Asymptotically Toeplitz Hessenberg operators and the Riemann mapping

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OPSFA 2011

Outline of talk

Preliminaries

Let μ be a **positive measure in $\mathbb{C}$ with compact support Ω**. Let $\mathcal{P}$ be the space of polynomials.

1. The **hermitian moment matrix** $M = (c_{jk})_{j,k=0}^{\infty}$ given by

$$c_{jk} = \int_{\Omega} z^j \overline{z}^k d\mu, \quad j, k \in \mathbb{Z}_+$$

   is the matrix of the inner product in the canonical basis.
2. There exists a unique orthonormal polynomials sequence (ONPS) $\{P_n(z)\}_{n=0}^{\infty}$ associated with the measure μ.
3. D is the infinite **upper Hessenberg matrix** of the multiplication by z operator in the basis of ONPS in $\mathcal{P}^2(\mu)$ (the closure of $\mathcal{P}$).
4. D is the natural generalization to the hermitian case of Jacobi matrix.

Preliminaries

In this work we consider:

1. **An analytic Jordan curve** Γ on the complex plane.
2. **A measure** $\mu \in \mathbb{S}(\Gamma)$ **belonging to the Szegö class for** Γ, i.e.

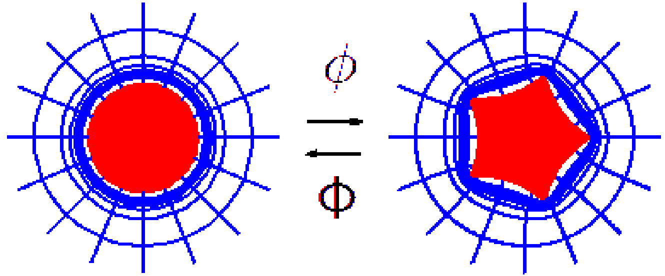
$$\mathrm{supp}(\mu) = \Gamma \text{ and } \int_\Gamma \log \mu'(z)\, |\Phi'(z)||dz| > -\infty.$$

3. **The upper Hessenberg matrix** D associated with μ.
4. **A Toeplitz matrix** $T = (c_{j,k} = c_{k-j})_{j,k=0}^{\infty}$ and its **symbol** f given by the Laurent series $f(e^{i\theta}) = \sum_{j,k=0}^{\infty} c_{k-j} e^{i(k-j)\theta}$.
5. **A weakly asymptotically Toeplitz operator** A on ℓ^2. Definition [Feintuch 1989, Barría-Halmos 1982]: if there exists a Toeplitz bounded operator T on ℓ^2 such that

$$\lim_{n\to\infty} \langle S_L^n A S_R^n u, v\rangle = \langle Tu, v\rangle, \text{ for every } u, v \in \updownarrow^2.$$

Riemann mapping theorem

Riemann mapping theorem states that there is a unique conformal map $\phi : \mathbb{C}_\infty \setminus \overline{\mathbb{D}} \to \mathbb{C}_\infty \setminus \Omega$ (where $\mathbb{D}$ the unit disk and Ω is a compact, which is not a point with $\mathbb{C}_\infty \setminus \Omega$ simply connected) such that $\phi(\infty) = \infty$ and $\phi'(\infty) = \text{cap}(\Omega)$. If $\Gamma = \partial\Omega$ is a Jordan curve, ϕ is continuous in $\mathbb{T}$. We denote by Φ the inverse mapping ϕ^{-1}.



Introduction

Rakhmanov's theorem [Rakhmanov 1977, 1983]. Let μ be a Borel measure with $\mathrm{supp}(\mu) = [-1,1]$ and $\mu' > 0$ almost everywhere in $[-1,1]$, then

$$a_n \to \frac{1}{2} \text{ and } b_n \to 0, \text{ where } J = \begin{pmatrix} b_0 & a_1 & 0 & \dots \\ a_1 & b_1 & a_2 & \dots \\ 0 & a_2 & b_2 & \dots \\ 0 & 0 & a_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that in this case

$$\phi(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}z + 0 + \frac{1}{2}\frac{1}{z}.$$

