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On the allowance for support costs in Prager-Rozvany' optimal layout theory

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Abstract A recent study by Rozvany and Sokół discussed an important topic in structural design: the allowance for support costs in the optimization process. This paper examines a frequently used kind of support —that of simple foundation with horizontal reaction by friction— that appears no covered for the Authors' approach. A simple example is examined to illustrate the case and to try of applying the Authors' method; also some solutions from standard standard design method are included.

Keywords Support cost · Michell trusses · fixed boundary problems · free loading problems

1 Introduction

The fixed boundary class of problems was exactly defined by Cox (1965:116-117), as a different class of problems than those covered by Maxwell & Michell design theory. In his remarkable book, Cox shown that although the theories to tackle with these two classes have different optimality criteria, both they lead to optimal layouts on the basis of Hencky-Prandtl nets (1965:96). The optimality criteria for the former class was after formulated by Hemp (1973) in detailed form and by many other authors.

The fixed boundary theory has a well-known drawback: "the reactions such as those at [fixed supports], are in any case carried by some other bodies acting as structures and the true picture of the economy achieved should include the abutments." (Owen, 1965:64). A conspicuous extension of this theory for the allowance of support cost has been derived by Rozvany and Sokół (2012:§3–7) from the optimal layout theory of Prager and Rozvany (1977), "which is based on 'optimal plastic design'" of Prager and Shield (1967). It could hope that with this extension the shortcoming noted by Cox and Owen could be filled.

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2 Foundation with friction

The frequently used foundation with friction (see e.g. Bow, 1873) is composed by a prismatic body with square base $A = a \times a$ and height h. For an allowable soil stress σ_S, $A = |Y|/\sigma_S$, being Y the vertical reaction to be exerted to the structure by the foundation (see Fig. 1a). If the static friction coefficient between foundation and soil is μ, the foundation can bear too an horizontal reaction X such that $|X| \leq -\mu Y$ providing $Y \leq 0$ accordingly with the sign convention of Fig. 1 of Rozvany and Sokół (2012), i.e., the soil under the foundation will be compressed under Y-action. For a given foundation material, the total volume of the foundation can be expressed as λa^3, being λ a given constant dependent of foundation material and soil properties. Let us select the foundation volume as the cost following Authors', then the function costs can be expressed as follows:

$$C(Y) = \begin{cases} \lambda \left(\frac{-Y}{\sigma_S}\right)^{\frac{3}{2}} & \text{if } Y \leq 0 \\ \infty & \text{if } Y > 0 \end{cases} \tag{1}$$

$$C(X) = \begin{cases} 0 & \text{if } Y \leq 0 \text{ and } |X| \leq -\mu Y \\ \infty & \text{if } Y > 0 \text{ or } |X| > -\mu Y \end{cases} \tag{2}$$

The infinity values mean that the corresponding value of the reaction can not be borne by the foundation. Accordingly with Authors' notation, we have:

$$\forall X, Y: \; R(X,Y) \in \left\{ \lambda \left(\frac{-Y}{\sigma_S} \right)^{\frac{3}{2}}, \infty \right\} \tag{3}$$

3 The Authors' method

For the allowance of support cost, the Authors propose to add to the optimality criterion on 'adjoint' strains, new conditions on 'adjoint' displacements at supports points. These displacements are given by generalised gradients of reaction cost functions, i.e., $\partial R/\partial X$ and $\partial R/\partial Y$—replaced by subgradients when appropriated.

