\phi - v_1\{\mathfrak{f}\} , \qquad (8)$$

where

$$\delta\phi = \gamma(z,\theta,t,r,\mathfrak{f}(z,\theta,t,r))$$

and v_1 is the projection of v onto the space of independent variables,

$$v_1 = \alpha^1(z,\theta,t,r,\mathfrak{f})\frac{\partial}{\partial z} + \alpha^2(z,\theta,t,r,\mathfrak{f})\frac{\partial}{\partial\theta} +$$
$$+ \tau(z,\theta,t,r,\mathfrak{f})\frac{\partial}{\partial t} + \beta(z,\theta,t,r,\mathfrak{f})\frac{\partial}{\partial r} .$$

In order to calculate the action of the group on the derivatives of ϕ which appear in Eqs. 1 to 3, let δ_k represent k times derivation with respect to independent variables. Application of Eq. 8 to

$$\delta_k\phi = \delta_k\mathfrak{f}(z,\theta,t,r)$$

yields

$$\delta(\delta_k\mathfrak{f}) = \delta(\delta_k\phi) - v_1\{\delta_k\mathfrak{f}\}, \qquad (9)$$

whereas, by derivating k times

$$\delta_k(\delta\mathfrak{f}) = \delta_k[\delta\phi - v_1\{\mathfrak{f}\}] . \qquad (10)$$

Since the left hand sides of Eqs. 9 and 10 are equal,

$$\delta(\delta_k\phi) = \delta_k[\delta\phi - v_1\{\mathfrak{f}\}] + v_1\{\delta_k\mathfrak{f}\} . \qquad (11)$$

Eq. 11 will be incorporated into the formula of prolongation (Ref. 3, pp. 41 and ff),

$$prv = v + \delta\phi_k\frac{\partial}{\partial\phi_k} , \qquad (12)$$

where the subscript $k \geq 1$ denotes all the k-th and lower order derivatives of ϕ with respect to independent variables. Eq. 12 represents the tangent vector field prolonged up to order k to the space of ϕ derivatives.

3.1 Invariance of the differential equation 1

The infinitesimal criterion for invariance of Eq. 1, i.e.,

$$prv\{\Delta\phi\} = 0 \quad \text{whenever } \Delta\phi = 0 ,$$

will be explicited by taking into account Eqs. 6, 11 and 12, and by expressing ϕ_r in terms of second order derivatives through Eq. 1. This will lead to and expression which should vanish no matter the values of ϕ_k. Then the coefficients of the monomials involving ϕ_k or products of ϕ_k must vanish.

Table 1 below gives the coefficients of those monomials which, not being identically nil a priori, provide information on the functions α^1, α^2, τ, β and γ.

Table 1.

Terms Appearing in prv$\{\Delta\phi\} = 0$

Monomial	Coefficient
$\phi_z\phi_{zz}, \phi_r\phi_{zr}, \phi_\theta\phi_{z\theta}/r^2$	$-2\alpha^1_\phi$
$\phi_z\phi_{z\theta}, \phi_r\phi_{\theta r}, \phi_\theta\phi_{\theta\theta}/r^2$	$-2\alpha^2_\phi$
$\phi_z\phi_{zt}, \phi_r\phi_{tr}, \phi_\theta\phi_{\theta t}/r^2$	$-2\tau_\phi$
$\phi_z\phi_{zr}, \phi_r\phi_{rr}, \phi_\theta\phi_{\theta r}/r^2$	$-2\beta_\phi$
$\phi_z^2\,,\ \phi_\theta^2\,,\ \phi_r^2$	$\gamma_{\phi\phi}$
ϕ_{zr}	$-2(\alpha^1_r + \beta_z)$
$\phi_{z\theta}/r^2$	$-2(\alpha^1_\theta + r^2\alpha^2_z)$
$\phi_{\theta r}/r^2$	$-2(r^2\alpha^2_r + \beta_\theta)$
ϕ_{zt}	$-2\tau_z$
$\phi_{\theta t}/r^2$	$-2\tau_\theta$
ϕ_{rt}	$-2\tau_r$
ϕ_{zz}	$-2\alpha^1_z + \frac{\beta}{r} + r\Delta\beta - 2r\gamma_{\phi r}$
$\phi_{\theta\theta}/r^2$	$-2\{\alpha^2_\theta + \frac{\beta}{r}\} + \frac{\beta}{r} + r\Delta\beta - 2r\gamma_{\phi r}$
ϕ_{rr}	$-2\beta_r + \frac{\beta}{r} + r\Delta\beta - 2r\gamma_{\phi r}$
ϕ_z	$-\Delta\alpha^1 + 2\gamma_{\phi z}$
ϕ_θ/r^2	$-r^2\Delta\alpha^2 + 2\gamma_{\phi\theta}$
ϕ_t	$-\Delta\tau$
	$\Delta\gamma$