Introduction

Under the conditions of Rakhmanov's theorem but with

$$\mathrm{supp}(\mu) = [a, b],$$

then the above limits are

$$a_n \to \frac{b-a}{4} \quad \text{and} \quad b_n \to \frac{a+b}{2}.$$

In this case

$$\phi(z) = \frac{b-a}{4} z + \frac{a+b}{2} + \frac{b-a}{4} \frac{1}{z}.$$

Introduction

In this work we show that:

the **upper Hessenberg matrix** D associated with the **measure** $\mu \in \mathbb{S}(\Gamma)$ is **weakly asymptotically Toeplitz** to a certain **Toeplitz matrix** T such that its **symbol** agrees with the restriction to the unit circle (note that it exits by Caratheodory Theorem) of the **Riemann mapping** ϕ, which applies conformally the exterior of the unit disk in the exterior of Γ, the support of the measure μ, whenever Γ is an **analytic Jordan curve**.

Unit circle $\mathbb{T}$

In the unit circle we have the Hessenberg matrix

$$D_n = \begin{pmatrix} -\Phi_1\overline{\Phi}_0 & \frac{-\kappa_0}{\kappa_1}\Phi_2\overline{\Phi}_0 & \frac{-\kappa_0}{\kappa_2}\Phi_3\overline{\Phi}_0 & \cdots & \frac{-\kappa_0}{\kappa_{n-2}}\Phi_{n-1}\overline{\Phi}_0 & \frac{-\kappa_0}{\kappa_{n-1}}\Phi_n\overline{\Phi}_0 \\ \frac{\kappa_0}{\kappa_1} & -\Phi_2\overline{\Phi}_1 & \frac{-\kappa_1}{\kappa_2}\Phi_3\overline{\Phi}_1 & \cdots & \frac{-\kappa_1}{\kappa_{n-2}}\Phi_{n-1}\overline{\Phi}_1 & \frac{-\kappa_1}{\kappa_{n-1}}\Phi_n\overline{\Phi}_1 \\ 0 & \frac{\kappa_1}{\kappa_2} & -\Phi_3\overline{\Phi}_2 & \cdots & \frac{-\kappa_2}{\kappa_{n-2}}\Phi_{n-1}\overline{\Phi}_2 & \frac{-\kappa_2}{\kappa_{n-1}}\Phi_n\overline{\Phi}_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\Phi_{n-1}\overline{\Phi}_{n-2} & \frac{-\kappa_{n-2}}{\kappa_{n-1}}\Phi_n\overline{\Phi}_{n-2} \\ 0 & 0 & 0 & \cdots & \frac{\kappa_{n-2}}{\kappa_{n-1}} & -\Phi_n\overline{\Phi}_{n-1} \end{pmatrix}$$

Geronimous (1971): $\mu \in \mathbb{S}(\mathbb{T}) \Leftrightarrow \sum_{n=0}^{\infty} |\Phi_n(\mu, 0)|^2 < +\infty$.
Then,

$$d_1 = \lim_{n\to\infty} d_{n+1,n} = \lim_{n\to\infty} \frac{\kappa_{n-1}}{\kappa_n} = 1, d_{-k} = \lim_{n\to\infty} d_{n,n+k} = 0, k = 0, 1, 2, \ldots$$

Note that these limits agree with the Laurent coefficients of the Riemann mapping function

$$\phi(z) = z.$$

The diagonals theorem for analytic Jordan curves

Theorem (The diagonals theorem for analytic Jordan curves, EGST2011)

Let Γ be an analytic Jordan curve on the complex plane and let μ be a measure belonging to the Szegö class for Γ.
Let $\phi : \overline{\mathbb{C}} \setminus \mathbb{D} \to \overline{\mathbb{C}} \setminus \Omega \, (\Gamma = \partial\Omega)$ be the Riemann mapping

$$\phi(z) = c_1 z + c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \dots .$$

Denote by $D = (d_{i,j})_{i,j=1}^{\infty}$ the upper Hessenberg matrix associated with the measure μ. Then, D is weakly asymptotically Toeplitz, with symbol the Riemann mapping ϕ restricted to the unit circle $\mathbb{T}$, i.e.,

$$\lim_{n\to\infty} d_{n-j,n} = c_{-j} \qquad \text{for all } j = -1, 0, 1, 2, \dots$$

