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A Rakhmanov-like theorem for orthogonal polynomials on Jordan arcs in the complex plane

C. Escribano[1], M. A. Sastre[1], A. Giraldo[1] and E. Torrano[1]

[1] *Departamento de Matemática Aplicada, Facultad de Informática, Universidad Politécnica de Madrid*

emails: cescribano@fi.upm.es, masastre@fi.upm.es, agiraldo@fi.upm.es, emilio@fi.upm.es

Abstract

Rakhmanov's theorem establishes a result about the asymptotic behavior of the elements of the Jacobi matrix associated with a measure μ which is defined on the interval $\mathcal{I} = [-1, 1]$ with $\mu' > 0$ almost everywhere on $\mathcal{I}$. In this work we give a weak version of this theorem, for a measure with support on a connected finite union of Jordan arcs on the complex plane, in terms of the Hessenberg matrix, the natural generalization of the tridiagonal Jacobi matrix to the complex plane.

Key words: Hessenberg matrix, regular measures, Riemann map.

1 Introduction

In this paper, we consider regular Borel measures μ defined on subsets of the complex plane which are Jordan arcs, or connected finite union of Jordan arcs, and we show how the support of μ is determined by the entries of the Hessenberg matrix D associated with μ. The Hessenberg matrix is the natural generalization of the tridiagonal Jacobi matrix to the complex plane and, in the particular case of measures with support the unit circle $\mathbb{T}$, the Hessenberg matrix is a Toeplitz matrix.

Our result represents a broader, although weaker, extension of Rakhmanov's theorem to $\mathbb{C}$. In the real case, Rakhmanov's theorem [15, 16] states that, if the support of a Borel measure is $[-1, 1]$ and $\mu' > 0$ almost everywhere in $[-1, 1]$, then $a_n \to \frac{1}{2}$ and $b_n \to 0$, where a_n are the sequences of elements in the subdiagonal and superdiagonal, and b_n are the sequences of elements in the diagonal, in the Jacobi matrix J associated with μ. Moreover, if the support of μ is the interval $[-2a + b, b + 2a]$, then the above limits are, respectively, $a_n \to a$ y $b_n \to b$. Conversely, if we know that $\mu' > 0$ and that the support of μ is a compact connected set of $\mathbb{R}$, knowing the limits of the diagonals

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of J we could obtain the support of μ, i.e., if $a_n \to a$ y $b_n \to b$ then the support is $[-2a+b, b+2a]$.

Generalizations of Rakhmanov's theorem to orthogonal polynomials and to orthogonal matrix polynomials on the unit circle has been given in [13] and [22]. The case of orthogonal polynomials in an arc of circumference has been studied in [2]).

There exist some previous results relating the properties of D and the support of μ. For example, if the Hessenberg matrix D defines a subnormal operator [12] in ℓ^2, then the closure of the convex hull of its numerical range agrees with the convex hull of its spectrum. On the other hand, the spectrum of the matrix D contains the spectrum of its minimal normal extension $N = \mathrm{men}(D)$ which is precisely the support of the measure [6].

In this work we show that, in the case of regular measures μ whose support is a Jordan arc or a connected union of Jordan arcs in the complex plane $\mathbb{C}$, the limits of the values at the diagonals of the Hessenberg matrix D of μ, supposing those limits exist, determine the terms of the coefficients of the series expansion of the Riemann map $\phi(z)$ (see [20]) which applies conformally the exterior of the unit disk in the exterior of the support of the measure. As a consequence, the support of μ can be determined just knowing the limits of the values at the diagonals of its Hessenberg matrix D.

For general information on the theory of orthogonal polynomials, we recommend the books [4, 20] by T. S. Chihara and G. Szegö, respectively, and the survey [11] by Golinskii and Totik.

2 Main result

Let $\mu(z)$ be a regular positive Borel measure with compact support Ω in the complex plane. Let $\mathcal{P}$ be the space of polynomials. The associated inner product is given by the expression

$$\langle Q(z), R(z)\rangle_\mu = \int_{\mathrm{supp}(\mu)} Q(z)\overline{R(z)}d\mu(z),$$

for $R, S \in \mathcal{P}$. Then there exists a unique orthonormal polynomials sequence (ONPS) $\{P_n(z)\}_{n=0}^{\infty}$ associated to the measure μ (see [4], [8] or [20]).

