THE ZERO-REMOVING PROPERTY AND LAGRANGE-TYPE INTERPOLATION SERIES

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□ *The classical Kramer sampling theorem, which provides a method for obtaining orthogonal sampling formulas, can be formulated in a more general nonorthogonal setting. In this setting, a challenging problem is to characterize the situations when the obtained nonorthogonal sampling formulas can be expressed as Lagrange-type interpolation series. In this article a necessary and sufficient condition is given in terms of the zero removing property. Roughly speaking, this property concerns the stability of the sampled functions on removing a finite number of their zeros.*

Keywords Analytic Kramer kernels; Lagrange-type interpolation series; Zero-removing property.

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1. STATEMENT OF THE PROBLEM

The classical Kramer sampling theorem provides a method for obtaining orthogonal sampling theorems [5, 13, 15, 21]. The statement of this general result is as follows. Let K be a complex function defined on $D \times I$, where $I \subset \mathbb{R}$ is an interval and D is an open subset of $\mathbb{R}$, and such that for every $t \in D$ the sections $K(\cdot, t)$ are in $\mathcal{L}^2(I)$. Assume that there exists a sequence of distinct real numbers $\{t_n\} \subset D$, indexed by a subset of $\mathbb{Z}$, such that $\{K(x, t_n)\}$ is a complete orthogonal sequence of functions for $\mathcal{L}^2(I)$. Then for any f of the form

$$f(t) = \int_I F(x)K(x,t)\,dx \quad t \in D, \tag{1}$$

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where $F \in \mathscr{L}^2(I)$, we have

$$f(t) = \sum_n f(t_n) S_n(t), \quad t \in D, \tag{2}$$

with

$$S_n(t) := \frac{\int_I K(x, t)\overline{K(x, t_n)}\, dx}{\int_I |K(x, t_n)|^2\, dx}. \tag{3}$$

The series in (2) converges absolutely and uniformly on subsets of D where $\|K(\cdot, t)\|_{\mathscr{L}^2(I)}$ is bounded.

For instance, taking $I = [-\pi, \pi]$, $K(x, t) = \mathrm{e}^{\mathrm{i}tx}$ and $\{t_n = n\}_{n\in\mathbb{Z}}$, we get the well–known Whittaker–Shannon–Kotel'nikov sampling formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t - n)}{\pi(t - n)}, \quad t \in \mathbb{R},$$

for functions in $L^2(\mathbb{R})$ whose Fourier transform has support in $[-\pi, \pi]$.

Now, if we take $I = [0, 1]$, $K(x, t) = \sqrt{xt}J_\nu(xt)$ and $\{t_n\}$, the sequence of the positive zeros of the Bessel function J_ν of νth order with $\nu > -1$, then

$$f(t) = \sum_n f(t_n) \frac{2\sqrt{t_n t}J_\nu(t)}{J_\nu'(t_n)(t^2 - t_n^2)}, \quad t \in \mathbb{R},$$

for every f of the form $f(t) = \int_0^1 F(x)\sqrt{xt}J_\nu(xt)dx$, where $F \in L^2(0, 1)$ (see [13, p. 83]).

The Kramer sampling theorem has played a very significant role in sampling theory, interpolation theory, signal analysis and, generally, in mathematics (see, e.g., the survey articles [3, 4]).

In [6], an extension of the Kramer sampling theorem has been obtained to the case when the kernel is analytic in the sampling parameter $t \in D \subseteq \mathbb{C}$. Namely, assume that the Kramer kernel K is an entire function for any fixed $x \in I$, and that the function $h(t) = \int_I |K(x, t)|^2 dx$ is locally bounded on $D \subseteq \mathbb{C}$. Then any function f defined by (1) is an entire function, as are all the sampling functions (3).

A straightforward discrete version of Kramer's theorem can be obtained. Namely, let $K(n, z)$ be a kernel such that, as function of n, the sequence $\{K(n, z)\} \in \ell^2(\mathbb{I})$ for any $z \in D \subseteq \mathbb{C}$, where $\mathbb{I}$ is a countable index set. Assume that, for a suitable sequence $\{z_n\} \subset D$, the sequence $\{K(\cdot, z_n)\}$ is an orthogonal basis for $\ell^2(\mathbb{I})$. Then, any function of the form $f(z) = \sum_{n\in\mathbb{I}} c_n K(n, z)$, where $\{c_n\} \in \ell^2(\mathbb{I})$, can be expanded by means of a sampling series like (2) (see [8]). As examples of discrete kernels for which a sampling formula works we can consider discrete kernels

$K(n, z) := P_n(z)$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}$, where $\{P_n(z)\}_{n\in\mathbb{N}_0}$ denotes a sequence of orthonormal polynomials associated with an indeterminate Hamburger or Stieltjes moment problem (see [8, 9] for the details).

The Kramer sampling theorem has been the cornerstone for a significant mathematical literature of sampling theory associated with differential or difference problems. See, among others, [1, 5, 8, 9, 13, 21] and the references therein.

Thus an abstract analytic formulation of the Kramer sampling theorem raises in a natural way: Let $\mathcal{H}$ be a complex, separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, and let $\{x_n\}_{n=1}^{\infty}$ be a Riesz basis for $\mathcal{H}$. Suppose K is a $\mathcal{H}$-valued function defined on $\mathbb{C}$. For each $x \in \mathcal{H}$, define the function $f_x(z) = \langle K(z), x\rangle_{\mathcal{H}}$ on $\mathbb{C}$, and let $\mathcal{H}_K$ denote the collection of all such functions f_x. Furthermore, each element in $\mathcal{H}_K$ is an entire function if and only if K is analytic on $\mathbb{C}$. In this setting, an abstract version of the analytic Kramer theorem is obtained assuming the existence of two sequences, $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ and $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C}\setminus\{0\}$, such that $K(z_n) = a_n x_n$ for each $n \in \mathbb{N}$. Namely, for any $f_x \in \mathcal{H}_K$ we have

$$f_x(z) = \sum_{n=1}^{\infty} f_x(z_n) \frac{S_n(z)}{a_n}, \quad z \in \mathbb{C},$$

where $S_n(z) = \langle K(z), y_n\rangle$, $n \in \mathbb{N}$, being $\{y_n\}_{n=1}^{\infty}$ the dual Riesz basis of $\{x_n\}_{n=1}^{\infty}$ (see sections 2 and 4 infra for all the details).

A challenging problem is to give a necessary and sufficient condition to ensure that the above sampling formula can be written as a Lagrange-type interpolation series, that is

$$f_x(z) = \sum_{n=1}^{\infty} f_x(z_n) \frac{P(z)}{(z - z_n)P'(z_n)}, \quad z \in \mathbb{C},$$

where P denotes an entire function having only simple zeros at all the points of the sequence $\{z_n\}_{n=1}^{\infty}$. Roughly speaking, the aforesaid necessary and sufficient condition concerns the stability of the functions belonging to the space $\mathcal{H}_K$ on removing a finite number of their zeros; this is an ubiquitous algebraic property in the mathematical literature (see section 3 infra) and it will be called the zero-removing property along the article.

