Small eigenvalues of large Hermitian moment matrices

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A B S T R A C T

We consider an infinite Hermitian positive definite matrix M which is the moment matrix associated with a measure μ with infinite and compact support on the complex plane. We prove that if the polynomials are dense in $L^2(\mu)$ then the smallest eigenvalue λ_n of the truncated matrix M_n of M of size $(n+1)\times(n+1)$ tends to zero when n tends to infinity. In the case of measures in the closed unit disk we obtain some related results.

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Measures
Approximation by polynomials

1. Introduction

Let $M=(c_{i,j})_{i,j=0}^{\infty}$ be an infinite Hermitian matrix, i.e., $c_{i,j}=\overline{c_{j,i}}$ for all i,j non-negative integers. We say that the matrix M is *positive definite* (in short, an HPD matrix) if $|M_n|>0$ for each $n\geqslant 0$, where M_n is the truncated matrix of size $(n+1)\times(n+1)$ of M. An HPD matrix M defines an inner product in the vector space $\mathbb{P}[z]$ of all polynomials with complex coefficients in the following way: if $p(z)=\sum_{k=0}^{n}a_k z^k$ and $q(z)=\sum_{k=0}^{m}b_k z^k$ then

$$\langle p,q\rangle=\begin{pmatrix}a_0 & a_1 & \dots & a_n & 0 & \dots\end{pmatrix}\begin{pmatrix}c_{0,0} & c_{0,1} & \cdots & \cdots\\ c_{1,0} & c_{1,1} & \cdots & \cdots\\ \vdots & \vdots & \ddots & \vdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}\overline{b_0}\\ \overline{b_1}\\ \vdots\\ \overline{b_m}\\ 0\\ \vdots\end{pmatrix}. \tag{1}$$

Given an HPD matrix $M=(c_{i,j})_{i,j=0}^{\infty}$, the *complex moment problem* (see e.g. [3,18,19]) entails finding a positive measure μ on $\mathbb{C}$ such that for all $i,j\geqslant 0$

$$c_{i,j}=\int z^i\bar{z}^j\,d\mu(z). \tag{2}$$

The measure μ is called a *representing* measure and $M(\mu)=(c_{i,j})_{i,j=0}^{\infty}$ is the associated moment matrix. This problem has been considered by Atzmon [1] and others [17,12]. Atzmon [1] characterized the moment matrices associated with measures

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in the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C}:\ |z| \leqslant 1\}$ in terms of a property for a bi-sequence $\{c_{i,j}\}_{i,j=0}^{\infty}$ to be *positive definite*. The complex moment problem when a representing measure with compact support exists is completely characterized in [17] and in a general theorem of Berg and Maserick [4]. See also the treatment in [3].

We here always consider positive Borel measures μ, compactly supported with an infinite number of points in its support. The moment matrix associated with μ, denoted by $M(\mu)$, is in this case an HPD matrix and the inner product (1) induced by $M(\mu)$ in $\mathbb{P}[z]$ coincides with the inner product in $L^2(\mu)$:

$$\int p(z)\overline{q(z)}\,d\mu = \langle p(z), q(z)\rangle.$$

Note that $M(\mu)$ is the Gram matrix of the above inner product with respect to $\{z^n\}_{n=0}^{\infty}$, i.e., $M(\mu) = (\langle z^i, z^j\rangle)_{i,j=0}^{\infty}$. Let $\{P_n(z)\}_{n=0}^{\infty}$ denote the sequence of orthonormal polynomials with respect to μ, uniquely determined by the requirements that $P_n(z) = \sum_{k=0}^{n} v_{k,n} z^k$, with positive leading coefficient $v_{n,n}$, and the orthonormality condition:

$$\langle P_n(z), P_m(z)\rangle = 0 \quad \text{if } n \neq m \quad \text{and} \quad \langle P_n(z), P_n(z)\rangle = 1.$$

Let $\{\Phi_n(z)\}_{n=0}^{\infty}$, where $\Phi_n(z) = \frac{1}{v_{n,n}} P_n(z)$, denote the sequence of monic orthogonal polynomials . Recall that, for each $n \in \mathbb{N}$, the n-reproducing kernel at $z, w \in \mathbb{C}$ is defined as $K_n(z,w) = \sum_{k=0}^{n} P_k(z)\overline{P_k(w)}$.