In the space $\mathcal{P}^2(\mu)$, closure of the polynomials space $\mathcal{P}$ in $L^2_\mu(\Omega)$, we consider the multiplication by z operator. Let $D = (d_{jk})_{j,k=0}^{\infty}$ be the infinite upper Hessenberg matrix of this operator in the basis of ONPS $\{P_n(z)\}_{n=0}^{\infty}$, hence

$$zP_n(z) = \sum_{k=0}^{n+1} d_{k,n}P_k(z), \quad n \geq 0, \tag{1}$$

with $P_0(z) = 1$ when $c_{00} = 1$.

It is a well known fact that the monic polynomials are the characteristic polynomials of the finite sections of D.

In order to state our main result, we will need that the measure μ is regular with support a connected finite union of Jordan arcs, and we will also need to consider an auxiliar Toeplitz matrix. We next recall the definitions of all these notions.

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A Jordan arc in $\mathbb{C}$ is any subset of $\mathbb{C}$ homeomorphic to the closed interval $[0,1]$ on the real line.

A measure μ is regular if $\lim_{n\to\infty} \frac{1}{\sqrt[n]{\gamma_n}} = \operatorname{cap}(\operatorname{supp}(\mu))$, the capacity of the support of μ, where the γ_n are the conductor coefficients of the orthonormal polynomials, i.e., $P_n(z) = \gamma_n z^n + \ldots$.

An infinite matrix $T = (a_{i,j})_{i,j=0}^{\infty}$ is a Toeplitz matrix if each descending diagonal from left to right is constant, i.e, there exists $(a_i)_{i\in\mathbb{Z}}$ such that $a_{i,j} = a_{i-j}$, for every $i,j \in \mathbb{N}\cup\{0\}$. Given a Toeplitz matrix T, the Laurent series whose coefficients are the entries a_i defines a function known as the symbol of T.

We are now in a position to state and prove the main result of the paper.

Theorem 1. *Let $D = (d_{ij})_{i,j=1}^{\infty}$ be a Hessenberg matrix associated with a measure μ with compact support on the complex plane. Assume that:*

1. *The measure μ is regular with support $\operatorname{supp}(\mu)$ a Jordan arc or a connected finite union of Jordan arcs Γ such that $\mathbb{C}\setminus\Gamma$ is a simply connected set of the Riemann sphere $\mathbb{C}_\infty$.*
2. *There exists a Hessenberg-Toeplitz matrix T such that $D-T$ defines a compact operator in ℓ^2 with its rows in ℓ^1.*

Then, the symbol of T is the Riemann function $\phi : \mathbb{C}_\infty \setminus \overline{\mathbb{D}} \to \mathbb{C}_\infty \setminus \Gamma$

Proof. Since $\operatorname{supp}(\mu) = \Gamma$ is a compact set and $\mathbb{C}_\infty\setminus\Gamma$ is connected, we can apply Merguelyan's theorem [9, p.97] which asserts that every continuous function in Γ can be uniformly approximated by polynomials. Since the set of continuous functions with compact support is dense in $L^2_\mu(\Gamma)$, then $L^2_\mu(\Gamma) = P^2_\mu(\Gamma)$. Therefore, D defines a normal operator in ℓ^2, hence $\sigma(D) = \Gamma$ [5, 21]. Since

$$\sigma(D)\setminus\sigma_{ess}(D) = \{\lambda \mid \lambda \text{ isolated eigenvalue if finite multiplicity}\},$$

where $\sigma_{ess}(D)$ is the essential spectrum of D (see, for example, [6] for its definition), and the support is connected, then it has not isolated points, and $\Gamma = \sigma(D) = \sigma_{ess}(D)$.