Let us consider the following toy example: Given a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ in $\mathbb{C}^2$, for the kernel $K(z) := z^2(\mathbf{e}_2 - \mathbf{e}_1) + \mathbf{e}_1$ consider the corresponding space $\mathcal{H}_K$, which coincides with $\{az^2 + b \mid a, b \in \mathbb{C}\}$. Obviously, this space has not the zero-removing property: if we remove a zero from an element in $\mathcal{H}_K$ the resulting polynomial does not belong to $\mathcal{H}_K$. Besides, the sampling formula $f(z) = f(0)(1 - z^2) + f(1)z^2$, which holds in $\mathcal{H}_K$ cannot be written as a Lagrange interpolation formula. The study of all these topics will be carried out throughout the remaining sections.

2. SOME PRELIMINARIES ON THE SPACE $\mathcal{H}_K$

Suppose we are given a separable complex Hilbert space $\mathcal{H}$ and an abstract kernel K which is nothing but a $\mathcal{H}$-valued function on $\mathbb{C}$. Set $f_x(z) := \langle K(z), x\rangle_{\mathcal{H}}$ and denote by $\mathcal{H}_K$ the collection of all such functions f_x, $x \in \mathcal{H}$. It is a reproducing kernel Hilbert space (RKHS) coming from the transforms $K(z)$, $z \in \mathbb{C}$, and corresponding to the reproducing kernel $(z, w) \mapsto \langle K(z), K(w)\rangle_{\mathcal{H}}$. Notice that the mapping $\mathcal{T}$ given by

$$\mathcal{H} \ni x \overset{\mathcal{T}}{\longmapsto} f_x \in \mathcal{H}_K \tag{4}$$

is an antilinear mapping from $\mathcal{H}$ onto $\mathcal{H}_K$ (henceforth we omit the subscript x for denoting the elements in $\mathcal{H}_K$). The mapping $\mathcal{T}$ is injective if and only if the set $\{K(z)\}_{z\in\mathbb{C}}$ is a complete set in $\mathcal{H}$. In particular, if there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ such that $\{K(z_n)\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$, then $\mathcal{T}$ is an antilinear isometry from $\mathcal{H}$ onto $\mathcal{H}_K$. Recall that a Riesz basis in a separable Hilbert space $\mathcal{H}$ is the image of an orthonormal basis by means of a boundedly invertible operator. Any Riesz basis $\{x_n\}_{n=1}^{\infty}$ has a unique biorthonormal (dual) Riesz basis $\{y_n\}_{n=1}^{\infty}$, i.e., $\langle x_n, y_m\rangle_{\mathcal{H}} = \delta_{n,m}$, such that the expansions

$$x = \sum_{n=1}^{\infty} \langle x, y_n\rangle_{\mathcal{H}}\, x_n = \sum_{n=1}^{\infty} \langle x, x_n\rangle_{\mathcal{H}}\, y_n$$

hold for every $x \in \mathcal{H}$ (see [20] for more details and proofs).

The convergence in the norm $\|\cdot\|_{\mathcal{H}_K}$ implies pointwise convergence which is uniform on those subsets of $\mathbb{C}$ where the function $z \mapsto \|K(z)\|_{\mathcal{H}}$ is bounded.

Like in the classical case the following result holds: The space $\mathcal{H}_K$ is a RKHS of entire functions if and only if the kernel K is analytic in $\mathbb{C}$ [19, p. 266]. Another characterization of the analyticity of the functions in $\mathcal{H}_K$ is given in terms of Riesz bases. Suppose that a Riesz basis $\{x_n\}_{n=1}^{\infty}$ for $\mathcal{H}$ is given and let $\{y_n\}_{n=1}^{\infty}$ be its dual Riesz basis; expanding $K(z)$, for each fixed $z \in \mathbb{C}$, with respect to the basis $\{x_n\}_{n=1}^{\infty}$ we obtain

$$K(z) = \sum_{n=1}^{\infty} \langle K(z), y_n\rangle_{\mathcal{H}}\, x_n,$$

where the coefficients $\langle K(z), y_n\rangle_{\mathcal{H}}$ as functions in z are in $\mathcal{H}_K$. The following result holds: The space $\mathcal{H}_K$ is a RKHS of entire functions if and only if all the functions

$$S_n(z) := \langle K(z), y_n\rangle_{\mathcal{H}}, \quad z \in \mathbb{C} \tag{5}$$

are entire and $\|K(\cdot)\|_{\mathcal{H}}$ is bounded on compact sets of $\mathbb{C}$ (see [11]).

3. THE ZERO-REMOVING PROPERTY

In this section, we introduce the zero-removing property for classes of entire functions.

Definition 1 (Zero-Removing Property). A set $\mathscr{A}$ of entire functions has the zero-removing property (ZR property hereafter) if for any $g \in \mathscr{A}$ and any zero w of g the function $g(z)/(z-w)$ belongs to $\mathscr{A}$.

The ZR property is ubiquitous in mathematics; for instance, the set $\mathscr{P}_N(\mathbb{C})$ of polynomials with complex coefficients of degree less or equal N has the ZR property. Another more involved examples sharing this property are:

- The entire functions in the Pólya class have the ZR property [2, p. 15]. Recall that an entire function $E(z)$ is said to be of Pólya class if it has no zeros in the upper half-plane, if $|E(x-iy)| \le |E(x+iy)|$ for $y > 0$, and if $|E(x+iy)|$ is a nondecreasing function of $y > 0$ for each fixed x.
- The entire functions in the Paley–Wiener class PW_π of bandlimited functions to $[-\pi, \pi]$, that is, $PW_\pi := \{f \in \mathscr{L}^2(\mathbb{R}) \cap C(\mathbb{R}) : \operatorname{supp}\hat{f} \subseteq [-\pi, \pi]\}$, where $\hat{f}$ stands for the Fourier transform of f, satisfy the ZR property; it follows from the classical Paley–Wiener theorem [20, p. 101], which says that this space can be written as $PW_\pi = \{f \text{ entire function} : |f(z)| \le A\,e^{\pi|z|}, f|_{\mathbb{R}} \in \mathscr{L}^2(\mathbb{R})\}$. From this characterization the ZR property immediately comes out.
- In general, de Branges spaces $\mathscr{H}(E)$ with strict de Branges function E have the ZR property [2, p. 52]. Let E be an entire function verifying $|E(x-iy)| < |E(x+iy)|$ for all $y > 0$. The de Branges space $\mathscr{H}(E)$ is the set of all entire functions F such that

$$\|F\|_E^2 := \int_{-\infty}^{\infty} \left|\frac{F(t)}{E(t)}\right|^2 dt < \infty,$$

and such that both ratios F/E and F^*/E, where $F^*(z) := \overline{F(\bar{z})}$, are of bounded type and of non-positive mean type in the upper half-plane. The structure function or de Branges function E has no zeros in the upper half plane. A de Branges function E is said to be strict if it has no zeros on the real axis. We require that F/E and F^*/E be of bounded type and nonpositive mean type in $\mathbb{C}^+$. A function is of bounded type if it can be written as a quotient of two bounded analytic functions in $\mathbb{C}^+$ and it is of nonpositive mean type if it grows no faster than $e^{\varepsilon y}$ for each $\varepsilon > 0$ as $y \to \infty$ on the positive imaginary axis $\{iy : y > 0\}$. Note that the Paley–Wiener space PW_π is a de Branges space for the structure function $E_\pi(z) = \exp(-i\pi z)$.