The differential equations which result from equating to zero the different coefficients are fulfilled by the following set of functions:

$$\alpha^1(z,\theta,t,r) = a^o(t) - b^1(t)z + c^1(\theta,t)r - 2c^2(\theta,t)zr \quad , \quad (13)$$

$$\alpha^2(z,\theta,t,r) = b^o(t) - \frac{\partial}{\partial\theta}\{c^o(\theta,t)\frac{1}{r} + c^1(\theta,t)\frac{z}{r} - c^2(\theta,t)\frac{z^2+r^2}{r}\},$$

$\tau(t)$,

$$\beta(z,\theta,t,r) = -c^o(\theta,t) - c^1(\theta,t)z - b^1(t)r + c^2(\theta,t)(z^2-r^2),$$

$$\gamma(z,\theta,t,r,\phi) = \phi\{c^2(\theta,t)r + C_{13}\} + \gamma^2(z,\theta,t,r) \quad ,$$

with

$$c^i(\theta,t) = C^i(t)\cos\theta + S^i(t)\sin\theta \quad , \ i=0,1,2.$$

C_{13}, as well as the C_j used below, are integration constants. Subscripts and superscripts are arranged so that a cross-checking with the symmetries in the threedimensional case (Ref. 4) is facilitated as far as possible.

3.2 Invariance of the boundary condition Eq. 2

To express the invariance of the boundary conditions on $r = \eta(z,\theta,t)$ the vector field v must be prolonged to the derivatives of η on the free surface (Ref. 4).

$$\text{prv}_S = v_S + (\partial\phi_k)_S\frac{\partial}{\partial\phi_k} + \partial\eta_k\frac{\partial}{\partial\eta_k} \quad , \qquad (14)$$

where S indicates that functions involved are to be calculated on the free surface.

Now, following with the function h

$$\eta = h(z,\theta,t)$$

the same steps as in Eqs. 8 to 11 with ζ,

$$\partial\eta_k = \partial_k[\partial\eta - v_o\{h\}] + v_o\{h_k\} \quad , \qquad (15)$$

where

$$\partial\eta = \beta(z,\theta,t,\eta(z,\theta,t)) \quad ,$$

and

$$v_o = \alpha^1(z,\theta,t,\eta(z,\theta,t))\frac{\partial}{\partial z} + \alpha^2(z,\theta,t,\eta(z,\theta,t))\frac{\partial}{\partial\theta} + \tau(t)\frac{\partial}{\partial t} \ .$$

(Recall Table 1 in connection with α^1, α^2 and τ).

The invariance of boundary condition Eq. 2 provides a set of ordinary differential equations, most of them duplicating those already established. The only coefficients which are not identically nil, after taking into account the results from Table 1, are summarized in Table 2.

Table 2.