We denote by λ_n the smallest eigenvalue of M_n. It is easy to check that the sequence $\{\lambda_n\}_{n=0}^{\infty}$ is a non-increasing positive sequence and therefore $\lim_{n\to\infty} \lambda_n$ exists.

In the case of Hankel positive definite matrices, which are moment matrices associated with positive measures on $\mathbb{R}$, the large n asymptotics of the smallest eigenvalue λ_n has been studied in the classical papers by Szegö [15] and Widom and Wilf [20]. More recently, Berg, Chen and Ismail [2] have proved that a measure μ on $\mathbb{R}$ is *determinate*, meaning that μ is the only measure with real support having the same moments as μ, if and only if $\lambda_n \to 0$ when n tends to infinity. This new criterion for the determinacy of a measure was our motivation to study the situation in the case of measures supported on $\mathbb{C}$. In this context the situation is completely different. Indeed, every measure with compact support on $\mathbb{C}$ is always determinate since, by Weierstrass theorem, polynomials in z and $\bar{z}$ are dense in the space $L^2(\mu)$. On the other hand, for the normalized Lebesgue measure on the unit circle $\mathbb{T}$, denoted by $\mathbf{m}$, the associated moment matrix is the identity matrix I and obviously $\lambda_n = 1$ for every $n \in \mathbb{N}$. The purpose of the present paper is to relate the asymptotic behaviour of the smallest eigenvalue λ_n with the problem of approximation by polynomials, i.e., the problem of when $P^2(\mu) = L^2(\mu)$, where $P^2(\mu)$ denotes the closure of $\mathbb{P}[z]$ in $L^2(\mu)$.

The paper is organized as follows: Section 2 is devoted to the proof of our main result, Theorem 9, which states that if μ is a positive measure on $\mathbb{C}$ with infinite and compact support such that $L^2(\mu) = P^2(\mu)$, then $\lim_{n\to\infty} \lambda_n = 0$. The converse is not true, as we show in Example 1.

In Section 3, we obtain several related results in the case of measures with support on the closed unit disk $\overline{\mathbb{D}}$. We found that for such measures, the large n asymptotics of the norm of the monic orthogonal polynomials and the smallest eigenvalue depend only on the corresponding large n asymptotics of the restriction to the unit circle $\mathbb{T}$ of the measure. This is not true for the large n asymptotics of the n-reproducing kernels at 0. We finish with some necessary conditions for polynomial approximation in the space $L^2(\mu)$ in terms of these asymptotics.

First, we introduce some notation and terminology. Let M_n be an HPD matrix of order $n+1$ and denote by $\{e_0, \ldots, e_n\}$ the canonical basis in $\mathbb{C}^{n+1}$. Denote by $\{v_0, \ldots, v_n\}$ the unique orthonormal basis in $\mathbb{C}^{n+1}$ with respect to the inner product induced by M_n in such a way that $v_i = (v_{0,i}, \ldots, v_{i,i}, 0, \ldots, 0)$ with $v_{i,i} > 0$. Denote by $\|v\|$ the norm induced by this inner product, i.e., $\|v\|^2 = vM_n v^*$. The vector space $\mathbb{P}_n[z]$ of all polynomials of degree $\leqslant n$ can be obviously identified with $\mathbb{C}^{n+1}$. In the case of M_n being the $(n+1)$ section of the moment matrix $M(\mu)$ associated with the measure μ, if $p(z) = a_0 + \cdots + a_n z^n$, $q(z) = b_0 + \cdots + b_n z^n$, $a = (a_i)_{i=0}^{n}$, $b = (b_i)_{i=0}^{n} \in \mathbb{C}^{n+1}$, we have

$$aM_n b^* = \int p(z)\overline{q(z)}\,d\mu.$$

In particular, if $q(z) = 1 + w_1 z + \cdots + w_n z^n$, we denote by $(1, w) \equiv (1, w_1, \ldots, w_n)$. With this notation:

$$\begin{pmatrix} 1 & w \end{pmatrix} M_n \begin{pmatrix} 1 \\ w^* \end{pmatrix} = \int |q(z)|^2\,d\mu.$$