Notation for the Proof

$$D = \begin{pmatrix} d_{0,0} & d_{0,1} & d_{0,2} & \dots \\ d_{1,0} & d_{1,1} & d_{1,2} & \dots \\ 0 & d_{2,1} & d_{2,2} & \dots \\ 0 & 0 & d_{3,2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad T = \begin{pmatrix} d_0 & d_{-1} & d_{-2} & \dots \\ d_1 & d_0 & d_{-1} & \dots \\ 0 & d_1 & d_0 & \dots \\ 0 & 0 & d_1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- $\phi(z) = c_1 z + c_0 + c_{-1}\frac{1}{z} + c_{-2}\frac{1}{z^2} + \cdots = \sum_{k=-1}^{\infty} c_{-k} z^{-k}$ is the Riemann function $\phi : \mathbb{C}_\infty \setminus \overline{\mathbb{D}} \to \mathbb{C}_\infty \setminus \Omega$ where $\Gamma = \partial\Omega = \mathrm{supp}(\mu)$
- $\Phi = \phi^{-1}$
- $d(z) = d_1 z + d_0 + d_{-1}\frac{1}{z} + d_{-2}\frac{1}{z^2} + \cdots = \sum_{k=-1}^{\infty} d_{-k} z^{-k}$ is the symbol of T
- L is the length of Γ.

Sketch of the Proof

- From the recurrence formula for ONPS $\widehat{P}_n$, we have

$$\frac{1}{L}\int_\Gamma z\widehat{P}_n(z)\overline{\widehat{P}_{n-j}(z)}v(z)|dz| = d_{n+1-j,n+1},\ j = -1, 0, 1, 2, \dots, n-1$$

- Using Szegö theorem [Szegö 1939]

$$\widehat{P}_n(z) = \left(\frac{L}{2\pi}\right)^{1/2}[\Delta_e(z)]^{-1}[\Phi'(z)]^{1/2}[\Phi(z)]^n + \mathcal{O}(h^n), 0 < h < 1,$$

in the above formula we obtain four summands, three of them go to zero.

- Using ϕ to take the measure from Γ to the unit circle, we have

$$\frac{1}{2\pi}\int_\Gamma z\,|\Delta_e(z)|^{-2}|\Phi'(z)|\Phi^n(z)\overline{\Phi}^{n-j}(z)\ v(z)|dz| = c_{-j}$$

Remark: Uniformly asymptotically Toeplitz

Barría-Halmos (1982):

A bounded operator A on ℓ^2 is uniformly asymptotically Toeplitz provided there is a (necessarily bounded and Toeplitz) operator T on ℓ^2 such that

$$\lim_{n\to\infty} ||S_L^n A S_R^n - T|| = 0$$

Feintuch's theorem (1989):

A is uniformly asymptotically Toeplitz $\Leftrightarrow$ A is Toeplitz + compact.

Note that, in the hypothesis of the theorem, whenever $D - T$ is a compact operator, we have that the convergence is uniform.
In particular, this is the case of matrices D with a finite number of non null diagonals.

Example I: Ellipse

- Consider the ellipse

$$z = [a \cos t, b \sin t].$$

  Let $c^2 = a^2 - b^2$ be the ellipse's focal distance.

- Consider the weight function

$$w(z, \tau) = |z^2 - c^2|^{\tau - 1/2} \text{ with } \tau \in [0, 1].$$

We will consider two cases

- $\tau = 1$ (Duren,1963).
- $\tau = 0$ (Walsh, 1934).