Terms Appearing in $\text{prv}_S\{\eta_t - \phi_z\eta_z - \frac{1}{\eta^2}\phi_\theta\eta_\theta + \phi_r\}$

Monomial	Coefficient
$\eta_\theta\phi_\theta/\eta^2\ ,\ \eta_z\phi_z\ ,\ \phi_r$	$\tau_t - 2\beta_r + \gamma_\phi$
η_z	$\alpha^1_t - \gamma_z$
η_θ/η^2	$\eta^2\alpha^2_t - \gamma_\theta$
	$\beta_t - \gamma_r$

The additional differential equations yield the following expressions for the components of v:

$$\alpha^1(z,\theta,t,r) = a^o(t) + \frac{C_{12}+C_{13}}{2}z + \{C_{10}\cos\theta + C_{11}\sin\theta\}r \quad , \qquad (16)$$

$$\alpha^2(z,\theta,t,r) = \{C^o(t)\sin\theta - S^o(t)\cos\theta\}\frac{1}{r} + C_2 + $$
$$+ \{C_{10}\sin\theta - C_{11}\cos\theta\}\frac{z}{r} ,$$

$$\tau(t) = C_{12}t + C_3 ,$$

$$\beta(z,\theta,t,r) = - \{C^o(t)\cos\theta + S^o(t)\sin\theta\} - \{C_{10}\cos\theta + $$
$$+ C_{11}\sin\theta\}z + \frac{C_{12}+C_{13}}{2} r ,$$

$$\gamma(z,\theta,t,r,\phi) = C_{13}\phi - \{C^o_t(t)\cos\theta + S^o_t(t)\sin\theta\}r + $$
$$+ a^o_t(t)z + \gamma^3(t) .$$

3.2 Invariance of the boundary condition Eq. 3

The process is quite similar to that pursued with Eq. 2.

The very involved expression of the curvature at the free surface, Eq. 4, greatly complicates the algebra. To circumvent this difficulty we first assume $\sigma = 0$ and calculate the symmetries, testing afterward which among the obtained symmetries also belong to the surface tension term.

The sole additional equation is

$$\gamma_t = g_z\alpha^1 + g_x\beta\cos\theta - g_x r\alpha^2\sin\theta + (C_{12}-C_{13})g_z z + $$
$$+ (C_{12}-C_{13})g_x r\cos\theta . \qquad (17)$$

After substitution of α^1, α^2 and β from Eqs. 16 into Eq. 17, the following system of ordinary differential equations result:

$$C^o_{tt} = -C_{10}g_z - \frac{3C_{12}-C_{13}}{2} g_x , \qquad (18)$$
$$S^o_{tt} = -C_{11}g_z + C_2 g_x ,$$
$$a^o_{tt} = \frac{3C_{12}-C_{13}}{2} g_z - C_{10}g_x ,$$
$$\gamma^3_t = a^o(t)g_z - C^o(t)g_x ,$$

which can be integrated both when the gs are constants or known functions of t.

In order to keep the analysis of the results within reasonable bounds we will restrict ourselves to the case

$$g_x = 0 , \; g_z = g \quad , \quad g \text{ time independent} ,$$

this furnishes the following expressions for the components of v:

$$\alpha^1(z,\theta,t,r) = C_1 + C_9 t + \frac{3C_{12}-C_{13}}{2} gt^2 + \frac{C_{12}+C_{13}}{2} z + $$
$$+ \{C_{10}\cos\theta + C_{11}\sin\theta\}r \quad , \qquad (19)$$

$$\alpha^2(z,\theta,t,r) = C_2 - \{(\frac{C_{10}}{2} gt^2 - C_7 t - C_5)\sin\theta - $$
$$- (\frac{C_{11}}{2} gt^2 - C_8 t - C_6)\cos\theta\}\frac{1}{r} + $$
$$+ \{C_{10}\sin\theta - C_{11}\cos\theta\}\frac{z}{r} ,$$