Consider now $K = D - T$ which, by hypothesis is a compact operator. Then all its diagonals converge to 0 [1] and hence the limits

$$\lim_n d_{n-k,n} = d_{-k}, \quad k = -1, 0, 1, 2, \ldots$$

exist, and the matrix T is

$$T = \begin{pmatrix} d_0 & d_{-1} & d_{-2} & \cdots \\ d_1 & d_0 & d_{-1} & \cdots \\ 0 & d_1 & d_0 & \cdots \\ 0 & 0 & d_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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Since the essential spectrum is invariant via compact perturbations [5], we have that $\sigma_{ess}(D) = \sigma_{ess}(T)$. Moreover, T is bounded in ℓ^2 and hence the rows and columns of T are in ℓ^2. Therefore, $(d_1, d_0, d_{-1}, d_{-2}, \ldots) \in \ell^2$.

The elements $d_{n,n-1}$ of the subdiagonal of the matrix D agree with the quotients γ_{n-1}/γ_n. Since $\lim_{n\to\infty} d_{n+1,n} = d_1$, then

$$d_1 = \lim_{n\to\infty} \frac{\gamma_{n-1}}{\gamma_n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\gamma_n}}.$$

On the other hand, since μ is regular, then [19, p.100]

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{\gamma_n}} = \operatorname{cap}(\operatorname{supp}(\mu)).$$

Therefore, $d_1 = \operatorname{cap}(\operatorname{supp}(\mu))$.

Consider now the Laurent series

$$d(z) = d_1 z + d_0 + \frac{d_{-1}}{z} + \frac{d_{-2}}{z^2} + \cdots$$

We see now that the fact that $(d_1, d_0, d_{-1}, \ldots) \in \ell^2$ implies that $d(z)$ is analytic for every z such that $1 < |z| < \infty$.

If $|z| > 1$, then $\dfrac{1}{|z|} < 1$ and

$$\sum_{k=-1}^{\infty} |d_{-k} z^{-k}| \le \sqrt{\sum_{k=-1}^{\infty} |d_{-k}|^2} \sqrt{\sum_{k=-1}^{\infty} |z^{-k}|^2} < +\infty.$$

Therefore $d(z)$ converges absolutely for every $1 < |z| < \infty$. To see that $d(z)$ is analytic we have just to show that $d'(z)$ exists for every $|z| > 1$. But

$$d\,'(z) = d_1 - \sum_{k=1}^{\infty} k \frac{d_{-k}}{z^{k+1}}$$

where

$$\sum_{k=1}^{\infty} k \left|\frac{d_{-k}}{z^{k+1}}\right| \le \sqrt{\sum_{k=1}^{\infty} |d_{-k}|^2} \sqrt{\sum_{k=1}^{\infty} \frac{k^2}{|z|^{2k+2}}} < +\infty$$

if $|z| > 1$. Hence $d'(z)$ exists for every $|z| > 1$.

Since $(d_1, d_0, d_{-1}, \ldots) \in \ell^1$, then $(d|_{\mathbb{T}})(z)$ is continuous (where $\mathbb{T}$ is the unit circle) and [3, p.10]

$$\Gamma = \sigma_{ess}(T) = d(\mathbb{T}) = \{d_1 w + d_0 + \frac{d_{-1}}{w} + \frac{d_{-2}}{w^2} + \ldots \mid w \in \mathbb{T}\}.$$

We can now apply Theorem 1.1 in [14] to conclude that

$$d : \mathbb{C}_\infty \setminus \operatorname{cl}(\mathbb{D}) \to \mathbb{C}_\infty \setminus \Gamma,$$

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(where $\mathbb{D}$ the unit disk) is an univalent map and, being also analytic, is conformal in $\mathbb{C}_\infty \setminus \mathrm{cl}(\mathbb{D})$.

Consider now the Riemann map

$$\phi(z) = c_1 w + c_0 + \frac{c_{-1}}{w} + \frac{c_2}{w^2} + \dots$$

in $\mathbb{C}_\infty \setminus \Gamma$ which is the unique conformal map which applies the exterior of the unit disk in the exterior of $\Gamma = \mathrm{supp}(\mu)$, which preserves the point at infinity and the direction therein, and which also satisfies $\mathrm{cap}(\Gamma) = c_1$ [20]. The map d satisfies that $d_1 = \mathrm{cap}(\Gamma)$. Moreover, since $d'(\infty) = \phi'(\infty) = d_1 = c_1$, then $d(z)$ preserves the point at infinity and the direction therein. Therefore $d = \phi$. □

3 Examples

As an illustration of the previous theorem we consider the following examples.