Example I: $\tau = 1$

(Duren, 1963): The monic orthogonal polynomials are analogous to the Tchebyschev polynomials of second kind on the interval $[-c, c]$, but with different norms:

$$\|\widetilde{P}_n(z;0)\|^2 = 2\pi \left[(\frac{a+b}{2})^{2(n+1)} + (\frac{a-b}{2})^{2(n+1)} \right], \quad n = 0, 1, 2, \ldots$$

The norms of these polynomials and the recurrence relation allow us to obtain the Hessenberg matrix D

$$\begin{pmatrix} 0 & \frac{a^2-b^2}{2}\sqrt{\frac{(a+b)^2+(a-b)^2}{(a+b)^4+(a-b)^4}} & 0 & 0 & \cdots \\ \frac{1}{2}\sqrt{\frac{(a+b)^4+(a-b)^4}{(a+b)^2+(a-b)^2}} & 0 & \frac{a^2-b^2}{2}\sqrt{\frac{(a+b)^4+(a-b)^4}{(a+b)^6+(a-b)^6}} & 0 & \cdots \\ 0 & \frac{1}{2}\sqrt{\frac{(a+b)^6+(a-b)^6}{(a+b)^4+(a-b)^4}} & 0 & \frac{a^2-b^2}{2}\sqrt{\frac{(a+b)^6+(a-b)^6}{(a+b)^8+(a-b)^8}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example I: $\tau = 0$

(Walsh, 1934): The monic orthogonal polynomials are analogous to the Tchebyschev polynomials of first kind on the interval $[-c, c]$, but with different norms:

$$\|\widetilde{P}_n(z;1)\|^2 = 2\pi\left[\left(\frac{a+b}{2}\right)^{2n} + \left(\frac{a-b}{2}\right)^{2n}\right], \quad n = 1, 2, \ldots$$

$$\|\widetilde{P}_0(z;1)\|^2 = 2\pi a.$$

The norms of these polynomials and the recurrence relation allow us to obtain the Hessenberg matrix D

$$\begin{pmatrix} 0 & \frac{a^2-b^2}{\sqrt{(a+b)^2+(a-b)^2}} & 0 & 0 & \cdots \\ \frac{\sqrt{(a+b)^2+(a-b)^2}}{2} & 0 & \frac{a^2-b^2}{2}\sqrt{\frac{(a+b)^4+(a-b)^4}{(a+b)^6+(a-b)^6}} & 0 & \cdots \\ 0 & \frac{1}{2}\sqrt{\frac{(a+b)^6+(a-b)^6}{(a+b)^4+(a-b)^4}} & 0 & \frac{a^2-b^2}{2}\sqrt{\frac{(a+b)^6+(a-b)^6}{(a+b)^8+(a-b)^8}} & \cdots \\ 0 & 0 & \frac{1}{2}\sqrt{\frac{(a+b)^8+(a-b)^8}{(a+b)^6+(a-b)^6}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Remark: Ellipse

For the ellipse

$$z = [a\cos t, b\sin t],$$

the Riemann mapping is given by

$$\phi(z) = \frac{a+b}{2}z + \frac{a-b}{2}\frac{1}{z}$$

Note that in the two cases studied above

$$\begin{aligned}
\lim_{n\to\infty} d_{n,n+1} &= \frac{a^2-b^2}{2}\frac{1}{\sqrt{(a+b)^2}} = \frac{a-b}{2}, \\
\lim_{n\to\infty} d_{n+1,n} &= \frac{1}{2}\sqrt{(a+b)^2} = \frac{a+b}{2}, \\
\lim_{n\to\infty} d_{n,n+k} &= 0 \quad \text{for every} \quad k \neq -1, 1,
\end{aligned}$$

as the theorem asserts.

Moreover, since $D - T$ is compact the convergence is uniform.

Remark: Degenerate cases of ellipse

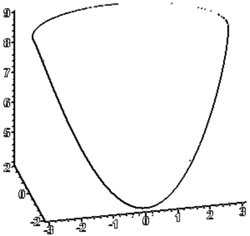
- For $a = b$ we have a circle of radius a. In both cases, D_τ is the shift right multiply by a.
- For $b = 0$ we have the interval $[-a, a]$ and we have the following Hessenberg matrices