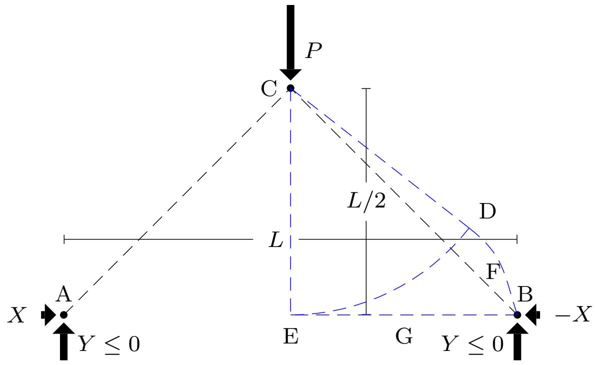
From (3), it is clear that R is a "homogeneous" function of order $\frac{3}{2}$ (Rozvany, 1976:40). But from its "constituents", (1) and (2), it is clear also that the cost function in this case has not a well defined gradient in the X, Y-space, neither fulfils the assumptions of the Prager-Shield optimality criterion (1976:40–48). We have of course well-defined derivatives for some of the constituents, but $R(X,Y) =$ constant does not define a closed, continuous surface, neither $R(X,Y) \leq$ constant does a convex domain—in 'plastic' jargon, perhaps it can be said that we have no "flow rule" in this case. Hence, the case can be tackled neither with Authors' Eqs. (3) and (4) nor with the rules for the special cases 'with non-separable variables' or 'with slope discontinuities' considered by the Authors. Therefore, it is not covered by the Authors' approach in the Writer's view. Of course, this fact does not point out any mistake in Authors', only a key shortcoming in the underlying theories—of equal nature that the drawback noted by Cox and Owen.

Against this conclusion, it can be argued that I have not taken into account the self-weight of the foundation itself and that if I would have done so the function R would have been continuous in half of its domain at least. I do not follow this approach for two main reasons: (i) I want to show a simple example without technical complexity; and (ii) to take into account the self-weight of the foundation would require to make the same with the structure itself, coming us very far of the realm of the Authors' paper.

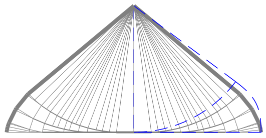
Perhaps the Authors can enlightened the Writer with some other rule that the latter is not aware of.

4 Standard design method

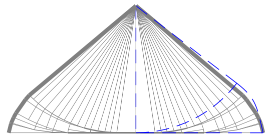
With the standard design method (see e.g. Cox, 1965; Owen, 1965: or Cervera Bravo (2008) for a contemporary writting), each value of $X = -\nu Y$ with $|\nu| \leq \mu$ and $Y \leq 0$ leads to a given Maxwell's problem without any kinematic support conditions (this is the Michell's approach: the "free loading" defined by Cox). All these problems form a family —dependent on ν— that covers all possible solutions for the given design problem. The designer's target is to find the optimal solution within this family.



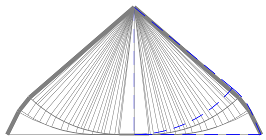
(a) The solution domain is the half plane over AB line. The given load is P. The foundations are under A y B.



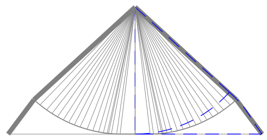
(b) Solution for $\mu = 0$: $\mathcal{V} = 1.435PL$



(c) Solution for $\mu = 0.25$: $\mathcal{V} = 1.310PL$



(d) Solution for $\mu = 0.50$: $\mathcal{V} = 1.190PL$



(e) Solution for $\mu = 0.75$: $\mathcal{V} = 1.084PL$

With $\mu = 1$, $\min \mathcal{V} = PL$ for the layout ACB.

Fig. 1 An illustrative example (Cox, 1965:127).

For the sake of brevity, let us consider an illustrative example, see Fig. 1a. The vertical reactions are statically determined as $|Y| = P/2$. The horizontal ones are restricted to maximum friction force, i.e., to $|\nu| \leq \mu$ for the given friction coefficient μ.

By elementary calculations, the minimal internal force in E with direction AB will be a traction $P/2$ without horizontal reactions, hence the designer would wish $X = P/2$ that corresponds to $\mu = 1$ —and to the optimal solution ACB. Although this value is not attainable with normal soil conditions, this fact leads to the conclusion that we must select $|\nu| = \mu$ when $\mu < 1$, i.e., to design accounting for all the friction force at our disposal as X is free-cost anyway. To fix $|\nu|$ is the same to select one problem from the family.