Example 1. Consider Γ the segment $[-1, 1]$ in $\mathbb{C}$. The Riemann map ϕ which applies the exterior of the unit disk in the exterior of Γ is

$$\phi(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

By Rakhmanov's theorem, if μ is a Borel measure is $[-1, 1]$ and $\mu' > 0$ almost everywhere in $[-1, 1]$, then $a_n \to \frac{1}{2}$ and $b_n \to 0$, where a_n are the sequences of elements in the subdiagonal and superdiagonal, and b_n are the sequences of elements in the diagonal, in the Jacobi matrix J associated with μ. Note that these are the coefficients of the Riemann map ϕ. Although Theorem 1 does not guarantee the existence of the limits of the diagonals of the Jacobi matrix in any case, in the case that those limits exist, they must agree with the coefficients of μ, even if μ is not absolutely continuous.

Example 2. Let Γ be a cross-like set, and μ the uniform measure on γ. The Riemann map is

$$\phi(z) = \frac{\sqrt{a^2(z^2+1)^2 + b^2(z^2-1)^2}}{2z},$$

where a and b are the length of the horizontal and vertical semi-axis, respectively. In the particular case of $a = b$,

$$\phi(z) = \frac{a\sqrt{2}}{2z}\sqrt{z^4+1}.$$

The series expansion of ϕ is

$$\phi(z) = \frac{\sqrt{a^2+b^2}}{2}\, z + \frac{-2\,b^2 + 2\,a^2}{4\sqrt{a^2+b^2}}\,\frac{1}{z} + \frac{\sqrt{a^2+b^2}\left(\frac{1}{2} - \frac{(-2\,b^2+2\,a^2)^2}{8\,(a^2+b^2)^2}\right)}{2\,z^3} + \mathrm{O}\left(\frac{1}{z^5}\right),$$

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where the first coefficient $\dfrac{\sqrt{a^2+b^2}}{2}$ agrees with the capacity of the support. If $a = b$, the series expansion is

$$\phi(z) = \frac{a\sqrt{2}}{2}\,z + \frac{a\sqrt{2}}{4}\frac{1}{z^3} + \mathrm{O}\left(\frac{1}{z^5}\right).$$

The image under ϕ of the unit circle is shown in Figure 1, where we have included on the right the same result with an interpolation with less steps to give a better insight of the Riemann map.

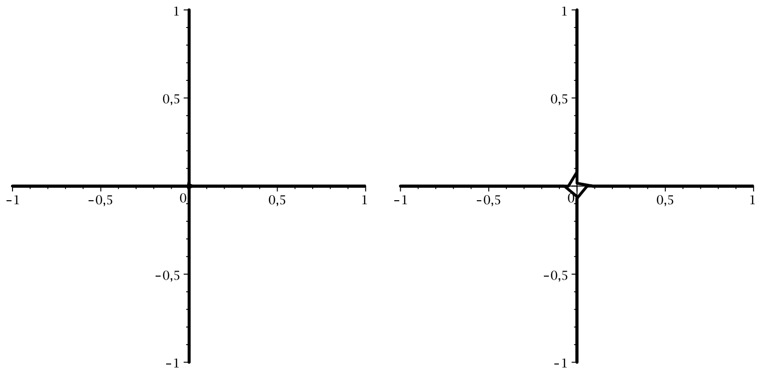

Figure 1: $\phi(\mathbb{T})$ for a cross-like set

There are many instances, however, when the Hessenberg matrix can not computed completely, but only finite sections of it, and it is not possible to compute the limits of the diagonals of D. In this case, it is still possible to compute approximations of the support of the measure μ obtained computing the image of the unit circle under suitable approximations of the Riemann map. Specifically, since the coefficients of the Riemann map are the limits of the elements in each of the diagonals of the Hessenberg matrix, we may consider, as approximations of the Riemann map ϕ, the functions

$$\phi_k(z) = d_{k,k-1}z + d_{k,k} + \sum_{i=1}^{k-1} \frac{d_{k-i,k}}{z^i},$$

where $D = (d_{i,j})$ is the Hessenberg matrix of μ [7].