$$\tau(t) = C_3 + C_{12}t ,$$

$$\beta(z,\theta,t,r) = \{(\frac{C_{10}}{2} gt^2 - C_7 t - C_5)\cos\theta + (\frac{C_{11}}{2} gt^2 - $$
$$- C_8 t - C_6)\sin\theta\} - \{C_{10}\cos\theta + $$
$$+ C_{11}\sin\theta\}z + \frac{C_{12}+C_{13}}{2} r ,$$

$$\gamma(z,\theta,t,r,\phi) = C_4 + C_1 gt + \frac{C_9}{2} gt^2 + \frac{3C_{12}-C_{13}}{2} g^2t^3 + $$
$$+ \frac{3C_{12}-C_{13}}{2} gtz + C_9 z + \{(C_{10}gt-C_7)\cos\theta + $$
$$+ (C_{11}gt-C_8)\sin\theta\}r + C_{13}\phi .$$

4. RESULTING SYMMETRIES

The Lie algebra of infinitesimal symmetries can be obtained by asigning arbitrary values to the integration constants, C_1, C_2, ..., in turn, with the rest equal to zero.

In the case of axial, constant, gravity these symtries are:

1 $C_1 = 1$ $\quad z^* = z + \varepsilon$,
$\phi^* = \phi + \varepsilon gt$,

whereas the other variables remain unchanged. This symmetry represents an axial translation.

2 $C_2 = 1$ $\quad \theta^* = \theta + \varepsilon$,
Rotation around the z axis.

3 $C_3 = 1$ $\quad t^* = t + \varepsilon$,
Time translation.

4 $C_4 = 1$ $\quad \phi^* = \phi + \varepsilon$,
Variation of ϕ level.

5 $C_5 = -1$ $\quad r^*\cos\theta^* = r\cos\theta + \varepsilon$,
$r^*\sin\theta^* = r\sin\theta$,
x - translation.

6 $C_6 = -1$ $\quad r^*\cos\theta^* = r\cos\theta$,
$r^*\sin\theta^* = r\sin\theta + \varepsilon$,
y - translation.

7 $C_7 = -1$ $\quad r^*\cos\theta^* = r\cos\theta + \varepsilon t$,
$r^*\sin\theta^* = r\sin\theta$,
$\phi^* = \phi + \varepsilon r\cos\theta + \frac{1}{2}\varepsilon^2 t$,
x - Galilean boost.

8 $C_8 = -1$ $\quad r^*\cos\theta^* = r\cos\theta$,
$r^*\sin\theta^* = r\sin\theta + \varepsilon t$,
$\phi^* = \phi + \varepsilon r\sin\theta + \frac{1}{2}\varepsilon^2 t$,
y - Galilean boost.

9 $C_9 = 1$ $\quad z^* = z + \varepsilon$,
$\phi^* = \phi + \varepsilon(z + \frac{1}{2} gt^2) + \frac{1}{2}\varepsilon^2 t$,
z - Galilean boost.

10 $C_{10} = 1$ $\quad r^*\cos\theta^* = r\cos\theta\cos\varepsilon + (\frac{1}{2}gt^2 - z)\sin\varepsilon$,
$r^*\sin\theta^* = r\sin\theta$,
$z^* = z\cos\varepsilon + r\cos\theta\sin\varepsilon + \frac{1}{2}gt^2(1-\cos\varepsilon)$,
$\phi^* = \phi + gt\{r\cos\theta\sin\varepsilon + (\frac{1}{2}gt^2 - z)(1-\cos\varepsilon)\}$,
Gravity-compensated rotation around y-axis.

$$dT_1 = gtd\eta + \psi d\eta_z + \eta_z d\psi - \frac{d}{dz}\{\psi d\eta\} .$$

Terms like $d\{\psi d\eta\}/dz$ will have influence only on the right hand side of Eq. 23, through integration by parts, and then can be left aside

$$T_1 = gt\eta + \eta_z\psi . \quad (25)$$

If we take $C_2 = -1$, $C_{j\neq 2} = 0$ or $C_4 = -1$, $C_{j\neq 4} = 0$, we arrive, respectively, to

$$T_2 = \eta_\theta\psi \quad , \; T_4 = \eta .$$

With $C_3 = -1$, $C_{j\neq 3} = 0$ the following expression results:

$$dT_3 = -\psi_t d\eta + \eta_t d\psi .$$

This equation, together with Eqs. 20, indicates that T_3 is the hamiltonian density function the volume integral of which equals the energy, E, of the system.