$$D_1 = \begin{pmatrix} 0 & \frac{a}{2} & 0 & 0 & \cdots \\ \frac{a}{2} & 0 & \frac{a}{2} & 0 & \cdots \\ 0 & \frac{a}{2} & 0 & \frac{a}{2} & \cdots \\ 0 & 0 & \frac{a}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, D_0 = \begin{pmatrix} 0 & \frac{a}{\sqrt{2}} & 0 & 0 & \cdots \\ \frac{a}{\sqrt{2}} & 0 & \frac{a}{2} & 0 & \cdots \\ 0 & \frac{a}{2} & 0 & \frac{a}{2} & \cdots \\ 0 & 0 & \frac{a}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

which are respectively, the Jacobi matrices for the Tchebyschev polynomials of second and the first kind on the interval $[-a, a]$. Note that in this case the above theorem can not applied.

Example: Ellipse with a non tridiagonal Hessenberg matrix

We consider the ellipse $z = [a\cos t, b\sin t]$ with the weight function



$$w(z) = \begin{cases} a^2 & t \in [0,\pi] \\ a^2\cos(t)^2 + b^2\sin(t)^2 & t \in [\pi, 2\pi] \end{cases}$$

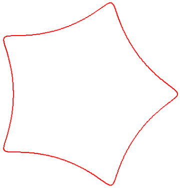
For $a = 3, b = 2$, the Riemann mapping function is $\phi(z) = \frac{5}{2}z + \frac{1}{2}\frac{1}{z}$.

By numerical computation we have the next section of D

$$\begin{pmatrix} 0.40420303\, i & 0.7689189591 & -0.1079262724\, i & -0.009644390418 & -0.01286235839\, i & -0.002008602409 \\ 2.457437865 & 0.045995046\, i & 0.5291737642 & -0.00920619945\, i & 0.001872767863 & -0.001869959504\, i \\ 0.0 & 2.485334997 & 0.01209088066\, i & 0.5024826317 & 0.0008939829463\, i & 0.0005005965148 \\ 0.0 & 0.0 & 2.496997905 & 0.001173415825\, i & 0.4995460938 & 0.0002345682020\, i \\ 0.0 & 0.0 & 0.0 & 2.499730018 & 0.0000299448513\, i & 0.4999222186 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.499941936 & 0.0000109476858\, i \end{pmatrix}$$

Example: Hypocycloid

We consider Γ as a the following hypocicloid



$$\Gamma = \phi(e^{i\theta}) = e^{i\theta} + \frac{1}{6}\frac{1}{e^{4i\theta}}.$$

which is an analytic Jordan curve.

The Riemann mapping is

$$\phi_\Gamma(z) = z + \frac{1}{6}\frac{1}{z^4}, \quad |z| > 1.$$

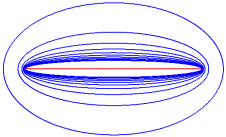
For every $\mu \in \mathbb{S}(\Gamma)$ we can apply the diagonals theorem, then

$$\lim_{n\to\infty} d_{n+1,n} = 1, \; \lim_{n\to\infty} d_{n+4,n} = \frac{1}{6} \; \lim_{n\to\infty} d_{n-k,n} = 0, \; \forall k \neq -1, 4.$$

Non analytic Jordan curves and rectifiable arcs

In the interval we can not apply our theorem.

The interval $[-1,1]$ is a degenerate case of ellipse. The image of a circle of radius $r>1$ by the Riemann mapping ϕ for the interval is the ellipse



$$\phi(rz)=\frac{1}{2}(rz+\frac{1}{rz}),\quad |z|=1$$

Applying the theorem

$$\lim_{n\to\infty} d_{n,n+1}=\frac{1}{2r},\ \lim_{n\to\infty} d_{n+1,n}=\frac{r}{2},\ \lim_{n\to\infty} d_{n,n+k}=0,\ \forall k\neq -1,1$$

Taking $r=1$ we obtain that the limits agree with the Laurent coefficients of the Riemann mapping ϕ.

Lemma for non analytic Jordan curves and rectifiable arcs

Let Γ be a rectifiable Jordan arc or a connected finite union of Jordan arcs, such that $\mathbb{C}_\infty \setminus \Gamma$ is a simply connected set of the Riemann sphere $\mathbb{C}_\infty$. Let $\phi(z) = c_1 z + c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \dots$ be the Riemann mapping for Γ.

