Partial Differential Equations in Mathematical Biology

On the Keller-Segel System with External Application of Chemoattractant

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First Problem: A System with external application of chemoattractant

$$\frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v), \qquad x \in I\!R^2 \quad t > 0$$

$$-\Delta v = u + 2\pi F_0 \delta(x) \qquad x \in I\!R^2$$

Second Problem: On a Chemotaxis system with logistic term

$$\frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + \lambda u(1-u), \qquad x \in \Omega \quad t > 0$$

$$-\Delta v + v = u \qquad x \in \Omega$$

First Part

1-. Introduction

2-. Auxiliary Problem

$$W(s,t) := \frac{1}{\pi}\int_{B_{\sqrt{s}}(0)} \rho u(\rho,t)d\rho, \qquad s>0,\ t>0,$$

3-. Instantaneous Blow up

4-. Case

$$\int_{\mathbb{R}^2} u_0 > 8\pi - 4\pi F_0$$

5-. Case

$$\int_{\mathbb{R}^2} u_0 < 8\pi - 4\pi F_0$$

1 Introduction

Chemotaxis is the ability of microorganisms to respond to chemical signals by moving along the gradient of the chemical substance, either toward the higher concentration (positive taxis) or away from it (negative taxis).

- Dictyostelium discoideum Bacteria. See Keller-Segel [1970], [1971]
- Tumour-induced angiogenesis. See Anderson-Chaplain [1997], [1998] A. Kubo in the conference (Tuesday Afternoon)
- Astrophysics and gravitational interaction of particles. See Biler [1995], Biler-Hilhorst-Nadzieja [1994]
- Morphogenesis formation of the embryo. See Merkin-Needham-Sleeman [2005], Bollenbach-Kruse-Pantazis-Gonzalez Gaitan-Julicher [2007], C. Stinner in the Poster Sesion of the conference.

Mathematical models of chemotaxis

Keller and Segel [1970], [1971] (after Patlak [1953])

u = bacteria, w = chemoattractant

$$\frac{\partial u}{\partial t} = Q(u,v) + \nabla \cdot (D(u,v)\nabla u - u\chi(v)\nabla v),$$
$$\frac{\partial v}{\partial t} = d\Delta v + h(u,v) \qquad x \in \Omega \quad t > 0$$

We consider $\Omega = R^2$, fast diffusion, and the simplified system

$$\frac{\partial u}{\partial t} = \Delta u - \chi\nabla \cdot (u\nabla v), \qquad x \in I\!R^2 \quad t > 0$$

$$-\Delta w = u + f(x) \qquad x \in I\!R^2$$

+initial datum for u

where f is an external application of chemoattractant.

Jäger and Luckhaus [1992], $\Omega \subset I\!R^2$

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v), \qquad \Omega \times (0, T)$$

$$-\Delta v = u - 1 \qquad \Omega \times (0, T)$$

zero flux on the boundary

- if $\int_\Omega u_0 < 8\pi$, then global solutions exist
- if $\int_\Omega u_0 > 8\pi$, solution blows up in finite time
- See also Herrero-Velazquez [1996]

Case $-\Delta v = u$ in $I\!\!R^2$. See Biler [1995a] [1995b], Blanchet-Dolbeault-Perthame [2006] Blanchet-Carrillo-Masmoundi [2008], Naito-Suzyki [2004], Velázquez [2002], [2004].

Under assumptions

$$\int_{I\!\!R^2} (1+|x|^2) u_0 < \infty, \qquad \int_{I\!\!R^2} u_0 \log u_0 < \infty$$

- The subcritical case $\int_{I\!\!R^2} u_0 < 8\pi \longrightarrow$ global existence. See Blanchet-Dolbeault-Perthame [2006].

- The supercritical case $\int_{I\!\!R^2} u_0 > 8\pi$ solutions blows up as a dirac functions. Velázquez [2002]

- The critical case $\int_{I\!\!R^2} u_0 = 8\pi$. The blow up at $t = \infty$ with similar profile than in the supercritical case. Blanchet-Carrillo-Masmoundi [2008].