The result of approximating $\mathrm{supp}(\mu)$ using this method, for $k = 30$, $k = 40$ and $k = 50$, respectively, is shown in Figure 1.

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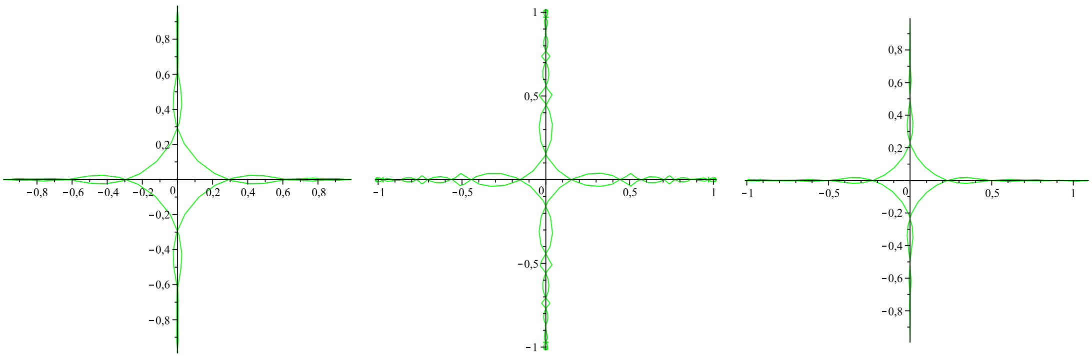

Figure 2: $\phi_k(\mathbb{T})$ for $k = 30$, $k = 40$ and $k = 50$, respectively

Example 3. Consider now Γ an arc of circumference. In this case [10] (see also [17, 18]), there exists a measure for which the diagonals of the Hessenberg matrix stabilize from the second element on. The monic orthogonal polynomials associated to this measure satisfy $\Phi_0(0) = 1$ and $\Phi_n(0) = \dfrac{1}{a}$ $(a > 1)$, if $n \geq 1$, and the corresponding Hessenberg matrix it the following unitary matrix:

$$D = \begin{pmatrix} -\dfrac{1}{a} & -\dfrac{(a^2-1)^{1/2}}{a^2} & -\dfrac{(a^2-1)^{2/2}}{a^3} & -\dfrac{(a^2-1)^{3/2}}{a^4} & -\dfrac{(a^2-1)^{4/2}}{a^5} & \cdots \\ \dfrac{(a^2-1)^{1/2}}{a} & -\dfrac{1}{a^2} & -\dfrac{(a^2-1)^{1/2}}{a^3} & -\dfrac{(a^2-1)^{2/2}}{a^4} & -\dfrac{(a^2-1)^{3/2}}{a^5} & \cdots \\ 0 & \dfrac{(a^2-1)^{1/2}}{a} & -\dfrac{1}{a^2} & -\dfrac{(a^2-1)^{1/2}}{a^3} & -\dfrac{(a^2-1)^{2/2}}{a^4} & \cdots \\ 0 & 0 & \dfrac{(a^2-1)^{1/2}}{a} & -\dfrac{1}{a^2} & \dfrac{(a^2-1)^{1/2}}{a^3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence we know the limits of the diagonals, and we can obtain the sum of these limits. It is easy to check that $D - T$ is compact and that the rows of T are in ℓ^1, and hence the expression of the Riemann map is

$$\begin{aligned} \phi(z) &= \frac{z\left(a - \sqrt{a^2-1}\, z\right)}{\sqrt{a^2-1} - az} \\ &= \frac{\sqrt{a^2-1}}{a} z - \frac{1}{a^2} - \frac{\sqrt{a^2-1}}{a^3 z} - O\left(\frac{1}{z^2}\right), \end{aligned}$$

and we can compute the image under ϕ of the unit circle. The result is shown in Figure 1, where we have included on the right the same result with an interpolation with less steps to give a better insight of the Riemann map.

