The logarithmic spiral, autoisoptic curve

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ABSTRACT

In the Line of Investigation that in the department of "Technical Drawing" in the School of Agriculture Engineering of Madrid, we carry out on the study of The Technical Curves and his singularities, we demonstrate an interesting property of the Logarithmic Spiral.

The demonstrated property consists of which the logarithmic spiral is a autoisoptic curve, that is to say that if from a point P anyone of the spiral tangent straight lines draw up to the previous arc, these form a constant angle α. This demonstration is novel and in addition we get to contribute a method to calculate the angle α given the equation of the spiral.

1. Introduction

We name Isoptic of a given curve, to the geometric place of the points from which one sees the curve one under constant angle. That is to say: both tangent straight lines we draw to the curve from any point of the isoptic, form always the same angle.

It's immediate to see, for example, that the isoptic of a circumference is another circumference. When the angle is a right angle the curve names ortoptic. We know that the *ortoptic* curve of a conic is a circumference.

It is not difficult to demonstrate that the isoptic curve of two circumferences is a Spiral of Pascal.

¿Are there auto-isoptic curves? In this paper we will see that,

1. The Logarithmic Spiral $\rho = a^{\omega}$ is autoisoptic.

2. We will give a method to calculate the isoptic angle α depending on a.

3. We will determine a for the spiral auto-ortoptic (α = 90º).

2. The logarithmic spiral is autoisoptic

We have A_1 a point of the logarithmic spiral $\rho = a^{\omega}$ (Figure 1), corresponding to the value $\omega = \omega_1$. The tangent in A_1 will form with the vector radio OA_1 a angle *V* given for the next equation

$$tgV = \frac{\rho}{\rho'} = \frac{1}{La} = cte \quad (1)$$

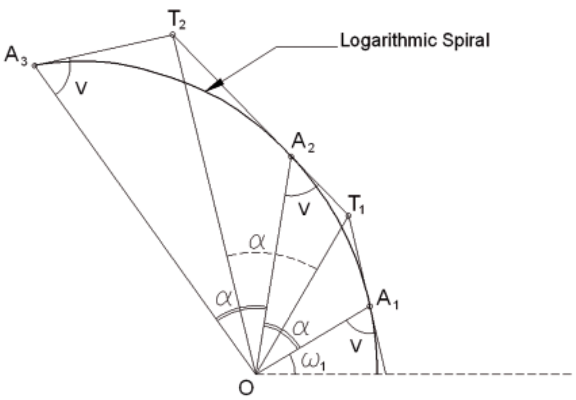

Figure 1. Graphical base for the analytic demonstration of which Logarithmic Spiral is autoisoptic

If α is an given constant angle, and we have A_2 the point in the curve corresponding to $\omega = \omega_1 + \alpha$. Then we will have:

$$OA_2 = a^{\omega 1 + \alpha} = OA_1 \cdot a^{\alpha} \quad (2)$$

And, analogously, the tangent in A_2 will form with the vector radio OA_2 the same angle *V*. T_1 is the point of intersection of the tangents to the spiral in A_1 y A_2

In the point A_3 corresponding to $\omega_1 + \alpha + \alpha$ we'll have:

$$OA_3 = OA_2 \cdot a^{\alpha} \quad (3)$$

And we'll name T_2 to the point of intersection of the tangents in A_2 y A_3.

Is evidente that the quadrangles $OA_1T_1A_2$ and $OA_2T_2A_3$ are similar (both quadrangles have the equal angles and two proportional sides) and the angle $A_1T_1A_2$ is equal to the angle $A_2T_2A_3$, and equal, both, to 180º – α.

But, besides,

$$T_1OT_2 = \alpha \quad (4)$$

and

$$OT_2 = a^{\alpha} \cdot OT_1 \quad (5)$$

That is to say: T_1 y T_2 describe a spiral of the form $\rho = b^{\omega}$ too. The value of b depend, of course, of the value α. Therefore, if we varythe angle α, always there will be some value where T_1, T_2,...will be PRECISELY in $\rho = a^{\omega}$, consequently this curve is AUTOISOPTIC.

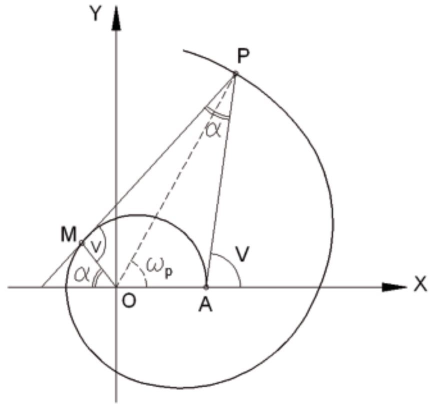

Figure 2. Graphical base for the calculation of the autoisoptic angle of the Logarithmic Spiral.