The mathematical model

- u concentration of living organisms

- v concentration of chemoattractant substance

- we assume $\chi = 1$

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v) \qquad x \in \mathbb{R}^2,\ t > 0$$
$$-\Delta v = u + f(x), \qquad x \in \mathbb{R}^2,$$

where

$$f(x) = \lim_{\varepsilon \to 0} \frac{f_0}{|\omega_\varepsilon|} I_{\omega_\varepsilon} = f_0 \delta(x)$$

where $f_0 \geq 0$.

We introduce higher gradient of v . We dont increse the mass of u.

2 Auxiliary Problem

Radially symmetric solution $u = u(r,t)$

$$W(s,t) := \frac{1}{\pi}\int_{B_{\sqrt{s}}(0)} \rho u(\rho,t)d\rho, \qquad s>0,\ t>0,$$

which satisfies

$$\begin{cases} W_t = 4sW_{ss} + WW_s + 2F_0W_s, & s>0,\ t>0, \\ W(0,t)=0, \quad \lim_{s\to\infty} W(s,t) = \frac{\mu}{\pi}, & t>0, \\ W(s,0) = W_0(s), \qquad s>0, \end{cases}$$

with $F_0 := \frac{f_0}{2\pi}$ and

$$W_0(s) := 2\int_0^{\sqrt{s}} \rho u_0(\rho)d\rho, \qquad s>0.$$

We may consider the cut-off function

$$\chi \in C^\infty([0,\infty)), \quad \chi \equiv 0 \quad \text{on} \quad \left[0, \frac{1}{2}\right],$$

$$\chi \equiv 1 \quad \text{on} \quad [1,\infty), \quad \chi' \geq 0 \quad \text{on } [0,\infty)$$

$$\varepsilon \in (0,1) \qquad \chi^{(\varepsilon)}(s) := \chi\left(\frac{s}{\varepsilon}\right), \qquad s \geq 0.$$

Then

$$\begin{cases} W_t^{(\varepsilon)} = 4sW_{ss}^{(\varepsilon)} + \chi^{(\varepsilon)}(s)W^{(\varepsilon)}W_s^{(\varepsilon)} + 2F_0\chi^{(\varepsilon)}(s)W_s^{(\varepsilon)}, & s>0,\ t>0, \\ W^{(\varepsilon)}(0,t) = 0, \quad \lim_{s\to\infty} W^{(\varepsilon)}(s,t) = \frac{\mu}{\pi}, & t>0, \\ W^{(\varepsilon)}(s,0) = W_0(s), & s>0, \end{cases}$$

- There exists a unique solution $W^{(\varepsilon)}$ to the ε-problem
- $W^{(\varepsilon)}$ is non-increasing with respect to ε
- $W^{(\varepsilon)} \to W$ pointwise and C^2 in Compact sets.
- $W \in L^\infty((0,\infty) \times (0,\infty))$ is a weak solution of the problem in the following sense

$$W(s,t) \to \frac{\mu}{\pi} \ \text{ as } \ s \to \infty \text{ for all } t > 0$$

$$-\int_0^\infty \int_0^\infty \zeta_t W - \int_0^\infty \zeta(\cdot,0) W_0 = 4 \int_0^\infty \int_0^\infty (s\zeta)_{ss} W - \frac{1}{2} \int_0^\infty \int_0^\infty \zeta_s W^2 - \frac{f_0}{2\pi} \int_0^\infty \int_0^\infty \zeta_s W$$

for $\zeta \in C_0^\infty([0,\infty) \times [0,\infty))$,

where $W_0(s) := \frac{1}{\pi} \int_{B_{\sqrt{s}}(0)} u_0(x) dx$ for $s \geq 0$.