Assume that the space $\mathcal{H}_K$ in section 2 comes from a polynomial kernel K with coefficients in $\mathcal{H}$; concerning the ZR property in $\mathcal{H}_K$, the following result holds:

Theorem 1. *The space $\mathcal{H}_K$ associated with a polynomial kernel $K(z) := \sum_{n=0}^{N} p_n z^n$, where $p_n \in \mathcal{H}$ and $p_N \neq 0$, has the ZR property if and only if the set $\{p_0, p_1, \dots, p_N\}$ is linearly independent in $\mathcal{H}$.*

Proof. Consider $f(z) = a_N z^N + \cdots + a_1 z + a_0 \in \mathcal{H}_K$ with $a_N \neq 0$; there exists $x \in \mathcal{H}$ such that $f(z) = \langle K(z), x\rangle$ and, consequently, $a_j = \langle p_j, x\rangle$ for $j = 0, 1, \dots, N$. If the space $\mathcal{H}_K$ has the ZR property and $\alpha_0, \alpha_1, \dots, \alpha_N$ are the roots of the polynomial f then the constant a_N and the polynomials $a_N(z - \alpha_N), a_N(z - \alpha_N)(z - \alpha_{N-1}), \dots, a_N(z - \alpha_N)(z - \alpha_{N-1}) \cdots (z - \alpha_1)$ belong to $\mathcal{H}_K$. Let $b_0, b_1, \dots, b_N \in \mathbb{C}$ such that

$$b_N p_N + b_{N-1} p_{N-1} + \cdots + b_0 p_0 = 0. \tag{6}$$

The vector $(b_N, \dots, b_0)$ is orthogonal in $\mathbb{C}^{N+1}$ to any vector $(c_N, \dots, c_0) \in \mathbb{C}^{N+1}$ with $c_N z^N + \cdots + c_0 \in \mathcal{H}_K$. As a consequence, since $a_N \in \mathcal{H}_K$, $b_0 a_N = 0$, which implies that $b_0 = 0$. Analogously, since $a_N(z - \alpha_N)$ belongs to $\mathcal{H}_K$ we have that $a_N b_1 - (a_N \alpha_N) b_0 = 0$ and consequently $b_1 = 0$. Proceeding iteratively it is straightforward to obtain that $b_2 = \cdots = b_{N-1} = 0$; finally, from (6) we conclude that $b_N = 0$.

Now suppose that the set $\{p_0, p_1, \dots, p_N\}$ is linearly independent in $\mathcal{H}$. In this case, the mapping $\Phi : \mathcal{H} \to \mathbb{C}^{N+1}$ given by $\Phi(x) = (\langle p_0, x\rangle, \dots, \langle p_N, x\rangle)$ is surjective. As a consequence, any complex polynomial of degree less than or equal to N belongs to $\mathcal{H}_K$. Let $f(z) = a_N z^n + \cdots + a_1 z + a_0 \in \mathcal{H}_K$ and let $w \in \mathbb{C}$ be a root of f. Hence, $f(z)/(z - w) = c_0 + c_1 z + \cdots + c_{N-1} z^{N-1}$ is a polynomial of degree less than or equal to $N - 1$. Since Φ is onto there exists $x \in \mathcal{H}$ such that $\Phi(x) = (c_0, c_1, \dots, c_{N-1}, 0)$. From the definition of Φ, we conclude that $f(z)/(z - w) = \langle K(z), x\rangle$, that is, the function $f(z)/(z - w) \in \mathcal{H}_K$. □

Giving a necessary and sufficient for a general analytic kernel K remains as an open problem. It is worth to mention that a straightforward application of Cauchy–Schwarz inequality shows that entire functions in $\mathcal{H}_K$ inherit the finite order and the type of the vector-valued entire function K provided it has finite order.

As examples of spaces $\mathcal{H}_K$ where the ZR property does not hold let us mention the following:

- Consider the spaces $\mathcal{H}_{K_i}, i = 1, 2$, associated with the analytic kernels $K_i : \mathbb{C} \to L^2[0, \pi]$ defined by $K_1(z)[x] := \sin zx$ and $K_2(z)[x] := \cos zx$. The space $\mathcal{H}_{K_1}$ corresponds to the space of odd bandlimited functions in PW_π

while $\mathcal{H}_{K_2}$ corresponds to the space of even bandlimited functions in PW_π. It is clear that the ZR property does not hold in these spaces.

- Let $K:\mathbb{C}\to\mathcal{H}$ be an analytic kernel such that $K(z_0)=0$ for some $z_0\in\mathbb{C}$. Then all the functions in the associated space $\mathcal{H}_K$ have a zero at z_0 and the ZR property does not hold in $\mathcal{H}_K$. Indeed, let f be a nonzero entire function in $\mathcal{H}_K$ and let r denote the order of its zero z_0. The function $f(z)/(z-z_0)^r$ is not in $\mathcal{H}_K$ since it does not vanish at z_0.
- A little more sophisticated example is the following: For $m\geq 2$ let $K_m:\mathbb{C}\to L^2[-\pi,\pi]$ be defined as $K_m(z)=\frac{1}{\sqrt{2\pi}}e^{iz^m\cdot}\in L^2[-\pi,\pi]$. It is straightforward to show that K_m is an analytic kernel; the corresponding space $\mathcal{H}_{K_m}$ does not have the ZR property. Indeed, expanding $K_m(z)$ as power series around the origin we obtain

$$[K_m(z)](x)=\sum_{k=0}^{\infty}\frac{(ix)^k z^{mk}}{k!}=1+ixz^m-\frac{x^2z^{2m}}{2!}-i\frac{x^3z^{3m}}{3!}+\cdots.$$

Thus, for any function $f(z)=\langle K_m(z),F\rangle$ with $F\in L^2[-\pi,\pi]$ we have

$$f(z)=\sum_{k=0}^{\infty}c_k z^{mk},$$

where $c_k=\langle (ix)^k/k!,F\rangle$, $k=0,1,\ldots$. Let $G\in L^2[-\pi,\pi]\backslash\{0\}$ be such that G is orthogonal to $K(0)$ and let $g(z)=\langle K_m(z),G\rangle$. Since $\langle K(0),G\rangle=0$ we have $g(0)=0$. Hence, the Taylor expansion of $g(z)/z$ around the origin has the form

$$\frac{g(z)}{z}=d_1z^{m-1}+d_2z^{2m-1}+\cdots$$

where $d_k=\langle (ix)^k/k!,G\rangle$, $k=1,2,\ldots$. Since G is not the zero function the function $g(z)/z$ does not belong to $\mathcal{H}_{K_m}$.