$$T_3 = \frac{1}{2}\psi\nabla\phi\cdot\vec{n}R - \frac{1}{2}gz\eta + \frac{\sigma}{\rho}R , \quad (26)$$

where $\vec{n}R = \vec{i}_r - (\eta_\theta/\eta)\vec{i}_\theta - \eta_z\vec{i}_z$, (27)

and R was given in connection with Eq. 4.

Taking $C_5 = -1$, $C_{j\neq 5} = 0$

$$dT_5 = d[\psi \frac{\partial}{\eta\partial\theta}\{\eta\sin\theta\}] - \frac{\partial}{\eta\partial\theta}[\psi d\{\eta\sin\theta\}] ,$$

then

$$T_5 = \psi \frac{\partial}{\eta\partial\theta}\{\eta\sin\theta\} \quad (28)$$

and, similarly

$$T_6 = -\psi\frac{\partial}{\eta\partial\theta}\{\eta\cos\theta\} .$$

T_7 to T_9 can be expressed as combinations of T_1, T_5 and T_6 as follows:

$$T_7 = tT_5 \quad , \quad T_8 = tT_6 \quad (29)$$

$$T_9 = tT_1 + \{z - \frac{1}{2}gt^2\}\eta .$$

Concerning T_{10} and T_{11},

$$dT_{10} = (z - \frac{1}{2}gt^2)dT_5 - \eta\cos\theta dT_1$$

$$dT_{11} = (z - \frac{1}{2}gt^2)dT_6 - \eta\sin\theta dT_1 ,$$

from which the following two conserved densities result,

$$T_{10} = (z-\frac{1}{2}gt^2)T_5 + \frac{1}{2}gt\eta^2\cos\theta + \frac{1}{2}\psi(\eta^2)_z\cos\theta , \quad (30)$$

$$T_{11} = (z-\frac{1}{2}gt^2)T_6 + \frac{1}{2}gt\eta^2\sin\theta + \frac{1}{2}\psi(\eta^2)_z\sin\theta .$$

Finally, when $C_{12} = 3/2$, $C_{13} = 1/2$, $C_j = 0$ otherwise, the resulting expressions of dT_{12} is not an exact derivative of a density.

5.3 Physical interpretation of the conserved densities

In order to look at the physical meaning of the conserved densities, the integral in the left hand side of Eq. 24 will be calculated over part of the free surface. To this aim, we introduce the simply-connected control volume, $\mathcal{D}$, sketched in Figure 2. This control volume is bounded by the part, S, of the free surface plus a permeable, fixed in space, control surface, Γ, within the liquid column. Since Γ can be chosen at convenience it is supposed to be normal to the undisturbed cylindrical free surface at the intersection.

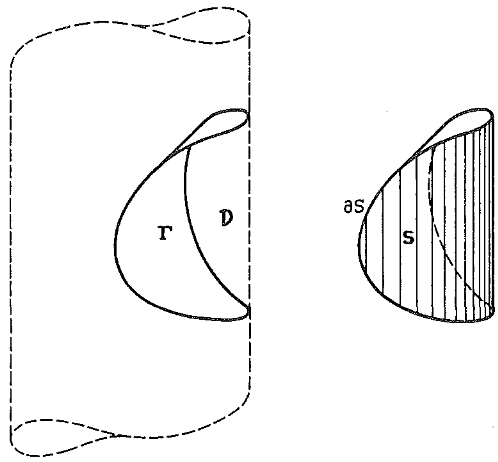

Figure 2. The control volume $\mathcal{D}$ bounded by a portion, S, of the free surface and an arbitrary surface, Γ.