Let's watch, finally, that when we obtain points of $\rho = a^{\omega}$ varying the angle α, they can be points of the next spiral arc or of another arc further, depending of the variation of the value.

3. Calculation of thr autoisoptic angle of the logarithmic spiral

As we have demonstrated, the tangents from any point of the logarithmic spiral $\rho = a^{\omega}$ to the next spire, form a constant angle (Figure 2).

If $\omega = 0$, $\rho = a^0 = 1$. As the equation (1) is correct, the equation of the tangent to the curve in the point *A(1,0)* is:

$$y = \frac{x-1}{La} \quad (6)$$

Now let's find the point of intersection *P* of this tangent with the next spire:

$$\rho = a^{\omega} \quad (7)$$

$$yLa = x - 1 \quad (8)$$

That is to say, $\rho \cdot sen\omega \cdot La = \rho \cdot cos\omega - 1$

If we replace the value ρ in the previous equation, we willl obtain:

$$a^{\omega} \cdot sen\omega \cdot La = a^{\omega} \cdot cos\omega - 1 \qquad (9)$$

Therefore:

$$a^{\omega} = \frac{1}{cos\omega - La \cdot sen\omega} \qquad (10)$$

As we want the FOLLOWING spire and $cos(\omega+2\pi) = cos\omega$ and $sen(\omega+2\pi) = sen\omega$, the result will be:

$$a^{\omega+2\pi} = \frac{1}{cos\omega - La \cdot sen\omega} \qquad (11)$$

Then:

$$\omega = \frac{L\,\frac{1}{cos\omega - La \cdot sen\omega}}{La} - 2\pi \qquad (12)$$

We can watch that the previous equation is like that $\omega = f(\omega)$. If we try solve it by iteration, that is to say obtaining from an approximate value $\omega = \omega_1$: $\omega_2 = f(\omega_1)$: $\omega_3 = f(\omega_2)$; ... Then it result that the succession ω_1, ω_2, ω_3, ... diverges. In this case we have to write the equation like that: $\omega = f^{-1}(\omega) = \varphi(\omega)$ and now the succession ω_1, $\omega_2 = \varphi(\omega_1)$, $\omega_3 = \varphi(\omega_2)$ ... certainly converges.

If we write:

$$a^{\omega+2\pi} = \frac{\frac{1}{\sqrt{(La)^2+1}}}{\frac{\cos\omega}{\sqrt{(La)^2+1}} - \frac{La \cdot sen\omega}{\sqrt{(La)^2+1}}} = \frac{\frac{1}{\sqrt{(La)^2+1}}}{sent \cdot \cos\omega - \cos t \cdot sen\omega} \qquad (13)$$

Being:

$$sent = \frac{1}{\sqrt{(La)^2+1}} \qquad (14)$$

and

$$cost = \frac{La}{\sqrt{(La)^2+1}} \qquad (15)$$

Then:

$$a^{\omega+2\pi} = \frac{\frac{1}{\sqrt{(La)^2+1}}}{sent(t-\omega)} \qquad (16)$$

And

$$sen(t-\omega) = \frac{1}{a^{\omega+2\pi} \cdot \sqrt{(La)^2+1}} \qquad (17)$$

Therefore

$$\omega = t - arcsen\,\frac{1}{a^{\omega+2\pi} \cdot \sqrt{(La)^2+1}} \qquad (18)$$

If we solve this equation by iteration we obtain ω_p. And $\rho_p = a^{\omega p + 2\pi}$. That is to say:

$$x_p = \rho_p \cdot \cos\omega_p \qquad (19)$$

$$y_p = \rho_p \cdot sen\omega_p \qquad (20)$$

These equations give us the point $A(x_p, y_p)$.

Now we are going to calculate the value of the autoisoptic angle α. If *M* is the corresponding point to $\omega = \pi - \alpha$, the tangent straight line in *M* will pass for p and then we will have:

$$\frac{y_P - y_M}{x_P - x_M} = tg\,(V-\alpha) \qquad (21)$$

$$\frac{y_p - a^{\pi-\alpha} \cdot \cos(\pi-\alpha)}{x_p - a^{\pi-\alpha} \cdot sen(\pi-\alpha)} = tg\,(V-\alpha) \qquad (22)$$

Consequently:

$$\alpha = V - arctg\,\frac{x_p - a^{\pi-\alpha} \cdot sen\alpha}{y_p - a^{\pi-\alpha} \cdot \cos\alpha} \qquad (23)$$

Equation that is solved by iteration.

4. Deduction of the auto-ortoptic logarithmic spiral

We have $\rho = a^{\omega}$ (Figure 3). For $\omega = \frac{\pi}{2} \Rightarrow \rho = a^{\frac{\pi}{2}}$.