The following result is trivial by uniqueness of the Riemann mapping

Lemma

Let $\phi(z)$ be the Riemann mapping from the complement of the unit disk to the complement of the Jordan curve or arc Γ, such that $\phi(\infty) = \infty$ with $\phi'(\infty) > 0$. If $r > 1$, $\Gamma_r = \phi(r\,\mathbb{T}) = \{rz \mid z \in \mathbb{T}\}$ is an analytic Jordan curve (Walsh 1969). Then the Riemann mapping $\phi_r(z)$ for Γ_r satisfies $\phi_r(z) = \phi(rz)$.

Lemma for non analytic Jordan curves and rectifiable arcs

In the above conditions:

- Γ a rectifiable Jordan curve or arc.
- $\phi(z)$ the Riemann from the complement of the unit disk to the complement of Γ.
- $\Gamma_r = \phi(r\,\mathbb{T}) = \{rz | z \in \mathbb{T}\}$ with $r > 1$

Corollary

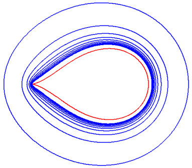
Let $\mu \in \mathbb{S}(\Gamma_r)$. Denote by $\{\widehat{P}_n(r,z)\}$ the ONPS associated with Γ_r and mu, and let $D(r) = (d_{i,j}(r))_{i,j=1}^{\infty}$ be the Hessenberg matrix associated to the multiplication by z operator on P^2_μ.
Then

$$\lim_n d_{n-j,n}(r) = -c_j\, r^{-j}, \quad j = -1, 0, 1, 2, \ldots$$

Example: Drop-like set

We take Γ as a drop-like set of parametric equation $z(t) = \frac{(e^{it})^2}{1+2e^{it}}$, which is not an analytic Jordan curve and its Riemann mapping is

$$\phi_\Gamma(z) = \sum_{k=-1}^{\infty} (-1)^{k+1} \frac{1}{2^{k+2}} z^{-k}, \ \ |z| > 1.$$

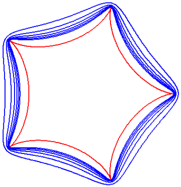


Now, $\Gamma_r = \phi_\Gamma(r\,\mathbb{T})$ is an analytic Jordan curve $\forall r > 1$. For every $\mu \in \mathbb{S}(\Gamma_r)$ we can apply the diagonals theorem, then

$$\lim_{n\to\infty} d_{n-k,n}(r) = (-1)^{k+1}\frac{1}{2^{k+2}} r^{-k}.$$

Example: Hypocycloid

We take Γ as a hypocicloid of parametric equation $z(t) = e^{it} + \frac{1}{4}\frac{1}{(e^{it})^4}$, which is not an analytic Jordan curve and its Riemann mapping is



$$\phi_\Gamma(z) = z + \frac{1}{4}\frac{1}{z^4}, \quad |z| > 1.$$

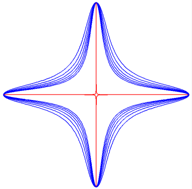
Now, $\Gamma_r = \phi_\Gamma(r\,\mathbb{T})$ is an analytic Jordan curve $\forall r > 1$. For every $\mu \in \mathbb{S}(\Gamma_r)$ we can apply the diagonals theorem, then

$$\lim_{n\to\infty} d_{n-1,n}(r) = r, \quad \lim_{n\to\infty} d_{n+4,n}(r) = \frac{1}{4}r^{-4} \quad \lim_{n\to\infty} d_{n-k,n}(r) = 0, \forall k \neq -1,$$