Let a vector $\vec{F}_i = F_i^r\vec{i}_r + F_i^\theta\vec{i}_\theta + F_i^z\vec{i}_z$, be deduced from

$$T_i = F_i^r - F_i^\theta\frac{\eta_\theta}{\eta} - F_i^z\eta_z . \quad (31)$$

By Gauss theorem

$$I_i = \int_{\partial\mathcal{D}} T_i\eta d\theta dz = \int \nabla\cdot\vec{F}_i dV , \quad (32)$$

where the divergence $\nabla\cdot\vec{F}_i$ is given by

$$\nabla\cdot\vec{F}_i = \frac{1}{r}\{\frac{\partial}{\partial r}(rF_i^r) + \frac{\partial}{\partial\theta}F_i^\theta + \frac{\partial}{\partial z}(rF_i^z)\} .$$

The resulting integrals of conserved densities, I_i, are listed in the following

$$I_1 = 2gtV - \int_{\mathcal{D}} V_z dV , \quad (33)$$

z-momentum.

$$I_2 = \int_{\mathcal{D}} rV_\theta dV , \quad (34)$$

z-angular momentum.

$$I_3 = \int_{\mathcal{D}}(\frac{1}{2}q^2 - gz)dV + \frac{\sigma}{\rho}\int_{\partial\mathcal{D}} R\eta d\theta dz , \quad (35)$$

kinetic, potential and surface energy.

$$I_4 = 2V , \qquad \text{mass.} \quad (36)$$

$$I_5 = \int_{\mathcal{D}} V_x dV , \quad I_6 = \int_{\mathcal{D}} V_y dV , \quad (37)$$

x and y momenta

$$I_7 = tI_5 \quad , \quad I_8 = tI_6 \ . \tag{38}$$

$$I_9 = 2\int_{\mathcal{D}} (z + \frac{1}{2} gt^2)dV - t\int_{\mathcal{D}} V_z dV \ . \tag{39}$$

$$I_{10} = \int_{\mathcal{D}} (z - \frac{1}{2}gt^2)V_x dV + \frac{3}{2}gt\int_{\mathcal{D}} xdV - \int_{\mathcal{D}} xV_z dV \ . \tag{40}$$

$$I_{11} = \int_{\mathcal{D}} (z - \frac{1}{2}gt^2)V_y dV + \frac{3}{2}gt\int_{\mathcal{D}} ydV - \int_{\mathcal{D}} yV_z dV \ . \tag{41}$$

The physical interpretations attached to these conserved densities, which are trivial at first, become more complicated in the last five cases, Eqs. 38 to 41. Nevertheless, some information can be deduced from the corresponding symmetries listed in §4. For instance, whereas Eqs. 37 come from time-independent x or y translations, Eqs. 38 concern a fluid domain observed in a frame of reference being translated with constant velocity in x or y directions. On the other hand, I_{10} and I_{11}, Eqs. 40 and 41, can be easily interpreted when g = 0 as representing y and x angular momenta, respectively. Clearly these angular momenta are observed in axes accelerating vertically to cancel gravity.

6. THE CONSERVATION LAWS

The last step in the search for conservation laws for liquid bridges will consist in calculating time derivatives of the integrals I_i. This will bring to light the dependence of the conserved densities on the fluid behavior at the boundaries of the control volume.

Let T_i be a conserved density not explicitly including the variable t, as is the case for T_2 to T_6. Differentiation under the volume-integral sign in Eq. 32, remembering that Γ is fixed in space whereas S is not, yields

$$\frac{dI_i}{dt} = \int_{\mathcal{D}} \nabla\cdot\vec{F}_{it} dV + \int_S \nabla\cdot\vec{F}_i(\nabla\phi\cdot\vec{n})R\eta d\theta dz \ . \tag{42}$$

Application of Gauss theorem to the first term in the right hand side of Eq. 42 results in:

$$\frac{dI}{dt} = \int_S (\vec{F}_{it} + \nabla\cdot\vec{F}_i\nabla\phi)\cdot\vec{n}R\eta d\theta dz + \int_\Gamma \vec{F}_{it}\cdot\vec{n}Rrd\theta dz \ . \tag{43}$$

Development of the cases i = 4 and i = 1, the final results of which are well known (conservation of mass and of z-momentum, respectively), could guide in the derivation of the different conservation laws.