4. LAGRANGE-TYPE INTERPOLATION SERIES

In this section, we introduce the analytic Kramer kernels K for which a nonorthogonal sampling theorem in $\mathcal{H}_K$ holds. We prove a converse result: From a sampling formula in $\mathcal{H}_K$ we deduce when K is an analytic Kramer kernel. Finally, we prove the main result: a necessary and sufficient condition ensuring that the Kramer sampling result can be expressed as a Lagrange-type interpolation series.

4.1. The Abstract Kramer Sampling Result

Consider the data

$$\{z_n\}_{n=1}^{\infty}\in\mathbb{C}\quad\text{and}\quad\{a_n\}_{n=1}^{\infty}\in\mathbb{C}\backslash\{0\}.\tag{7}$$

Definition 2 (Analytic Kramer Kernel). An analytic kernel $K : \mathbb{C} \longrightarrow \mathcal{H}$ is said to be an analytic Kramer kernel (with respect to the data (7)) if it satisfies $K(z_n) = a_n x_n$, $n \in \mathbb{N}$, for some Riesz basis $\{x_n\}_{n=1}^{\infty}$ of $\mathcal{H}$.

A sequence $\{S_n\}_{n=1}^{\infty}$ of functions in the space $\mathcal{H}_K$ is said to have the interpolation property (with respect to the data (7)) if

$$S_n(z_m) = a_n\, \delta_{n,m}. \tag{8}$$

Thus, an analytic kernel K is an analytic Kramer one if and only if the sequence of functions $\{S_n\}_{n=1}^{\infty}$ in $\mathcal{H}_K$ given by (5), where $\{y_n\}_{n=1}^{\infty}$ is the dual Riesz basis of $\{x_n\}_{n=1}^{\infty}$, has the interpolation property with respect to the same data (7).

Concerning the existence of analytic Kramer kernels, it has been proved in [11] that, associated with any arbitrary sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} |z_n| = +\infty$, there exists an analytic Kramer kernel K.

Under the notation introduced so far an abstract version of the classical Kramer sampling theorem sampling [15] holds in $\mathcal{H}_K$; this is a slight modification of a sampling result in [14]. For notational purposes we include its proof.

Theorem 2 (Kramer Sampling Theorem). *Let $K : \mathbb{C} \longrightarrow \mathcal{H}$ be an analytic Kramer kernel, and assume that the interpolation property* (8) *holds for some sequences $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ and $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C}\setminus\{0\}$. Let $\mathcal{H}_K$ be the corresponding RKHS of entire functions. Then any $f \in \mathcal{H}_K$ can be recovered from its samples $\{f(z_n)\}_{n=1}^{\infty}$ by means of the sampling series*

$$f(z) = \sum_{n=1}^{\infty} f(z_n)\, \frac{S_n(z)}{a_n}, \quad z \in \mathbb{C}, \tag{9}$$

where the reconstruction functions S_n are given in (5). *The series converges absolutely and uniformly on compact subsets of $\mathbb{C}$.*

Proof. First, notice that $\lim_{n\to\infty} |z_n| = +\infty$; otherwise the sequence $\{z_n\}_{n=1}^{\infty}$ contains a bounded subsequence and, hence, the entire function $S_n \equiv 0$ for all $n \in \mathbb{N}$, which contradicts (8). The anti-linear mapping $\mathcal{T}$ given by (4) is a bijective isometry between $\mathcal{H}$ and $\mathcal{H}_K$. As a consequence, the functions $\{S_n = \mathcal{T}(y_n)\}_{n=1}^{\infty}$ form a Riesz basis for $\mathcal{H}_K$; let $\{T_n\}_{n=1}^{\infty}$ be its dual Riesz basis. Expanding any $f \in \mathcal{H}_K$ in this basis we obtain

$$f(z) = \sum_{n=1}^{\infty} \langle f, T_n\rangle_{\mathcal{H}_K} S_n(z).$$

Moreover,

$$\langle f, T_n\rangle_{\mathcal{H}_K} = \overline{\langle x, x_n\rangle_{\mathcal{H}}} = \left\langle \frac{K(z_n)}{a_n}, x\right\rangle_{\mathcal{H}} = \frac{f(z_n)}{a_n}. \tag{10}$$

Since a Riesz basis is an unconditional basis, the sampling series will be pointwise unconditionally convergent and hence, absolutely convergent. The uniform convergence is a standard result in the setting of the RKHS theory since $z \mapsto \|K(z)\|_{\mathcal{H}}$ is bounded on compact subsets of $\mathbb{C}$. □

Riesz bases theory (see, e.g., [20]) assures the existence of two positive constants $0 < A \leq B$ such that

$$A\|f\|^2_{\mathcal{H}_K} \leq \sum_{n=1}^{\infty} |f(z_n)/a_n|^2 \leq B\|f\|^2_{\mathcal{H}_K} \quad \text{for all } f \in \mathcal{H}_K, \tag{11}$$

that is, $\|f\|_s := \left(\sum_{n=1}^{\infty} |f(z_n)/a_n|^2\right)^{1/2}$ defines an equivalent norm in $\mathcal{H}_K$. Following [12], we can say that the data (7) is a sampling set for $\mathcal{H}_K$; here the sequence of samples belongs to a weighted ℓ^2 space. In [12], the authors characterize the reproducing kernel Hilbert spaces having a fixed sampling set.

The Whittaker–Shannon–Kotel'nikov sampling formula in PW_π becomes a particular case of formula (9) in Theorem 2. Indeed, any $f \in PW_\pi$ can be written as

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(w)e^{izw}\, dw = \left\langle \frac{e^{izw}}{\sqrt{2\pi}}, \overline{\hat{f}} \right\rangle_{L^2[-\pi,\pi]}, \quad z \in \mathbb{C}.$$

The Fourier kernel $K(z) := \frac{e^{iz\cdot}}{\sqrt{2\pi}} \in L^2[-\pi,\pi]$ is an analytic Kramer kernel for the data $\{z_n = n\}_{n\in\mathbb{Z}}$ and $\{a_n = 1\}_{n\in\mathbb{Z}}$. In this case, as $\{e^{inw}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\pi,\pi]$ we get

$$S_n(z) = \frac{1}{2\pi}\langle e^{iz\cdot}, e^{in\cdot}\rangle_{L^2[-\pi,\pi]} = \frac{\sin \pi(z-n)}{\pi(z-n)}, \quad z \in \mathbb{C}.$$

As a consequence, we obtain the WSK sampling formula in PW_π:

$$f(z) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(z-n)}{\pi(z-n)}, \quad z \in \mathbb{C}. \tag{12}$$

The series converges absolutely and uniformly on horizontal strips of the complex plane.

It is worth to remark that a kernel K can be an analytic Kramer kernel with respect to different data (7). For instance, the Fourier kernel is also an analytic Kramer kernel with respect to the data $\{z_n = n+\alpha\}_{n\in\mathbb{Z}}$ where $\alpha \in \mathbb{R}$ and $\{a_n = 1\}_{n\in\mathbb{Z}}$. More generally, it is an analytic Kramer kernel with respect to any data $\{t_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ and $\{a_n = 1\}_{n\in\mathbb{Z}}$, where the points t_n satisfy Kadec's condition $\sup_n |t_n - n| < 1/4$ since the sequence $\{e^{it_n w}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi,\pi]$ [20, p. 42].