Given an HPD matrix M_n of order $n+1$ we denote by $\|M_n\|$ the norm of M_n as a linear mapping from $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n+1}$ with the euclidean norm $\|v\|_2$. In this case:

$$\|M_n\| = \sup\{\|M_n v\|_2:\ \|v\|_2 = 1,\ v \in \mathbb{C}^{n+1}\} = \sup\{vM_n v^*:\ v \in \mathbb{C}^{n+1},\ \|v\|_2 = 1\} = \beta_n,$$

where β_n is the largest eigenvalue of M_n. On the other hand,

$$\lambda_n = \inf\{vM_n v^*:\ v \in \mathbb{C}^{n+1},\ \|v\|_2 = 1\}.$$

Let M_1, M_2 be HPD matrices of size $n \times n$. We say that $M_1 \leqslant M_2$ if $vM_1 v^* \leqslant vM_2 v^*$, for every $v \in \mathbb{C}^n$. For infinite HPD matrices the ordering is defined in an analogous way replacing $\mathbb{C}^n$ by the space c_{00} of all complex sequences with only finitely many non-zero entries.

2. Polynomial approximation in $L^2(\mu)$ and asymptotic behaviour of the smallest eigenvalue λ_n

In order to prove the main result we need some lemmas. The following lemma states a certain minimization problem for infinite Hermitian matrices that can be of independent interest.

Lemma 1. *Let M be an HPD matrix and let M_n be the section of order $(n+1)$. Let $\{v_0, \dots, v_n\}$ be the orthonormal basis in $\mathbb{C}^{n+1}$ with respect the inner product induced by M_n with $v_i = (v_{0,i}, \dots, v_{i,i}, 0, \dots, 0)$ for $i = 0, \dots, n$ and $v_{i,i} > 0$. Then,*

$$\frac{1}{\sum_{i=0}^{n} |v_{0,i}|^2} = \inf\left\{ (1 \quad w) M_n \begin{pmatrix} 1 \\ w^* \end{pmatrix} : w \in \mathbb{C}^n \right\} = \frac{1}{e_0 M_n^{-1} e_0^*}, \tag{3}$$

and

$$\frac{1}{\sum_{i=0}^{\infty} |v_{0,i}|^2} = \inf\left\{ (1 \quad w) M \begin{pmatrix} 1 \\ w^* \end{pmatrix} : w \in c_{00} \right\}, \tag{4}$$

where the left side is zero if $\sum_{i=0}^{\infty} |v_{0,i}|^2 = \infty$.

Proof. Each vector $(1, w) \in \mathbb{C}^{n+1}$ can be expressed as $(1, w) = \sum_{i=0}^{n} a_i v_i$, with the additional condition $\sum_{i=0}^{n} a_i v_{0,i} = 1$. Hence,

$$\inf\left\{ (1 \quad w) M_n \begin{pmatrix} 1 \\ w^* \end{pmatrix} : w \in \mathbb{C}^n \right\} = \inf\left\{ \left(\sum_{i=0}^{n} a_i v_i \right) M_n \left(\sum_{i=0}^{n} a_i v_i \right)^* : \sum_{i=0}^{n} a_i v_{0,i} = 1 \right\},$$

and since $v_i M_n v_j^* = \delta_{i,j}$ for $i, j \leqslant n$:

$$= \inf\left\{ \sum_{i=0}^{n} |a_i|^2 : \sum_{i=0}^{n} a_i v_{0,i} = 1 \right\} = \frac{1}{\sum_{i=0}^{n} |v_{0,i}|^2}.$$

This shows the left side of equality (3). In order to prove the right side of equality (3), consider $\alpha_i = \frac{v_i}{v_{i,i}} = (\alpha_{0,i}, \alpha_{1,i}, \dots, 1, 0, \dots, 0)$, hence $\alpha_{0,i} = \frac{v_{0,i}}{v_{i,i}} = \|\alpha_i\| v_{0,i}$. Let C_n be the matrix given by

$$C_n = \begin{pmatrix} 1 & \overline{\alpha_{0,1}} & \dots & \overline{\alpha_{0,n}} \\ 0 & 1 & \dots & \overline{\alpha_{1,n}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

It is clear that $|C_n| = 1$ and C_n^{-1} exists. On the other hand, if we denote by $\mathcal{D}(\|\alpha_0\|^2, \dots, \|\alpha_n\|^2)$ the diagonal matrix of order $(n+1) \times (n+1)$ with entries $\|\alpha_i\|^2$, $i = 0, \dots, n$, then

$$C_n^* M_n C_n = \mathcal{D}\left(\|\alpha_0\|^2, \dots, \|\alpha_n\|^2\right).$$