When i = 4, Eq. 31 gives $\vec{F}_4 = \eta\vec{i}_r$ and $\nabla\cdot\vec{F}_4 = 2$.

On the other hand, $\vec{F}_{4t} = 0$ everywhere except on S where, according to Eq. 2,

$$\vec{F}_{4t} = (\nabla\phi\cdot\vec{n}R)\vec{i}_r \ .$$

Thus, Eq. 42, together with Gauss theorem, yields

$$\frac{dI_4}{dt} = -2\int_\Gamma \nabla\phi\cdot\vec{n}Rrd\theta dz \ , \tag{44}$$

which merely indicates that the volume changes in $\mathcal{D}$ are due to the fluid flowing through surface Γ.

The case i = 1 is a bit more complicated but it illustrates how to proceed in other cases.

Let us consider the combination $T_1 - gtT_4$. Equation 43 with $\vec{F}_1 = -\psi\vec{i}_z$, $\nabla\cdot\vec{F}_1 = -\psi_z$ becomes,

$$\frac{dI_1}{dt} - gI_4 - gt\frac{dI_4}{dt} =$$

$$= -\int_S (\psi_t\vec{i}_z + \psi_z\nabla\phi)\cdot\vec{n}R\eta d\theta dz - \int_\Gamma \psi_t\cdot\vec{n}Rrd\theta dz \ . \tag{45}$$

Bernoulli's equation reads

$$\psi_t = -\frac{1}{2}|\nabla\phi|^2 + gz - \frac{\sigma}{\rho}K \quad \text{on } S \tag{46}$$

and

$$\psi_t = -\frac{1}{2}|\nabla\phi|^2 + gz - \frac{p}{\rho} \quad \text{on } \Gamma \ .$$

The first of Eqs. 46 is again Eq. 3, with $g_x = 0$, rewritten in a form more convenient for present purposes.

Bringing Eqs. 46 to Eq. 45 and applying Gauss theorem to the solenoidal vector

$$\frac{1}{2}|\nabla\phi|^2\vec{i}_z - \phi_z\nabla\phi$$

yields

$$\frac{dI_1}{dt} - gI_4 - gt\frac{dI_4}{dt} = \int_\Gamma \phi_z\nabla\phi\cdot\vec{n}Rrd\theta dz -$$

$$- g\int_{\partial\mathcal{D}} z\vec{i}_z\cdot\vec{n}Rrd\theta dz + \frac{\sigma}{\rho}\int_S K\vec{i}_z\cdot\vec{n}R\eta d\theta dz +$$

$$+ \frac{1}{\rho}\int_\Gamma p\vec{i}_z\cdot\vec{n}Rrd\theta dz \ .$$

The first and last terms in the right hand side of Eq. 47 are, respectively, the z-momentum flux and z component of pressure forces on Γ . Application of Gauss theorem to the vector $z\vec{i}_z$ indicates that the second term is equal to $-gI_4/2$, partially cancelling a similar term in the left hand side. Finally, the third term is the z-component of the capillary pressure on S, which can be expressed as a line integral along the intersection of S and Γ by application of Green's formula (Ref. 7).

$$\int_S K\vec{i}_z\cdot\vec{n}R\eta d\theta dz = \int_{\partial S} Pd\theta + Qdz \ , \tag{48}$$

where P and Q are deduced from

$$\frac{\partial P}{\partial z} - \frac{\partial Q}{\partial \theta} = K\eta_z\eta \ .$$

Taking the mean curvature, K, from Eq. 4,

$$P = \frac{\eta^2 + \eta_\theta^2}{\eta R} \ , \quad Q = \frac{\eta_z\eta_\theta}{\eta R} \ . \tag{49}$$

The line integral in Eq. 47 should be taken such that the interior of the surface S lies on the left when moving along ∂S.