Example: Cross-like set

We take Γ as a cross-like set formed by the intervals $[-a, a]$ y $[-bi, bi]$ such that its Riemann mapping is

$$\phi(z) = \frac{\sqrt{a^2(z^2+1)^2 + b^2(z^2-1)^2}}{2z}, |z| > 1.$$



In the particular case of $a = b = 1$,

$$\begin{aligned} \phi(z) &= \frac{\sqrt{2}}{2z}\sqrt{z^4+1} = \\ & \frac{1}{2}\sqrt{2}z + \frac{1}{4}\frac{\sqrt{2}}{z^3} - \frac{1}{16}\frac{\sqrt{2}}{z^7} + \frac{1}{32}\frac{\sqrt{2}}{z^{11}} - \frac{5}{256}\frac{\sqrt{2}}{z^{15}} + \frac{7}{512}\frac{\sqrt{2}}{z^{19}} + \mathrm{O}\left(\frac{1}{z^{23}}\right) \end{aligned}$$

The diagonals Theorem can be applied when we consider a measure μ in the Szegö class for Γ_r for every $r > 1$.

The diagonals theorem for rectifiable Jordan arcs

Theorem (The diagonals theorem for rectifiable Jordan arcs EGST2010)

Let $D = (d_{ij})_{i,j=1}^{\infty}$ be a Hessenberg matrix associated with a measure μ with compact support on the complex plane. Assume that:

1. *The measure μ is regular with support $\operatorname{supp}(\mu)$ a Jordan arc or a connected finite union of Jordan arcs Γ such that $\mathbb{C} \setminus \Gamma$ is a simply connected set of the Riemann sphere $\mathbb{C}_\infty$.*
2. *There exists a Hessenberg-Toeplitz matrix T such that $D - T$ defines a compact operator in ℓ^2 with its rows in ℓ^1.*

Then, the symbol of T is the Riemann function

$$\phi : \mathbb{C}_\infty \setminus \overline{\mathbb{D}} \to \mathbb{C}_\infty \setminus \Gamma.$$

Example: Arcs of the unit circle $\mathbb{T}$

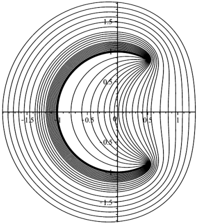
[Golinskii-Nevai-Van Assche, 1995] $\exists\mu$ regular measure on Γ, an arc of the unit circle $\mathbb{T}$, for which $\tilde{P}_0(0) = 1$ and $\tilde{P}_n(0) = \dfrac{1}{a}$ $(a > 1)$, if $n \geq 1$, and the Hessenberg matrix D is

$$\begin{pmatrix} -\frac{1}{a} & -\frac{(a^2-1)^{1/2}}{a^2} & -\frac{(a^2-1)^{2/2}}{a^3} & -\frac{(a^2-1)^{3/2}}{a^4} & -\frac{(a^2-1)^{4/2}}{a^5} & \cdots \\ \frac{(a^2-1)^{1/2}}{a} & -\frac{1}{a^2} & -\frac{(a^2-1)^{1/2}}{a^3} & -\frac{(a^2-1)^{2/2}}{a^4} & -\frac{(a^2-1)^{3/2}}{a^5} & \cdots \\ 0 & \frac{(a^2-1)^{1/2}}{a} & -\frac{1}{a^2} & -\frac{(a^2-1)^{1/2}}{a^3} & -\frac{(a^2-1)^{2/2}}{a^4} & \cdots \\ 0 & 0 & \frac{(a^2-1)^{1/2}}{a} & -\frac{1}{a^2} & \frac{(a^2-1)^{1/2}}{a^3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that $D - T = \begin{pmatrix} -\frac{a-1}{a^2} & -\frac{(a^2-1)^{1/2}(a-1)}{a^3} & -\frac{(a^2-1)(a-1)}{a^4} & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ is compact.

Example: Arcs of the unit circle $\mathbb{T}$

According to DT for arcs, the expression of the Riemann mapping as a Laurent series is



$$\phi(z) = \frac{z\left(a - \sqrt{a^2-1}\,z\right)}{\sqrt{a^2-1} - az}$$

$$= \frac{\sqrt{a^2-1}}{a} z - \frac{1}{a^2} - \frac{\sqrt{a^2-1}}{a^3 z} - O\left(\frac{1}{z^2}\right).$$

Moreover, we can apply DT for analytic Jordan curves in $\Gamma_r = \phi(r\,\mathbb{T})$ for all $\mu \in \mathbb{S}(\Gamma_r)$.