Therefore $M_n = (C_n^*)^{-1} \mathcal{D}(\|\alpha_0\|^2, \dots, \|\alpha_n\|^2) C_n^{-1}$ and thus M_n^{-1} can be expressed as

$$M_n^{-1} = C_n \mathcal{D}\left(\frac{1}{\|\alpha_0\|^2}, \dots, \frac{1}{\|\alpha_n\|^2} \right) C_n^*.$$

Therefore

$$e_0 M_n^{-1} e_0^* = e_0 C_n \mathcal{D}\left(\frac{1}{\|\alpha_0\|^2}, \dots, \frac{1}{\|\alpha_n\|^2} \right) C_n^* e_0^* = \sum_{k=0}^{n} \frac{|\alpha_{0,k}|^2}{\|\alpha_k\|^2},$$

thus

$$\frac{1}{\sum_{i=0}^{n} |v_{0,i}|^2} = \frac{1}{e_0 M_n^{-1} e_0^*}.$$

The infinite-dimensional version (4) is now an easy consequence. $\square$

If M is a moment matrix, Lemma 1 can be applied to obtain the extremal property of the n-reproducing kernels at 0 (see e.g. [19]).

Corollary 2. *Let $M(\mu)$ be an infinite HPD moment matrix associated with a positive measure μ with support on $\mathbb{C}$ and let $\{P_k(z)\}_{k=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to μ. Then, for every $n \in \mathbb{N}$:*

$$\frac{1}{\sum_{k=0}^{n} |P_k(0)|^2} = \min\left\{ \int |q_n(z)|^2 \, d\mu \colon q_n(z) \in \mathbb{P}_n[z],\ q_n(0) = 1 \right\} = \frac{1}{e_0 M_n^{-1} e_0^*}.$$

Proof. Note that, by using the matricial notation introduced above, we have

$$\min\left\{ \int |q_n(z)|^2 \, d\mu \colon q_n(z) \in \mathbb{P}_n[z],\ q_n(0) = 1 \right\} = \inf\left\{ (1 \quad w)\, M_n \begin{pmatrix} 1 \\ w^* \end{pmatrix} \colon w \in \mathbb{C}^n \right\}.$$

The result is a consequence of Lemma 1, and the fact that $v_{0,k} = P_k(0)$, for every k. $\square$

The following result is analogous to a result for Hankel matrices in [5].

Lemma 3. *Let $M(\mu)$ be an infinite HPD moment matrix associated with a positive measure μ with support on $\mathbb{C}$, let λ_n be the smallest eigenvalue of M_n and consider z_0 with $|z_0| < 1$. Then $\lambda_n \leqslant (\sum_{k=0}^{n} |z_0|^{2k})(\sum_{k=0}^{n} |P_k(z_0)|^2)^{-1}$, for each $n \in \mathbb{N}$. As a consequence,*

$$\lim_{n\to\infty} \lambda_n \leqslant \left((1 - |z_0|^2) \sum_{k=0}^{\infty} |P_k(z_0)|^2 \right)^{-1}.$$

Proof. Since M_n is Hermitian, we have that $\lambda_n = \frac{1}{\|M_n^{-1}\|}$. On the other hand, if $|z_0| < 1$ and $v = (1, z_0, \ldots, z_0^n)$, by some analogous results in [7, p. 52] and [14, p. 377], we have

$$\sum_{k=0}^{n} |P_k(z_0)|^2 = K_n(z_0, z_0) = v M_n^{-1} v^* \leqslant \frac{1}{\lambda_n} \left(\sum_{k=0}^{n} |z_0|^{2k} \right).$$

Therefore $\lambda_n \leqslant (\sum_{k=0}^{n} |z_0|^{2k})(\sum_{k=0}^{n} |P_k(z_0)|^2)^{-1}$. Taking limits when n tends to infinity the result is proved. $\square$

Remark 4. If M is a moment matrix, in general it is not true that $\lim_{n\to\infty} \lambda_n = ((1 - |z_0|^2) \sum_{k=0}^{\infty} |P_k(z_0)|^2)^{-1}$. Indeed, let η be the Lebesgue measure (uniform measure) on the disk $\overline{\mathbb{D}}$; it is well known that the moment matrix associated with η is the diagonal matrix with entries $c_{n,n} = \frac{\pi}{n+1}$ and $P_n(z) = \sqrt{\frac{n+1}{\pi}} z^n$. Consequently $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} c_{n,n} = 0 \neq ((1 - |z_0|^2) \sum_{k=0}^{\infty} |P_k(z_0)|^2)^{-1}$.