4.2. A Converse Result

An interesting converse problem is to decide whether a sampling formula as (9), pointwise convergent in $\mathcal{H}_K$, implies the Kramer kernel condition in definition 2 for K. From formula (9) in Theorem 2 we derive that:

- From (5), for each $z \in \mathbb{C}$, the sequence $\{S_n(z)\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$.
- The sequence $\{f(z_n)/a_n\}_{n=1}^{\infty}$ belongs to $\ell^2(\mathbb{N})$ for any $f \in \mathcal{H}_K$, and
- $\sum_{n=1}^{\infty} \alpha_n S_n(z) = 0$ for all $z \in \mathbb{C}$ and $\{\alpha_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$ implies $\alpha_n = 0$ for all $n \in \mathbb{N}$, due to the uniqueness of a Riesz basis expansion in the RKHS $\mathcal{H}_K$.

It is worth to point out that these conditions are also sufficient to prove that K is an analytic Kramer kernel.

Theorem 3. *Let $\mathcal{H}_K$ be the range of a mapping $\mathcal{T}$ as in (4) considered as a RKHS with reproducing kernel $k(z,w) = \langle K(z), K(w)\rangle_{\mathcal{H}}$. Let $\{S_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}_K$ such that $\{S_n(z)\}_{n=1}^{\infty}$ belongs to $\ell^2(\mathbb{N})$ for each $z \in \mathbb{C}$. Suppose that the following conditions are fulfilled:*

(i) *$\sum_{n=1}^{\infty} \alpha_n S_n(z) = 0$ for all $z \in \mathbb{C}$ and $\{\alpha_n\}_{n=1}^{\infty}$ in $\ell^2(\mathbb{N})$ implies $\alpha_n = 0$ for all n.*

(ii) *There exist sequences $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ and $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C}\backslash\{0\}$ such that*

$$\left\{\frac{f(z_n)}{a_n}\right\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) \quad \text{and} \quad f(z) = \sum_{n=1}^{\infty} f(z_n)\frac{S_n(z)}{a_n}, \quad \text{for any } f \in \mathcal{H}_K,$$

where the sampling series is pointwise convergent in $\mathbb{C}$.

Then, the sequence $\{S_n\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}_K$ and the kernel K of the mapping $\mathcal{T}$ evaluated at $z \in \mathbb{C}$ can be expressed as $K(z) = \sum_{n=1}^{\infty} S_n(z)\, y_n$, where $\{y_n\}_{n=1}^{\infty}$ is the dual Riesz basis of the Riesz basis $\{x_n = \mathcal{T}^{-1}(S_n)\}_{n=1}^{\infty}$ in $\mathcal{H}$. In particular, $K(z_n) = a_n y_n$ for any $n \in \mathbb{N}$.

Proof. By defining $\tilde{k}(z,w) := \sum_{n=1}^{\infty} S_n(z)\overline{S_n(w)}$, we obtain a positive definite function which defines a RKHS $\widetilde{\mathcal{H}}$, such that $\widetilde{\mathcal{H}} \subseteq \mathcal{H}_K$. Condition (i) implies that the sequence $\{S_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\widetilde{\mathcal{H}}$ (see [17]).

Now we prove that $\widetilde{\mathcal{H}} = \mathcal{H}_K$ and that the identity mapping $\widetilde{\mathcal{H}} \hookrightarrow \mathcal{H}_K$ is continuous. Take $f \in \mathcal{H}_K$, by condition ii), the sequence $\{f(z_n)a_n^{-1}\}_{n=1}^{\infty}$ is in $\ell^2(\mathbb{N})$. As a consequence, the series $\sum_{n=1}^{\infty} f(z_n)a_n^{-1}S_n$ converges in the norm of $\widetilde{\mathcal{H}}$. By the reproducing kernel property, we have that the series $\sum_{n=1}^{\infty} f(z_n)a_n^{-1}S_n(z)$ is pointwise convergent. Comparing this with what we get from the sampling formula for f we deduce that $f = \sum_{n=1}^{\infty} f(z_n)a_n^{-1}S_n$, where the convergence is in $\widetilde{\mathcal{H}}$ and, consequently, $f \in \widetilde{\mathcal{H}}$.

Next we show the continuity of the identity mapping by applying the closed graph theorem. Indeed, let $\{f_n\}_{n=1}^{\infty}$ be a sequence such that $f_n \to f$ in $\widetilde{\mathcal{H}}$ and $f_n \to g$ in $\mathcal{H}_K$ as $n \to \infty$. Using the reproducing property in both $\mathcal{H}_K$ and $\widetilde{\mathcal{H}}$, for $z \in \mathbb{C}$ we have

$$|f_n(z) - f(z)| \le \|f_n - f\|_{\widetilde{\mathcal{H}}} \sqrt{\tilde{k}(z,z)};$$
$$|f_n(z) - g(z)| \le \|f_n - g\|_{\mathcal{H}_K} \sqrt{k(z,z)}.$$

Therefore, $\lim_{n\to\infty} f_n(z) = f(z) = g(z)$ for each $z \in \mathbb{C}$, and hence $f = g$.

Since it is also surjective, we infer that the norms $\|\cdot\|_{\mathcal{H}_K}$ and $\|\cdot\|_{\widetilde{\mathcal{H}}}$ are equivalent from the open mapping theorem. As a consequence, the orthonormal basis $\{S_n\}_{n=1}^{\infty}$ in $\widetilde{\mathcal{H}}$ is a Riesz basis for $\mathcal{H}_K$.

Assuming that the mapping $\mathcal{T}$ is one-to-one, the sequence $\{x_n = \mathcal{T}^{-1}(S_n)\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$; denote by $\{y_n\}_{n=1}^{\infty}$ its dual Riesz basis. Expanding $K(z)$ with respect to $\{y_n\}_{n=1}^{\infty}$, for each fixed $z \in \mathbb{C}$ we obtain

$$K(z) = \sum_{n=1}^{\infty} \langle K(z), x_n \rangle_{\mathcal{H}}\, y_n = \sum_{n=1}^{\infty} S_n(z)\, y_n,$$

that is, the required expansion for $K(z)$.

Notice that the interpolatory condition $S_n(z_m) = a_m \delta_{n,m}$ comes out of a direct application of condition (ii) to S_n, followed by condition (i).