3 Instantaneous Blow up

Lemma 1

Let $F_0 > 0$, $\delta > \frac{2-F_0}{2}$ such that $\delta \in (0,1)$. Then there exist positive constants a, b, ξ, k_0, K_0 such that for any $\gamma > 0$,

$$\varphi(s) := \begin{cases} \frac{a}{\gamma^\delta} s^{-\delta} - b & \text{if } 0 < s < \frac{\xi}{\gamma}, \\ e^{-\gamma s} & \text{if } s \geq \frac{\xi}{\gamma}, \end{cases}$$

$\varphi(s) \in W^{2,\infty}_{loc}((0,\infty))$

$4s\varphi_{ss} + (8 - 2F_0)\varphi_s \geq k_0 \gamma \varphi \qquad \text{a.e. in } (0,\infty)$

$$\int_0^\infty \varphi^2(s)|\varphi_s(s)|^{-1} ds \leq \frac{K_0}{\gamma^2}.$$

Theorem 1. Let $F_0 > 0$, and that W_0

-. $W_0 \in W^{1,\infty}((0,\infty))$,

-. $W_{0s} \geq 0$ in $(0,\infty)$

-. $W_0(s) \to \frac{\mu}{\pi}$ as $s \to \infty$ for some $\mu > 0$.

Then for any positive $\alpha > \frac{2-F_0}{2}$ and any $t_0 \geq 0$,

$$\sup_{s>0, t\in(t_0,t_0+\tau)} \frac{W(s,t)}{s^\alpha} = \infty \qquad \text{for all } \tau > 0.$$

In particular

$$\|W_s\|_{L^\infty((0,\infty)\times(t_0,t_0+\tau))} = \infty \qquad \text{for all } \tau > 0.$$

which implies

$$\|u\|_{L^\infty(\mathbb{R}^2\times(t_0,t_0+\tau))} = \infty \qquad \text{for all } \tau > 0.$$

Idea of the proof.

We consider

$$y(t) := \int_0^\infty \varphi(s) W(s,t)ds, \qquad t > 0,$$

and

$$y^{(\varepsilon)}(t) := \int_0^\infty \varphi(s)\chi_\varepsilon W(s,t)ds, \qquad t > 0,$$

multiply the $\varepsilon-$ problem by $\chi_\varepsilon \varphi$ and integrate by parts and take limits as $\varepsilon \longrightarrow 0$ after technical estimates we arrive to

$$y(t) \geq y(t_1) + k_0\gamma \int_{t_1}^t \int_0^\infty \varphi W - \frac{1}{2}\int_{t_1}^t \int_0^\infty \varphi_s W^2 \qquad \text{for all } t \in (t_1, t_0+\tau).$$

Notice that

$$y^2(t) = \Big(\int_0^\infty \varphi W\Big)^2 \leq \Big(\int_0^\infty \varphi^2 |\varphi_s|^{-1}\Big)\cdot\Big(\int_0^\infty |\varphi_s| W^2\Big) \leq \frac{K_0}{\gamma^2}\cdot\int_0^\infty |\varphi_s| W^2 \qquad \text{for } t > 0,$$

therefore we have

$$y(t) \geq y(t_1) + \int_{t_1}^t Ay(\bar{t}) + By^2(\bar{t})d\bar{t} \qquad \text{for all } t \in (t_1, t_0+\tau).$$

for

$$A := k_0\gamma \qquad \text{and} \qquad B := \frac{\gamma^2}{K_0}.$$

By comparison with the following equation

$$z' = Az + Bz^2 \qquad z(t_1) = y(t_1)$$

we have finite time blow up for $T \leq \frac{C}{\gamma}$.

4 Formation of Dirac-type singularities for $\mu > 8\pi - 4\pi F_0$

Theorem 2. Let $F_0 \geq 0$

$$\mu := \int_{I\!R^2} u_0(x)dx < \infty,$$

such that $\mu > 0$. Then, for $\mu > 8\pi - 4\pi F_0$, u satisfies

$$u(x,t) \to \mu\delta(x) \qquad \text{as } t \to \infty.$$

The Proof of the Theorem is given in 3 steps.