The following result is essentially contained in [2,6] in the case of positive measures on $\mathbb{R}$. The same result is true when positive measures on $\mathbb{C}$ are considered. We include it for the sake of completeness:

Lemma 5. *Let $M(\mu)$ be an infinite HPD matrix associated with a positive measure μ on $\mathbb{C}$ and let $\mathbf{m}$ be the normalized Lebesgue measure on $\mathbb{T}$. Then the following are equivalent:*

(1) $\lim_{n\to\infty} \lambda_n > 0$.
(2) *There exists $c > 0$ such that, for every polynomial $p(z)$,*

$$\int |p(z)|^2 \, d\mu \geqslant c \int_{\mathbb{T}} |p(z)|^2 \, d\mathbf{m}.$$

Proof. Note that, if $p(z) = a_0 + \cdots + a_n z^n$ and $a = (a_0, \ldots, a_n)$, then

$$\int |p(z)|^2 \, d\mu = a M_n a^* \geqslant \lambda_n \sum_{k=0}^{n} |a_k|^2 = \lambda_n \int_{\mathbb{T}} |p(z)|^2 \, d\mathbf{m},$$

and from this the conclusion follows. $\square$

Lemma 6. *Let $r \in (0, 1)$ and let $M(\mu)$ be an infinite moment matrix associated with a positive measure μ on $\overline{\mathbb{D}}(0; r)$. Then, for every $n \in \mathbb{N}$:*

$$\|M_n\| \leqslant \mu(\overline{\mathbb{D}}(0; r)) \frac{1}{1 - r^2}.$$

Proof. Note that if β_n is the largest eigenvalue of $M_n = (c_{i,j})_{i,j=0}^n$ then

$$\|M_n\| = \beta_n \leqslant Trace(M_n) = \sum_{i=0}^n c_{i,i} = \sum_{i=0}^n \int_{\overline{\mathbb{D}}(0;r)} z^i \overline{z}^i \, d\mu \leqslant \mu(\overline{\mathbb{D}}(0;r)) \sum_{i=0}^n r^{2i} \leqslant \mu(\overline{\mathbb{D}}(0;r)) \frac{1}{1-r^2}. \qquad \square$$

Following the ideas in [16] we have:

Lemma 7. *Let μ be a positive measure with compact support on $\mathbb{C}$. Assume $z_0 \notin$ Supp(μ), then*

$$\frac{1}{D^2 \sum_{k=0}^\infty |P_k(z_0)|^2} \leqslant dis^2\left(\frac{1}{z-z_0}, \mathbb{P}[z]\right) \leqslant \frac{1}{d^2 \sum_{k=0}^\infty |P_k(z_0)|^2},$$

where $d = \min\{|z-z_0|: z \in \text{Supp}(\mu)\}$, $D = \max\{|z-z_0|: z \in \text{Supp}(\mu)\}$.

Proof. We have that

$$\begin{aligned} dis^2\left(\frac{1}{z-z_0}, P^2(\mu)\right) &= \lim_{n\to\infty} \inf_{q_n(z)\in\mathbb{P}_n[z]} \int \left|\frac{1}{z-z_0} - q_n(z)\right|^2 d\mu \\ &= \lim_{n\to\infty} \inf_{q_n(z)\in\mathbb{P}_n[z]} \int \left|1-(z-z_0)q_n(z)\right|^2 \frac{d\mu}{|z-z_0|^2} \\ &= \lim_{n\to\infty} \inf_{q_{n+1}(z)\in\mathbb{P}_{n+1}[z],\ q_{n+1}(z_0)=1} \int \left|q_{n+1}(z)\right|^2 \frac{d\mu}{|z-z_0|^2}. \end{aligned}$$

By the extremal property of the n-reproducing kernel at z_0 (see [19]) we have that

$$\frac{1}{\sum_{k=0}^{n+1} |P_k(z_0)|^2} = \inf_{q_{n+1}(z)\in\mathbb{P}_{n+1}[z],\ q_{n+1}(z_0)=1} \int \left|q_{n+1}(z)\right|^2 d\mu$$

and from this the result follows. $\square$

As a consequence of Lemma 7 we have:

Corollary 8. *Let μ be a positive measure with infinite and compact support on $\mathbb{C}$. Assume that $z_0 \notin$ Supp(μ), then the following are equivalent:*

(1) *The function $\frac{1}{z-z_0} \in P^2(\mu)$.*
(2) $\sum_{n=0}^\infty |P_n(z_0)|^2 = \infty$.