As to the case when, a priori, $\mathcal{T}$ is not known to be one-to-one, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$ with $P(x_n) \neq 0$ for all n, where P denotes the orthogonal projection onto the closed subspace $(\operatorname{Ker}\mathcal{T})^{\perp}$. Consider $S_n = \mathcal{T}(x_n) \in \mathcal{H}_K$, and suppose that these functions satisfy the hypotheses in Theorem 3. In this case, $\{S_n\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}_K$. Consequently, since $S_n = \mathcal{T}[P(x_n)]$ and $\mathcal{T}|_{P(\operatorname{Ker}\mathcal{T})} = 0$, we obtain that $\{P(x_n)\}_{n=1}^{\infty}$ is a Riesz basis for $P(\mathcal{H}) = (\operatorname{Ker}\mathcal{T})^{\perp}$. The result comes out taking into account the orthogonal sum $\mathcal{H} = (\operatorname{Ker}\mathcal{T})^{\perp} \oplus (\operatorname{Ker}\mathcal{T})$. □

4.3. Lagrange-Type Interpolation Series

A more difficult question concerns whether the sampling expansion (9) can be written, in general, as a Lagrange-type interpolation series.

For instance, for $f \in PW_\pi$ the WSK formula (12) can be written as the Lagrange-type interpolation series

$$f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{P(z)}{(z-n)P'(n)}, \quad z \in \mathbb{C},$$

by taking $P(z) = (\sin \pi z)/\pi$, an entire function having only simple zeros at $\mathbb{Z}$.

The case where the sequence $\{x_n\}_{n=1}^\infty$ in Definition 2 is an orthonormal basis for $\mathcal{H}$ was studied in [7]: A necessary and suficient condition involves the ZR property. Next, we prove that the same necessary and sufficient condition holds in the general case of analytic Kramer kernels K involving Riesz bases.

Theorem 4. *Let $\mathcal{H}_K$ be a RKHS of entire functions obtained from an analytic Kramer kernel K with respect to the data $\{z_n\}_{n=1}^\infty \subset \mathbb{C}$ and $\{a_n\}_{n=1}^\infty \in \mathbb{C}\setminus\{0\}$, that is, $K(z_n) = a_n\, x_n$, $n \in \mathbb{N}$, for some Riesz basis $\{x_n\}_{n=1}^\infty$ for $\mathcal{H}$. Then, the sampling formula* (9) *for $\mathcal{H}_K$ can be written as a Lagrange-type interpolation series*

$$f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{P(z)}{(z-z_n)P'(z_n)}, \quad z \in \mathbb{C}, \tag{13}$$

where P denotes an entire function having only simple zeros at $\{z_n\}_{n=1}^\infty$ if and only if the space $\mathcal{H}_K$ satisfies the ZR property.

Proof. For the sufficient condition we have to prove that sampling formula (9) can be written as a Lagrange-type interpolation series (13) for some entire function P. First, we prove that the only zeros of the sampling function S_n are given by $\{z_r\}_{r\neq n}$. Suppose that $S_n(w) = 0$, then by hypothesis the function $S_n(z)/(z-w)$ is in $\mathcal{H}_K$. Hence, the function

$$\frac{z-z_n}{z-w} S_n(z) = S_n(z) + \frac{w-z_n}{z-w} S_n(z)$$

also belongs to $\mathcal{H}_K$. If $w \notin \{z_r\}_{r\neq n}$, the function $\frac{z-z_n}{z-w} S_n(z)$ in $\mathcal{H}_K$ vanishes at the sequence $\{z_r\}_{r=1}^\infty$ which implies that $S_n \equiv 0$, to give a contradiction. In addition, the zeros of S_n are simple; indeed, suppose that z_m is a multiple zero of S_n. Proceeding as above, the function $\frac{z-z_n}{z-z_m} S_n(z)$ belongs to $\mathcal{H}_K$ and vanishes at $\{z_r\}_{r=1}^\infty$ which again implies that $S_n \equiv 0$.

Consequently, choosing an entire function Q having only simple zeros at $\{z_n\}_{n=1}^\infty$, for each $n \in \mathbb{N}$ there exists an entire function A_n without zeros such that $(z-z_n)S_n(z) = Q(z)A_n(z)$, $z \in \mathbb{C}$. Next, we prove that there exists an entire function A without zeros and a sequence $\{\sigma_n\}_{n=1}^\infty$ in $\mathbb{C}\setminus\{0\}$ such that $A_n(z) = \sigma_n A(z)$ for all $z \in \mathbb{C}$. For $m \neq n$ the function $\frac{z-z_n}{z-z_m} S_n(z)$ in $\mathcal{H}_K$ has its zeros at $\{z_r\}_{r \neq m}$. Thus, the sampling formula (9) gives

$$\frac{z - z_n}{z - z_m} S_n(z) = [(z_m - z_n) S_n'(z_m)] \frac{S_m(z)}{a_m}, \quad z \in \mathbb{C}.$$

Fixing $m = 1$, we conclude that $A_n(z) = \sigma_n A(z)$ where $A = A_1$ and $\sigma_n = (z_1 - z_n) S_n'(z_1) \neq 0$ for $n \in \mathbb{N} \setminus \{1\}$ and $\sigma_1 = 1$. Hence, $S_n(z) = \frac{\sigma_n Q(z) A(z)}{z - z_n}$ for $z \neq z_n$ and $S_n(z_n) = a_n = \sigma_n Q'(z_n) A(z_n)$. Substituting in (9) we obtain the Lagrange-type interpolation series (13) where $P(z) = A(z) Q(z)$.

For the necessary condition, assume that the sampling formula in $\mathcal{H}_K$ takes the form of a Lagrange-type interpolation series (13). Given $g \in \mathcal{H}_K$, there exists $x \in \mathcal{H}$ such that $g(z) = \langle K(z), x \rangle$, $z \in \mathbb{C}$. Assuming that $g(w) = 0$, we have to prove that the function $g(z)/(z - w)$ belongs to $\mathcal{H}_K$. The sampling expansion for g at w gives

$$\sum_{n=1}^{\infty} g(z_n) \frac{P(w)}{(w - z_n) P'(z_n)} = 0. \tag{14}$$

We distinguish two cases:

(i) $w \in \mathbb{C} \setminus \{z_n\}_{n=1}^{\infty}$. As $P(w) \neq 0$, from (14) we obtain

$$\sum_{n=1}^{\infty} g(z_n) \frac{1}{(w - z_n) P'(z_n)} = 0.$$

Thus,

$$\begin{aligned} g(z) &= \sum_{n=1}^{\infty} g(z_n) \frac{P(z)}{(z - z_n) P'(z_n)} - \sum_{n=1}^{\infty} g(z_n) \frac{P(z)}{(w - z_n) P'(z_n)} \\ &= (z - w) \sum_{n=1}^{\infty} g(z_n) \frac{P(z)}{P'(z_n)} \frac{1}{(z - z_n)(z_n - w)}. \end{aligned}$$

Therefore, the entire function $G(z) := g(z)/(z - w)$ can be recovered from its samples at $\{z_n\}_{n=1}^{\infty}$ through the formula

$$G(z) = \sum_{n=1}^{\infty} G(z_n) \frac{P(z)}{(z - z_n) P'(z_n)}, \quad z \in \mathbb{C}. \tag{15}$$

Moreover, the function G is in $\mathcal{H}_K$ because $G(z) = \langle K(z), y \rangle_{\mathcal{H}}$, where $y \in \mathcal{H}$ has the expansion $y = \sum_{n=1}^{\infty} \langle y, x_n \rangle y_n$ with respect to the dual Riesz basis

$\{y_n\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$, where the coefficients are given by

$$\left\{\langle y, x_n\rangle := \frac{1}{\overline{z}_n - \overline{w}}\langle x, x_n\rangle\right\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}).$$

Indeed, sampling formula (13) for S_n gives $S_n(z) = a_n\frac{P(z)}{(z-z_n)P'(z_n)}$. Hence, by using the biorthogonality $\langle x_n, y_n\rangle = \delta_{n,m}$, we obtain

$$\langle K(z), y\rangle = \sum_{n=1}^{\infty} \frac{S_n(z)\overline{\langle x, x_n\rangle}}{w - z_n} = G(z), \quad z \in \mathbb{C},$$

where we have used (15), and the result that $\overline{\langle x, x_n\rangle} = g(z_n)/a_n$, $n \in \mathbb{N}$.