Lemma 3

Let $F_0 \geq 0$, and W_0 such that

$$\begin{cases} \text{(H1)} \;\; W_0 \in W^{1,\infty}((0,\infty)), \\ \text{(H2)} \;\; W_{0s} \geq 0 \text{ in } (0,\infty) \qquad \text{as well as} \\ \text{(H3)} \;\; W_0(s) \to \frac{\mu}{\pi} \qquad \text{as } s \to \infty. \end{cases}$$

Then for all $\hat{\mu} \in (8\pi - 4\pi F_0, \mu)$ there exist $s_0 > 0$ and $\underline{W}_0$

$$\underline{W}_0(s) := \begin{cases} 0 & \text{if } s \in [0, s_0], \\ a - \frac{1}{b + cs^\beta} & \text{if } s > s_0, \end{cases}$$

such that

$$\underline{W}_0 \in W^{1,\infty}((0,\infty)) \cap C^2([0,\infty) \setminus \{s_0\})$$

$$\liminf_{s \searrow s_0} \underline{W}_{0s}(s) > 0, \qquad \underline{W}_0(s) \to \frac{\hat{\mu}}{\pi} \text{ as } \; s \to \infty,$$

$$4s\underline{W}_{0ss} + \underline{W}_0 \underline{W}_{0s} + 2F_0 \underline{W}_{0s} = 0 \qquad \text{in } (s_0, \infty)$$

$$\underline{W}_0 \leq W_0 \qquad \text{in } (0,\infty)$$

For

$$a := \frac{\hat{\mu}}{\pi} \qquad b := \frac{1}{2(a + 2F_0 - 4)} \qquad \beta := \frac{a + 2F_0 - 4}{4} \qquad s_0 := \left(\frac{\frac{1}{a} - b}{c}\right)^{\frac{1}{\beta}} \geq s_1.$$

for some large $s_1 > 0$ such that

$$W_0(s) \geq a \qquad \text{for all } s \geq s_1,$$

and $c > 0$.

Lemma 4

Let $F_0 \geq 0$, and assume that $\psi \in C^2((0,\infty))$ is a nonnegative solution of

$$0 = 4s\psi_{ss} + \psi\psi_s + 2F_0\psi_s, \qquad s > 0,$$

with the additional properties $\psi_s \geq 0$ on $(0,\infty)$ and

$$\psi(s) \nearrow \frac{\mu}{\pi} \qquad \text{as } s \to \infty$$

with some $\mu \geq 0$. In that case, if

$$\mu > 8\pi - 4\pi F_0,$$

then

$$\psi \equiv \frac{\mu}{\pi} \qquad \text{in } (0,\infty).$$

Lemma 5

Let $F_0 \geq 0$, and for some

$$\mu > 8\pi - 4\pi F_0$$

we have

$$\begin{cases} \text{(H1)} \;\; W_0 \in W^{1,\infty}((0,\infty)), \\ \text{(H2)} \;\; W_{0s} \geq 0 \text{ in } (0,\infty) \qquad \text{as well as} \\ \text{(H3)} \;\; W_0(s) \to \frac{\mu}{\pi} \qquad \text{as } s \to \infty. \end{cases}$$

Then, W satisfies

$$W(s,t) \to \frac{\mu}{\pi} \qquad \text{as } t \to \infty,$$

the convergence being uniform on compact subsets of $(0, \infty)$.

Idea of the proof:

– Let $\underline{W}$ be the solution for initial data $\underline{W}_0$

– Since $\underline{W}_0 < W_0 \Longrightarrow \underline{W} < W$

– $\underline{W}_t \geq 0 \qquad \text{in } (0,\infty) \times (0,\infty).$

– $\underline{W}(s,t) \nearrow \psi(s) \qquad \text{as } t \to \infty$

– Lemma 4 ends the proof.