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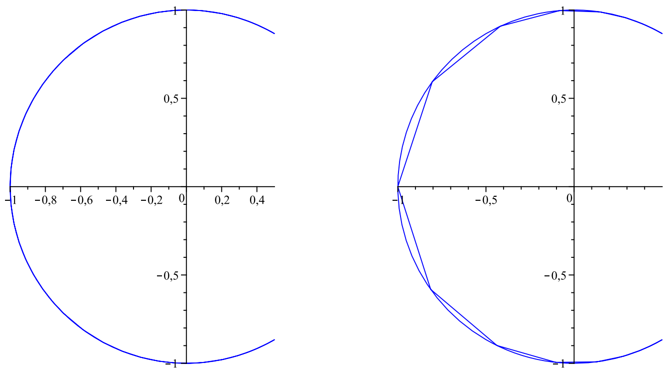

Figure 3: $\phi(\mathbb{T})$ for an arc of circumference

In the following figure we compute several approximations of the support of μ, for the particular case $a = 2$, using the above method, for $k = 10$, $k = 20$ and $k = 30$, respectively.

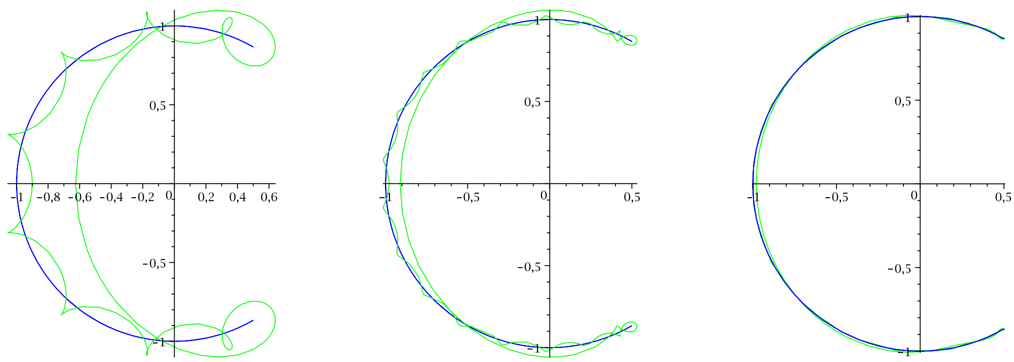

Figure 4: $\phi_k(\mathbb{T})$ for $k = 10$, $k = 20$ and $k = 30$, respectively

Example 4. In the following example we take Γ as the half part of a drop-like set of parametric equation

$$z(t) = \frac{(e^{it})^2}{1 + 2e^{it}}, t \in [0, \pi].$$

and μ the uniform measure on γ. In the following figure we show several approximations of the support of μ using this method, for $k = 5$, $k = 8$ and $k = 11$, respectively.

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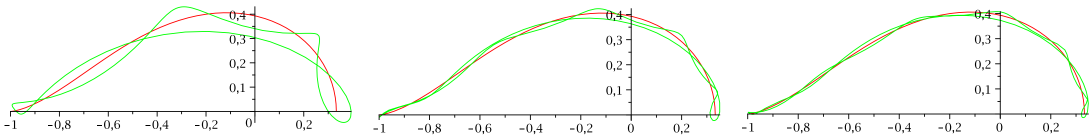

Figure 5: $\phi_k(\mathbb{T})$ for $k = 5$, $k = 8$ and $k = 11$, respectively

Example 5. For the last example we take Γ as the spiral with parametric equation

$$z(t) = t\frac{e^{it}}{6}, t \in [0, 2\pi]$$

and we consider μ the uniform measure on γ. In the following figure we show several approximations of the support of μ using this method, for $k = 1$ and $k = 12$, respectively.

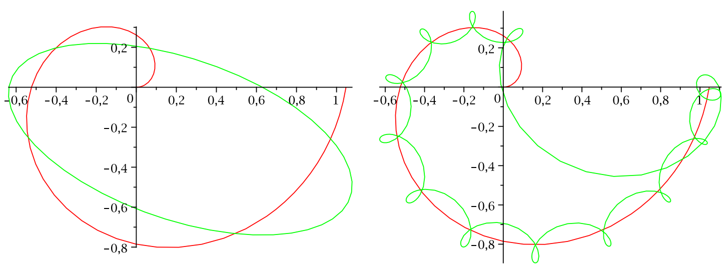

Figure 6: $\phi_k(\mathbb{T})$ for $k = 1$ and $k = 12$, respectively

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