The other conservation laws follow in a completely similar way.

7. CONCLUSIONS

Integral Conservation Laws for surface disturbances in inviscid liquid bridges have been systematically obtained. Although the aim of the present exercise has been mastering the mathematical routine through the obtention of physically palatable results, the same technique can be used in more complicated cases, for instance, when the gravity field, although still parallel, is no longer time independent in both intensity and direction.

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APPENDIX I

Derivation of Eqs. 20

Let E_k, V_g and V_σ be respectively the kinetic, potential and surface energy of a fluid volume, D_η, limited by a free surface η. The total energy of the system, E, will be:

$$E = E_k + V_g + V_\sigma$$

1. Variation of the kinetic energy, E_k.

$$E_k = \frac{1}{2}\int_{D_\eta} |\nabla\phi|^2 dV ,$$

$$\delta E_k = \int_{D_\eta} \nabla\delta\phi\cdot\nabla\phi dV + \frac{1}{2}\int_\eta |\nabla\phi|^2 \delta\eta\ \eta d\theta dz .$$

Application of Gauss theorem to the first integral in the right hand side yields

$$\delta E_k = \int_\eta (\delta\phi \frac{\partial\phi}{\partial n} R + \frac{1}{2}|\nabla\phi|^2 \delta\eta)\eta d\theta dz .$$

Being ψ the velocity potential, ϕ, particularized on η,

$$\delta\psi = \delta\phi + \frac{\partial\phi}{\partial r}\delta\eta ,$$

then

$$\delta E_k = \int_\eta [\frac{\partial\phi}{\partial n} R\delta\psi + (\frac{1}{2}|\nabla\phi|^2 - \frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial n} R)\delta\eta]\eta d\theta dz .$$

and, finally

$$\frac{\delta E_k}{\delta\psi} = \frac{\partial\phi}{\partial n} R ,$$

$$\frac{\delta E_k}{\delta\eta} = \frac{1}{2}|\nabla\phi|^2 - \frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial n} R .$$

2. Variation of the potential energy, V_g.

$$V_g = - g_z\int_{D_\eta} dV - g_x\int_D dV .$$

Following with the time-independent vector

$$- g_z z\vec{i}_z - g_x x\vec{i}_x$$

the same steps as with $\phi\nabla\phi$ above, we reach

$$\frac{\delta V_g}{\delta\eta} = - g_z z - g_x x$$

3. Variation of the surface energy, V_σ.

$$V_\sigma = \frac{\sigma}{\rho}\int_\eta R\eta d\theta dz .$$

Now the relevant vector is $\vec{n}$, which on the free surface, η, is related to the mean curvature, K, through $\nabla\cdot\vec{n} = K$ (Eq. 4). Then

$$\frac{\delta V_\sigma}{\delta\eta} = \frac{\sigma}{\rho} K$$

Adding up the several contributions to the total energy

$$\frac{\delta E}{\delta\psi} = \frac{\partial\phi}{\partial n} R$$

$$\frac{\delta E}{\delta\eta} = \frac{1}{2}|\nabla\phi|^2 - \frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial n} R - g_z z - g_x x + \frac{\sigma}{\rho} K .$$

On the other hand, Eq. 2 indicates that

$$\frac{\partial\phi}{\partial n} R = \eta_t ,$$

whereas Eq. 3,

$$\frac{1}{2}|\nabla\phi|^2 - \frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial n} R - g_z z - g_x x + \frac{\sigma}{\rho} K = - \frac{\partial\phi}{\partial t} - \frac{\partial\phi}{\partial r}\eta_t ,$$

from which Eqs. 20 follow.

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