5 Emergence of mild singularities for $\mu < 8\pi - 4\pi F_0$

Theorem 3: Let $F_0 \geq 0$, and W_0 such that

$$\begin{cases} \text{(H1)} \;\; W_0 \in W^{1,\infty}((0,\infty)), \\ \text{(H2)} \;\; W_{0s} \geq 0 \text{ in } (0,\infty) \qquad \text{as well as} \\ \text{(H3)} \;\; W_0(s) \to \frac{\mu}{\pi} \qquad \text{as } s \to \infty. \end{cases}$$

for $\mu < 8\pi - 4\pi F_0$. Then, for all $\tau > 0$ there exists $C > 0$ such that

$$W_s(s,t) \leq C(1 + s^{-\frac{F_0}{2}}) \qquad \text{for all } s > 0 \text{ and any } t > \tau.$$

therefore

$$u(x,t) \leq C|x|^{-F_0} \qquad \text{for all } x \in \mathbb{R}^2 \text{ and } t \geq \tau.$$

Moreover, for $p \in [1, \frac{2}{F_0})$ and $\tau > 0$

$$\|u(\cdot,t)\|_{L^p(B_1(0))} \leq C \qquad \text{for all } t \geq \tau.$$

Second Part

Contents

1.- Global bounded solutions

2.- Weak global solutions

3.- Steady states

4.- Asymptotic behaviour

The mathematical model

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u\chi\nabla v) + \lambda u(1-u), \qquad x \in \Omega,\ t > 0$$

$$-\Delta v + v = u, \qquad x \in \Omega,$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \qquad x \in \partial\Omega,$$

$$u(x,0) = u_0(x) \geq 0, \qquad x \in \Omega,$$

Global bounded solutions

– Assumptions:

- $0 \leq u_0 \leq c < \infty$;
- $\lambda > 0$ for $n = 1, 2$;
- $\lambda > \frac{n-2}{n}\chi$ for $n \geq 3$.

$\Longrightarrow$ Global Classical Solution exists

$\lambda = 0$ blows up for $n \geq 3$ and for $n = 2$ if $\int_\Omega u_0 > c(\Omega)$ (Herrero-Velázquez 96)

The result is valid if we replace $u(1-u)$ by $h(u)$ satisfying $h(u) \leq s_0 - s_1 u^2$.

2.-Global weak solutions for arbitrary $\lambda > 0$

(u, v) is a *weak solution* to the problem in $(0, T)$ if

$$u \in L^1((0,T); W^{1,1}(\Omega)), \qquad v \in L^1((0,T); W^{1,1}(\Omega));$$

such that

$$u\nabla v \in L^1((0,T); L^1(\Omega)), \qquad \lambda u(1-u) \in L^1((0,T); L^1(\Omega))$$

and

$$-\int_0^T \int_\Omega u\varphi_t + \int_0^T \int_\Omega \nabla u \cdot \nabla \varphi - \chi \int_0^T \int_\Omega u \nabla v \cdot \nabla \varphi = \int_\Omega u_0 \varphi(0) + \int_0^T \int_\Omega \lambda u(1-u)\varphi;$$

$${}_0^T \int_\Omega \nabla v \cdot \nabla \psi + \int_0^T \int_\Omega v\psi = \int_0^T \int_\Omega u\psi \tag{1}$$

for $\varphi, \psi \in C_0^\infty(\bar{\Omega} \times [0, T))$.

If $u_0 \in L^\gamma(\Omega)$ for $\gamma \in (1, \frac{\chi}{(\chi-\lambda)_+})$ then there exists a global weak solution satisfying

$$u \in L^\infty((0,\infty); L^\gamma(\Omega)) \ \cap \ L^{\gamma+1}_{loc}([0,\infty); L^{\gamma+1}(\Omega)) \ \cap \ L^p_{loc}([0,\infty); W^{1,p}(\Omega)),$$

$$\nabla u^{\frac{\gamma}{2}} \in L^2_{loc}([0,\infty); L^2(\Omega))$$

$$v \in L^\infty((0,\infty); W^{2,\gamma}(\Omega)) \ \cap \ L^{\gamma+1}_{loc}([0,\infty); W^{2,\gamma+1}(\Omega))$$

for

$$p \in (1, \tfrac{2}{3}(1 + \min\{2, \tfrac{\chi}{(\chi-\lambda)_+}\})).$$

3.- Steady states

$$\begin{cases} 0 = \Delta u - \chi \nabla \cdot (u \nabla v) + \lambda u(1-u) & \text{in } \Omega, \\ 0 = \Delta v - v + u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & \text{in } \partial\Omega \end{cases}$$