(ii) $w = z_m$ for some $m \in \mathbb{N}$. As $g(z_m) = 0$, the sampling expansion for g reads

$$g(z) = \sum_{\substack{n=1\\n\neq m}}^{\infty} g(z_n)\frac{P(z)}{(z - z_n)P'(z_n)}, \quad z \in \mathbb{C}.$$

Setting $P(z) = (z - z_m)Q_m(z)$ we have $P'(z) = Q_m(z) + (z - z_m)Q_m'(z)$ and, hence,

$$P'(z_k) = \begin{cases} (z_k - z_m)Q_m'(z_k) & \text{if } k \neq m\\ Q_m(z_m) & \text{if } k = m\end{cases}$$

Hence,

$$\frac{g(z)}{z - z_m} = \sum_{\substack{n=1\\n\neq m}}^{\infty} \frac{g(z_n)}{z_n - z_m}\frac{Q_m(z)}{(z - z_n)Q_m'(z_n)}, \quad z \in \mathbb{C}. \tag{16}$$

Using the uniform convergence of the series in (16) we deduce that this series defines a continuous function. Hence, taking the limit as $z \to z_m$ we obtain

$$g'(z_m) = \sum_{\substack{n=1\\n\neq m}}^{\infty} \frac{g(z_n)}{z_n - z_m}\frac{Q_m(z_m)}{(z_m - z_n)Q_m'(z_n)} \tag{17}$$

Now we prove that

$$\frac{g(z)}{z - z_m} = \sum_{\substack{n=1\\n\neq m}}^{\infty} \frac{g(z_n)}{z_n - z_m}\frac{P(z)}{(z - z_n)P'(z_n)} + g'(z_m)\frac{P(z)}{(z - z_m)P'(z_m)}. \tag{18}$$

Indeed, substituting (17) into (18) we obtain

$$\sum_{\substack{n=1\\n\neq m}}^{\infty}\left[\frac{g(z_n)}{z_n-z_m}\frac{P(z)}{(z-z_n)P'(z_n)}+\frac{g(z_n)}{z_n-z_m}\frac{Q_m(z)}{(z_m-z_n)Q_m'(z_n)}\right]$$

$$=\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{g(z_n)}{z_n-z_m}\frac{Q_m(z)}{Q_m'(z_n)}\left[\frac{z-z_m}{(z_n-z_m)(z-z_n)}-\frac{1}{z_n-z_m}\right]$$

$$=\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{g(z_n)}{z_n-z_m}\frac{Q_m(z)}{(z-z_n)Q_m'(z_n)}$$

$$=\frac{g(z)}{z-z_m}.$$

Thus, defining $y\in\mathcal{H}$ by the expansion $y=\sum_{n=1}^{\infty}\langle y,x_n\rangle y_n$ where the coefficients $\{\langle y,x_n\rangle\}_{n=1}^{\infty}$ in $\ell^2(\mathbb{N})$ are given by

$$\langle y,x_n\rangle:=\begin{cases}\dfrac{\langle x,x_n\rangle}{\overline{z_n-z_m}} & \text{if } n\neq m\\[2mm] \dfrac{\overline{g'(z_m)}}{a_m} & \text{if } n=m\end{cases}$$

and proceeding as in case (i), it may be shown that

$$\frac{g(z)}{z-z_m}=\langle K(z),y\rangle,\quad z\in\mathbb{C},$$

which proves that the function $g(z)/(z-z_m)$ belongs to $\mathcal{H}_K$. This concludes the proof of the theorem. □

Some comments concerning Theorem 4 are in order:

1. In the proof of Theorem 4 we have found that the entire function P satisfies:

$$(z-z_n)S_n(z)=\sigma_nP(z),\quad z\in\mathbb{C},$$

for some sequence $\{\sigma_n\}_{n=1}^{\infty}\in\mathbb{C}\setminus\{0\}$. In the case where P can be factorized as $P(z)=A(z)Q(z)$, where Q denotes a canonical product having its simple zeros at $\{z_n\}_{n=1}^{\infty}$ and A is an entire function without zeros, then the Lagrange-type interpolation series (13) can be expressed as

$$f(z)=\sum_{n=1}^{\infty} f(z_n)\frac{A(z)}{A(z_n)}\frac{Q(z)}{(z-z_n)Q'(z_n)}, \quad z\in\mathbb{C}.$$

2. In particular, as de Branges space satisfy the ZR property the orthogonal sampling formulas in these spaces, first proved in [16], can be expressed as Lagrange-type interpolation series (see [11] for some nontrivial examples).
3. It is worth to mention that if one particular sampling formula (9) can be written as a Lagrange-type interpolation formula, then the same occurs for all the sampling formulas (9) obtained from other compatible data (7). Besides, if the space $\mathcal{H}_K$ does not satisfy the ZR property, we conclude that it does not exist any data (7) for which the kernel K is an analytic Kramer kernel and the associated sampling formula (9) can be written as a Lagrange-type interpolation series.

4.4. Some Illustrative Examples

Closing the article, we show some examples illustrating Theorems 2 and 4.

4.4.1. *Classical Polynomial Interpolation*

Let $\mathcal{P}_N(\mathbb{C})$ be the set of polynomials with complex coefficients of degree less or equal N. As we proved in Theorem 1, $\mathcal{P}_N(\mathbb{C})$ coincides with the corresponding $\mathcal{H}_K$ space where $K(z):=\sum_{n=0}^{N}\mathbf{p_n}z^n$ being $\{\mathbf{p_0},\mathbf{p_1},\ldots,\mathbf{p_N}\}$ any basis for the euclidean space $\mathcal{H}:=\mathbb{C}^{N+1}$. Consider $N+1$ different points $\{z_n\}_{n=0}^{N}$ in $\mathbb{C}$; it is easy to prove that K is an analytic Kramer kernel with respect the data $\{z_n\}_{n=0}^{N}$ and $\{a_n=1\}_{n=0}^{N}$. Indeed, the set $\{K(z_n)=\mathbf{q_n}\}_{n=0}^{N}$ is linearly independent in $\mathbb{C}^{N+1}$ by using Vandermonde determinants, that is, it forms a (Riesz) basis for $\mathbb{C}^{N+1}$. Thus, Theorems 2 and 4 give, for any $f\in\mathcal{P}_N(\mathbb{C})$

$$f(z)=\sum_{n=0}^{N} f(z_n)S_n(z)=\sum_{n=0}^{N} f(z_n)\frac{P(z)}{(z-z_n)P'(z_n)}, \quad z\in\mathbb{C},$$

where $S_n(z)=\langle K(z),\mathbf{q_n^*}\rangle$, being $\{\mathbf{q_n^*}\}_{n=0}^{N}$ the dual basis of $\{\mathbf{q_n}\}_{n=0}^{N}$ in $\mathbb{C}^{N+1}$, and $P(z)=\prod_{n=0}^{N}(z-z_n)$.