Once we have one Maxwell's problem, we can search for the optimal layout that minimise the Michell's functional, the "quantity" of structure $\mathcal{V}$ (1904:Eq. (3)). We can compute as an alternative the volume of the structure —to remain within Authors' realm— with the well-known formula (Cox, 1965:87, Eq. (121); Owen, 1965:53, Eq. (18); Barnett, 1966:20, Eq. (5)):

$$V = \frac{1}{2}\left\{\left(\frac{1}{\mathrm{P}}+\frac{1}{\mathrm{Q}}\right)\cdot\mathcal{V}+\left(\frac{1}{\mathrm{P}}-\frac{1}{\mathrm{Q}}\right)\cdot\mathrm{C}\right\} \quad (4)$$

where —following Michell's notation— P, Q are the allowable stresses in tension and compression, respectively; and C is the "static constant" of Owen (1965)—see Michell (1904:Eq. (1)): $\mathrm{C} = -\frac{1}{2}PL(1+\mu)$ in this example. Notice that in doing so we do not account any cost for vertical and horizontal reactions because these reactions are invariable—their cost are constant and finite for all the solutions in the search space. Of course, once a solution is selected as the appropriated one, the designer adds to the cost of the structure the cost of the foundations.

It should be noted too that the optimal layout will be independent of P, Q for any given μ—but, nevertheless, volume varies as indicated by (4). This fact is important and its proof is simple. Let us consider the variation of V within the feasible solution space:

$$\delta V = \frac{1}{2}\left(\frac{1}{\mathrm{P}}+\frac{1}{\mathrm{Q}}\right)\cdot\delta\mathcal{V} \quad (5)$$

as $\delta C = 0$. Hence $\forall \mathrm{P}, \mathrm{Q} : \delta V = 0 \Leftrightarrow \delta\mathcal{V} = 0$ and any optimal layout for $\mathcal{V}$ will be optimal too for V for any P, Q couple.

The non-optimal solutions in Fig. 1 were obtained by a simulated annealing code (Vázquez Espí, 1995). All they suggest the existence of Michell solutions that fulfil his second theorem. These solutions seem to correspond to a fan CDE (a **T**-region following Authors'), and a region EDB, see Fig. 1a. The latter will be composed of a **T**-region (EDFG) and a **R**-region (GFB). With $\mu = 0$, GFB area leads to zero; on the other hand CDE and EDFG areas will be zero with $\mu = 1$. With this hypothesis, the internal force in the arch CDFB will be constant and equal to $N_{\mathrm{CDFB}} = P\sqrt{1+\mu^2}/2$ and the angle of the arch in B with the vertical direction will be exactly $\arctan\mu$. The angle of the fan can then be obtained by elementary equilibrium calculations, and the frontier GF would be obtained by the minimum condition on $\mathcal{V}$. The dashed-lines in Fig. 1 represents the first part of these calculus.

It should be stressed that when a given optimal design —completely defined— will be analysed with any standard, suitable code, the Michell virtual displacement field will not be obtained, because the difference between this field and the actual field of the designed structure: in the latter the horizontal displacement of foundations is non-linear in respect to its contact with the ground—including a perfect rigid one—, the yield tension strain can be different than that of the compression —e.g., because difference in Young's Modulus yet when allowable stresses have equal value—, or the self-weight of both structure and foundations have to be accounted, etc. This fact is a consequence of the static approach of Maxwell and Michell ("free loading" without *any* kinematic support conditions): the displacement fields used in the theory are *virtual* ones ever, and that of the Michell's theorems is only required to be of bounded absolute strain and continuous.

This point is a key one to formulate any design theory, and it is worth of further analysis. The design problem with an unique load conditions is defined by given useful loads and some planes for support the structure (e.g., the soil plane for a standard building). Accordingly with Hemp (1958:1), "The theory [of structural design] ought to be in a position to tackle the design problem directly, that is, to begin with the given forces and to produce by calculation the best structure that will safely carry them". The first pass in the designing process is to look for some appropriate shapes. As at this very time the designer has not selected any structural material, the theory must be formulated without any constitutive equations. Moreover, the designers neither has any structure to be analysed, hence it makes no sense to use structural analysis models as rolled or pinned supports. The outstanding merit of Maxwell in 1870 was to formulate his design theory only with equilibrium equations so it could be useful for the designer. And the equal remarkable improvement of Michell was to use the compatibility equations only to derive his optimality criterion on optimal solutions for Maxwell's problems, being respectful with Maxwell's fundamental axioms.

5 Conclusion

The case of simple foundations with friction has solution within the standard design theory of Maxwell & Michell—the "free loading" of Cox—, but it is not obvious that it can be tackled with the method proposed by the Authors.

The Writer will be grateful to the Authors for further comments and criticism.

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