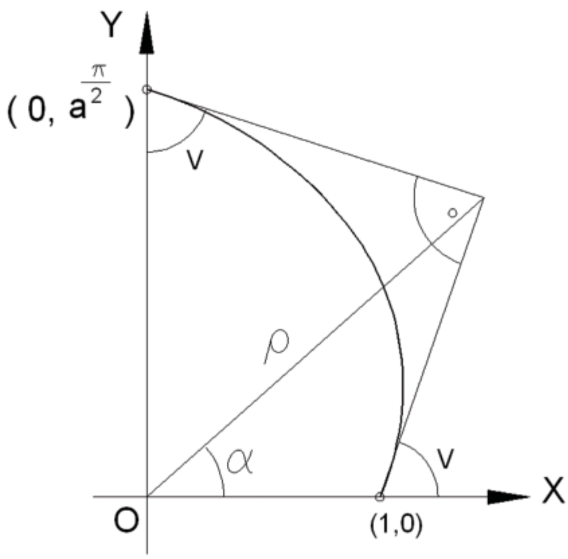

Figure 3. Graphical base for the deduction of the Auto-ortoptic Logarithmic Spiral

If $tgV = \frac{\rho}{\rho'} = \frac{1}{La}$, then the tangent in *(1,0)* is:

$y = \frac{1}{La}(x-1)$ and the tangent in $\left(0, a^{\frac{\pi}{2}}\right)$ is:

$y - a^{\frac{\pi}{2}} = -La \cdot x$

Let's find the point of intersection *P* of the tangent straight lines:

$$\frac{1}{La}(x-1) = a^{\frac{\pi}{2}} - La \cdot x \qquad (24)$$

$$x - 1 = a^{\frac{\pi}{2}} La - (La)^2 x \qquad (25)$$

$$x\left[1+(La)^2\right] = a^{\frac{\pi}{2}} La + 1 \tag{26}$$

$$\boxed{x = \frac{a^{\frac{\pi}{2}} La + 1}{1 + (La)^2}} \tag{27}$$

$$y = \frac{1}{La}\left(\frac{a^{\frac{\pi}{2}} La + 1}{1 + (La)^2} - 1\right) = \frac{1}{La} \cdot \frac{a^{\frac{\pi}{2}} La + 1 - 1 - (La)^2}{1 + (La)^2}$$

Consequently the value for y is:

$$\boxed{y = \frac{a^{\frac{\pi}{2}} La - 1}{1 + (La)^2}} \tag{28}$$

Then, from the equations (27) and (28) we can deduce:

$$x^2 + y^2 = \frac{\left(a^{\frac{\pi}{2}} La + 1\right)^2 + \left(a^{\frac{\pi}{2}} - La\right)^2}{\left[1 + (La)^2\right]^2} =$$

$$= \frac{a^{\pi}(La)^2 + 2a^{\frac{\pi}{2}} La + 1 - a^{\pi} - 2a^{\frac{\pi}{2}} La + (La)^2}{\left[1 + (La)^2\right]^2} =$$

$$= \frac{a^{\pi}\left[1 + (La)^2\right] + 1 + (La)^2}{\left[1 + (La)^2\right]^2} = \frac{a^{\pi} + 1}{1 + (La)^2} \tag{29}$$

Therefore:

$$\sqrt{x^2 + y^2} = \rho = \sqrt{\frac{a^{\pi} + 1}{1 + (La)^2}} \tag{30}$$

On the other hand:

$$tag\alpha = \frac{y}{x} = \frac{a^{\frac{\pi}{2}} - La}{a^{\frac{\pi}{2}} La + 1} \tag{31}$$

What turns out to be that:

$$\alpha = arctg\frac{a^{\frac{\pi}{2}} - La}{a^{\frac{\pi}{2}} La + 1} + 2\pi \tag{32}$$

(This if it is the following spire)

It will be necessary to verify, to obtain the auto-ortóptica, the following thing: $\rho - a^{\alpha} = 0$

For $\alpha = 1{,}05$ we have $\rho - a^{\alpha} > 0$ (we save the calculations)

For $\alpha = 1{,}06$ we have $\rho - a^{\alpha} < 0$

If we write:

$$La = \frac{L\rho}{\alpha} = \frac{\sqrt{\dfrac{a^{\pi} + 1}{1 + (La)^2}}}{arctg\dfrac{a^{\frac{\pi}{2}} - La}{a^{\frac{\pi}{2}} La + 1} + 2\pi} \tag{33}$$

For iteration we obtain (for example from α=1,055) with the calculator: $a = 1{,}056879$.

That is to say, $\rho = 1{,}056879^{\omega}$ is AUTO-ORTOPTIC.

5. Conclusions

1. The Logarithmic Spiral $\rho = a^{\omega}$ is autoisoptic.
2. We have given a method to calculate the isoptic angle α depending on a.
3. We have calculated a for the spiral auto-ortoptic ($\alpha = 90^{\circ}$).
4. The content of this paper we have not seen it in the existing bibliography relating to logarithmic spirals.

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