Definition. (u, v) is a steady state of the problem if $u, v \geq 0$;

$$u, v \in W^{1,1}(\Omega), \qquad u \nabla v \in L^1(\Omega), \qquad u(1-u) \in L^1(\Omega),$$

and satisfy the identities

$$\int_\Omega \nabla u \cdot \nabla \varphi - \chi \int_\Omega u \nabla v \cdot \nabla \varphi = \int_\Omega \lambda u(1-u)\varphi \qquad \text{and}$$

$$\int_\Omega \nabla v \cdot \nabla \psi + \int_\Omega v\psi = \int_\Omega u\psi$$

for all $\varphi \in C^\infty(\bar{\Omega})$ and $\psi \in C^\infty(\bar{\Omega})$.

Regularity of steady states

Lemma

i) Under assumption

$$\lambda > 0 \text{ and } n \le 4 \qquad \text{or} \qquad \lambda > \frac{n-4}{n-2}\chi \text{ and } n > 4;$$

the solution is bounded and $u, v \in C^{1,\alpha}(\overline{\Omega})$ for $\alpha \in (0,1)$.

ii) For any nonconstant solution (u, v), we have

$$exp\{\chi(\min_{x\in\Omega} v(x) - \max_{x\in\Omega} v(x))\} \le u(x) \le exp\{\chi(\max_{x\in\Omega} v(x) - \min_{x\in\Omega} v(x))\} \quad \text{in } \Omega.$$

In particular, if v is bounded $u, v \in C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$.

4.- Asymptotic behavior

Theorem. Under assumptions

$$\lambda > 2\chi, \qquad u_0 \in C_0(\Omega), \qquad 0 < u < c$$

the unique solution (u, v) satisfies

$$|u - 1|_{L^\infty(\Omega)} + |v - 1|_{L^\infty(\Omega)} \longrightarrow 0 \text{ as } t \longrightarrow \infty.$$

Idea of the proof:

$$u_t - \Delta u = -\chi \nabla u \cdot \nabla v + \chi u(u - v) + \lambda u(1 - u)$$

$$-\Delta v + v = u$$

We consider the following system of equations

$$\overline{u}_t = \chi\overline{u}(\overline{u} - \underline{u} + \frac{\lambda}{\chi}(1 - \overline{u})),$$

$$\underline{u}_t = \chi\underline{u}(\underline{u} - \overline{u} + \frac{\lambda}{\chi}(1 - \underline{u})).$$

Step 1.- $\underline{u}$ and $\overline{u}$ exist for $t \in (0, \infty)$;

Step 2.- $\underline{u} \leq \overline{u}$;

Step 3.- $0 < \underline{u} \leq 1 \leq \overline{u}$;

Step 4.- $\lim_{t\to\infty} |\overline{u} - \underline{u}| = 0$.

$$\frac{\overline{u}_t}{\overline{u}} = \chi(\overline{u} - \underline{u} + \frac{\lambda}{\chi}(1 - \overline{u})),$$
$$\frac{\underline{u}_t}{\underline{u}} = \chi(\underline{u} - \overline{u} + \frac{\lambda}{\chi}(1 - \underline{u})).$$

Subtracting

$$\frac{d}{dt}(Ln\frac{\overline{u}}{\underline{u}}) = \chi\left(2(\overline{u}-\underline{u}) + \frac{\lambda}{\chi}(\underline{u}-\overline{u})\right) \;=\; (2\chi-\lambda)(\overline{u}-\underline{u}).$$

Integrating

$$Ln\frac{\overline{u}}{\underline{u}} \leq e^{-\alpha\alpha_0 t}Ln\frac{\overline{u}_0}{\underline{u}_0}$$

and taking limits, we obtain

$$\lim_{t\to\infty} Ln\frac{\overline{u}}{\underline{u}} = 0.$$

Step 5.- (3) + (4) implies

$$|\overline{u}-1| + |\underline{u}-1| \to 0 \qquad \text{as } t\to\infty.$$

By comparison, if $\underline{u}_0 < u_0 < \overline{u}_0$, $\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution.

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