4.4.2. *The Paley–Wiener–Levinson Theorem Revisited*

Let $\{z_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{C}$ for which $\sup_n |\mathrm{Re}\, z_n - n| < 1/4$ and $\sup_n |\mathrm{Im}\, z_n| < \infty$. It is known that the system $\{e^{iz_n w}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi,\pi]$ (see [20, p. 196]). The Fourier kernel $K(z) = \frac{e^{iz\cdot}}{\sqrt{2\pi}} \in L^2[-\pi,\pi]$ is an analytic Kramer kernel for the data $\{z_n\}_{n\in\mathbb{Z}}$ and $\{a_n = 1\}_{n\in\mathbb{Z}}$. Thus, Theorems 2 and 4 give, for any $f \in PW_\pi$

$$f(z) = \sum_{n=-\infty}^{\infty} f(z_n) S_n(z) = \sum_{n=-\infty}^{\infty} f(z_n) \frac{P(z)}{(z - z_n)P'(z_n)}, \qquad z \in \mathbb{C},$$

where, for $n \in \mathbb{Z}$, the sampling function $S_n(z) = \langle K(z), h_n\rangle_{L^2[-\pi,\pi]}$, being $\{h_n(w)\}_{n\in\mathbb{Z}}$ the dual Riesz basis of $\{e^{iz_n w}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ in $L^2[-\pi,\pi]$, and P is the entire function having only simple zeros at $\{z_n\}_{n\in\mathbb{Z}}$. Since a result from Titchmarsh [18] assures that the functions in PW_π are completely determined by their zeros, we derive that, up to a constant factor, the entire function P coincides with the infinite product

$$(z - z_0) \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right)\left(1 - \frac{z}{z_{-n}}\right).$$

Indeed, the function $S_0 \in PW_\pi$ has only simple zeros at $\{z_m\}_{m\neq 0}$ $(S_0(z_m) = \delta_{0,m})$. Suppose on the contrary that $s \notin \{z_m\}_{m\neq 0}$ is a zero of S_0. According to the classical Paley–Wiener theorem, the function $S(z) := (z - z_0)S_0(z)/(z - s)$ belongs to PW_π and vanishes at every z_n. If we take into account the completeness of the Riesz basis $\{e^{iz_n w}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$, this implies that $S \equiv 0$, a contradiction. Therefore, by using the Titchmarsh's result, the function S_0 coincides, up to a constant factor, with the (convergent) product $\prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)\left(1 - \frac{z}{z_{-n}}\right)$. Since Theorem 4 gives $(z - z_n)S_n(z) = \sigma_n P(z)$ for all $n \in \mathbb{Z}$, we obtain the desired result.

4.4.3. *Finite Cosine Transform*

It is known that any function $f(z) = \langle \cos zx, F(x)\rangle_{L^2[0,\pi]}$, $z \in \mathbb{C}$, where $F \in L^2[0,\pi]$, can be expanded as the sampling formula [13, p. 5]

$$f(z) = f(0)\frac{\sin \pi z}{\pi z} + \frac{2}{\pi}\sum_{n=0}^{\infty} f(n) \frac{(-1)^n z \sin \pi z}{z^2 - n^2}, \qquad z \in \mathbb{C}.$$

This sampling formula cannot be expressed as a Lagrange-type interpolation series since, as we noticed in section 3, the corresponding $\mathcal{H}_K$ space does not satisfy the ZR property.

4.4.4. *An Example Involving a Sobolev Space*

Finally, we give an example taken from [10] of a RKHS $\mathcal{H}_K$, built from the Sobolev Hilbert space $\mathcal{H} := H^1(-\pi, \pi)$, where the ZR property fails. Namely, consider the Sobolev Hilbert space $H^1(-\pi, \pi)$ with its usual inner product

$$\langle f, g\rangle_1 = \int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\,dx + \int_{-\pi}^{\pi} f'(x)\,\overline{g'(x)}\,dx, \quad f, g \in H^1(-\pi, \pi).$$

The sequence $\{e^{inx}\}_{n\in\mathbb{Z}} \cup \{\sinh x\}$ forms an orthogonal basis for $H^1(-\pi, \pi)$: It is straightforward to prove that the orthogonal complement of $\{e^{inx}\}_{n\in\mathbb{Z}}$ in $H^1(-\pi, \pi)$ is a one-dimensional space for which $\sinh x$ is a basis. For a fixed $a \in \mathbb{C}\backslash\mathbb{Z}$ we define a kernel

$$\begin{aligned} K_a : \mathbb{C} &\longrightarrow H^1(-\pi, \pi) \\ z &\longrightarrow K_a(z), \end{aligned}$$

by setting

$$[K_a(z)](x) = (z - a)\, e^{izx} + \sin \pi z \, \sinh x, \quad \text{for } x \in (-\pi, \pi).$$

Clearly, K_a defines an analytic Kramer kernel. Expanding $K_a(z) \in H^1(-\pi, \pi)$ in the former orthogonal basis we obtain

$$K_a(z) = [1 - i(z - a)]\,\sin \pi z \,\sinh x + (z - a) \sum_{n=-\infty}^{\infty} \frac{1 + zn}{1 + n^2}\,\mathrm{sinc}(z - n) e^{inx}.$$

As a consequence, Theorem 2 gives the following sampling result in $\mathcal{H}_{K_a}$: Any function $f \in \mathcal{H}_{K_a}$ can be recovered from its samples $\{f(a)\} \cup \{f(n)\}_{n\in\mathbb{Z}}$ by means of the sampling formula

$$f(z) = [1 - i(z - a)]\,\frac{\sin \pi z}{\sin \pi a} f(a) + \sum_{n=-\infty}^{\infty} f(n)\,\frac{z - a}{n - a}\,\frac{1 + zn}{1 + n^2}\,\mathrm{sinc}(z - n).$$

The function $(z - a)\mathrm{sinc}\, z$ belongs to $\mathcal{H}_{K_a}$ since $(z - a)\mathrm{sinc}\, z = \langle K_a(z), 1/2\pi\rangle_1$ for all $z \in \mathbb{C}$. However, by using the sampling formula for $\mathcal{H}_{K_a}$ it is straightforward to check that the function $\mathrm{sinc}\, z$ does not belong to $\mathcal{H}_{K_a}$; as a consequence, the above sampling formula cannot be expressed as a Lagrange-type interpolation series.

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