Singularities around $w = -1$

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Sketch

1. Introduction
2. Models close to $w = -1$
3. Singularities at $w = -1$
4. Conclusions

Motivation

- Astronomical observations of luminosity distances derived from Type Ia supernovae, CMB spectrum and global matter distribution provide evidence of cosmic speed up of the Universe.
- Alternatively, cosmic acceleration might be due to an exotic fluid filling the Universe, known as dark energy.
- These have given rise to a collection of new cosmological evolutions, future singularites being the most perplexing ones (“big rip”, “sudden singularities”. . .).

Classification of singularities

- Big Bang / Crunch: zero a, divergent H, density and pressure. Strong.
- Type I: "Big Rip": divergent a. Strong.
- Type II: "Sudden": finite a, H, density, divergent $\dot{H}$ and pressure. Weak
- Type III: "Big Freeze": finite a, divergent H, density and pressure. Weak/Strong
- Type IV: "Big Brake": finite a, H, $\dot{H}$, density and pressure but divergent higher derivatives. Weak.
- Type V: Barotropic index singularities: finite a, ρ, p, but divergent barotropic index w.
- ...

Barotropic index w

- Different observations provide a value close to $w = -1$ for the barotropic index of the universe.
- We would like to analyse models with small deviations from $w = -1$.

Perturbed $w = -1$

- The barotropic index is defined as $w = \frac{p}{\rho}$.
- We assume $w = -1 + h(t), \qquad |h(t)| \ll 1$.
- In terms of the scale factor of the universe $a(t) = e^{f(t)}$,

$$w = -\frac{1}{3} - \frac{2}{3}\frac{a\ddot{a}}{\dot{a}^2} = -1 + \frac{2}{3}\left(\frac{1}{\dot{f}}\right)^{\cdot}.$$

- We may write the scale factor in terms of $h(t)$,

$$\dot{f} = \frac{2}{k + 3\int h} \Rightarrow f(t) = C + 2\int \frac{dt}{k + 3\int h(t)\,dt}.$$

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- One constant of integration is a change of scale,

$$f(t) \to f(t) + C \Rightarrow a(t) \to e^{C} a(t).$$

- The other one is fixed by the energy density,

$$\rho = \left(\frac{\dot{a}}{a}\right)^2 = \dot{f}^2 \Rightarrow \sqrt{\rho} = \frac{2}{k + 3\int h}.$$

Singular/nonsingular density at $w = -1$

- Assume $h(t) = \alpha(-t)^p$, $p > 0$ ($w(0) = -1$),

$$\sqrt{\rho(t)} = -\frac{2(p+1)}{3\alpha}\frac{1}{k + (-t)^{p+1}}.$$

- Two cases:
  - $k \neq 0$: $\sqrt{\rho}$ has simple poles out of $t = 0$ and $\rho \sim (t - t_0)^{-2}$, but $w = -1$ is regular.
  - $k = 0$: Singular density and scale factor at $w = -1$.

$$\rho(t) = \frac{4}{9}\left(\frac{p+1}{\alpha}\right)\frac{1}{t^{2p+2}},$$

$$a(t) = e^{-\beta/(-t)^p}, \quad \beta = \frac{2(p+1)}{3\alpha p}.$$

Singular density at $w = -1$

- Singular scale factor at $w = -1$,

$$a(t) = e^{-\beta/(-t)^p}, \quad \beta = \frac{2(p+1)}{3\alpha p}.$$

- The scale factor is non-analytical and has an essential singularity at $t = 0$.
- The energy density blow up as $1/t^{2p+2}$ instead of t^{-2}!

- For $\alpha > 0$, we have a *Great Crunch*: $a(t) \to 0$.
- For $\alpha < 0$, we have a *Great Rip*: $a(t) \to \infty$.

Geodesics near $w = -1$

- Equations for causal geodesics ($\delta = 0$ for null and $\delta = 1$ for timelike geodesics) parametrised by proper time τ, $d\tau = \sqrt{-g_{ij}dx^i dx^j}$, in a flat FLRW model reduce to

$$\frac{dt}{d\tau} = \sqrt{\delta + \frac{P^2}{a^2(t)}}, \qquad \frac{dr}{d\tau} = \pm\frac{P}{a^2(t)},$$

where P is a constant of motion.

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- Null geodesics: $\tau = P^{-1}\int_{t_0}^{0} a(t)\,dt$.
- It takes an infinite proper time to Great Rip ($a(0) = \infty$)!

Geodesics near $w = -1$

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  where P is a constant of motion.
- Timelike geodesics: $\tau = \int_{t_0}^{0} \frac{dt}{\sqrt{1 + P^2 a^{-2}(t)}}$.
- All geodesics reach $w = -1$ in finite proper time.
- $w = -1$ becomes singular but for null geodesics.

Strong singularities

- The idea of a strong singularity was first introduced by Ellis and Schmidt. A singularity is meant to be strong if tidal forces exert a severe disruption on finite objects falling into it.
- According to Tipler, the singularity is strong if the volume tends to zero as the geodesic approaches the value of proper time where it meets its end.
- Królak's definition just requires the derivative of the volume with respect to proper time to be negative.

Strong singularities

- There are necessary and sufficient conditions (Clarke and Królak) based on integrals of Riemann tensor along incomplete geodesic.
- A lightlike geodesic with velocity u meets a strong singularity, according to Tipler, at proper time τ_0 if and only if the integral $\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' R_{ij} u^i u^j$ diverges as τ tends to τ_0."
- With Królak's definition a lightlike geodesic meets a strong singularity at proper time τ_0 if and only if the integral $\int_0^{\tau} d\tau' R_{ij} u^i u^j$ diverges as τ tends to τ_0.

Strong timelike geodesics

- Conditions are not so simple for timelike geodesics.
- A timelike geodesic meets a strong singularity, according to Tipler if $\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' R_{ij}u^i u^j$ diverges as τ tends to τ_0.
- With Królak's definition, a timelike geodesic meets a strong singularity at τ_0 if $\int_0^{\tau} d\tau' R_{ij}u^i u^j$ diverges as τ tends to τ_0.
- For Tipler's definition, if a causal geodesic with velocity u meets a strong singularity,
  $I^i_j(\tau) = \int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' \left| R^i_{kjl}u^k u^l \right|$ diverges as τ tends to τ_0.
- Królak's definition just requires that
  $I^i_j(\tau) = \int_0^{\tau} d\tau' \left| R^i_{kjl}u^k u^l \right|$ diverges as τ tends to τ_0.

Strong singularities

- Singularities at $w = -1$ are checked to be strong according to both definitions!

Results

- There are two possible behaviours for models close to $w = -1$:
  - Regular crossing of $w = -1$.
  - Essential singularity at $w = -1$: Great Crunch / Rip.
- Density blows up at the singularity worse than t^{-2}.
- The singularities are strong.
- But null geodesics never reach the Great Rip.

Further reading

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Thank you all!