We now prove the main result:

Theorem 9. *Let $M(\mu)$ be the moment matrix associated with a positive measure μ with infinite and compact support on $\mathbb{C}$. If the polynomials are dense in $L^2(\mu)$, i.e., $P^2(\mu) = L^2(\mu)$, then*

$$\lim_{n\to\infty} \lambda_n = 0,$$

where λ_n is the smallest eigenvalue of the section of order $(n+1)$ of $M(\mu)$.

Proof. Consider $R > 0$ such that $\Omega = \text{Supp}(\mu) \subset \{z \in \mathbb{C}: |z| \leqslant R\}$. In order to prove the result we consider several cases:

First case: $0 \notin \text{Supp}(\mu)$. Since $1/z \in P^2(\mu)$, it follows, by Corollary 8, that $\sum_{n=0}^\infty |P_n(0)|^2 = \infty$ and then, by Lemma 3, $\lim_{n\to\infty} \lambda_n = 0$.

Second case: $0 \in \text{Supp}(\mu)$ and $\mu(\{0\}) = 0$. In this case $\lim_{r\to 0} \mu(\overline{\mathbb{D}}(0;r)) = 0$. Assume that $P^2(\mu) = L^2(\mu)$ and there is $\lambda > 0$ such that $\lambda_n \geqslant \lambda$, for all $n \in \mathbb{N}$. Consider r small enough to ensure that $\mu(\overline{\mathbb{D}}(0;r))\frac{1}{1-r^2} \leqslant \frac{\lambda}{2}$. We denote by μ_r^c the restriction to the set $\Omega \setminus \overline{\mathbb{D}}(0;r)$ of the measure μ, and μ_r the restriction to $\overline{\mathbb{D}}(0;r)$ of μ. Let $n \in \mathbb{N}$ be fixed and consider $v = (v_0, \ldots, v_n) \in \mathbb{C}^{n+1}$ with $\sum_{k=0}^n |v_k|^2 = 1$. Since $M_n(\mu) = M_n(\mu_r) + M_n(\mu_r^c)$ and $\|M_n(\mu_r)\| \leqslant \frac{\lambda}{2}$ by Lemma 6 it follows that

$$vM_n(\mu_r^c)v^* = vM_n(\mu)v^* - vM_n(\mu_r)v^* \geqslant vM_n(\mu)v^* - \frac{\lambda}{2}.$$

By taking the infimum in both sides of the inequality we obtain

$$\lambda_n(M(\mu_r^c)) \geqslant \lambda_n(M(\mu)) - \frac{\lambda}{2} \geqslant \frac{\lambda}{2}. \tag{5}$$

On the other hand, since $L^2(\mu) = P^2(\mu)$ we have that $L^2(\mu_r^c) = P^2(\mu_r^c)$. Now, by applying the first case to the measure μ_r^c we obtain that

$$\lim_{n\to\infty} \lambda_n(M(\mu_r^c)) = 0.$$

This contradicts (5).

Third case: $0 \in \mathrm{Supp}(\mu)$ and $\mu(\{0\}) > 0$. Assume again that $P^2(\mu) = L^2(\mu)$ and there is $\lambda > 0$ such that $\lambda_n \geqslant \lambda > 0$ for all $n \in \mathbb{N}$. By Lemma 5, for every polynomial $p(z)$, we have

$$\int |p(z)|^2 \, d\mu \geqslant \lambda \int_{\mathbb{T}} |p(z)|^2 \, d\mathbf{m}.$$

Let $\Omega_0 = \Omega \setminus \{0\}$ and let μ_0 be the restriction to this set of the measure μ. Consider a polynomial $q(z)$ then

$$R^2 \int_{\Omega_0} |q(z)|^2 \, d\mu_0 \geqslant \int_{\Omega} |zq(z)|^2 \, d\mu \geqslant \lambda \int_{\mathbb{T}} |zq(z)|^2 \, d\mathbf{m} = \lambda \int_{\mathbb{T}} |q(z)|^2 \, d\mathbf{m}.$$

Therefore

$$\int_{\Omega_0} |q(z)|^2 \, d\mu_0 \geqslant \frac{\lambda}{R^2} \int_{\mathbb{T}} |q(z)|^2 \, d\mathbf{m}.$$

This means that for every $n \in \mathbb{N}$ we have

$$\lambda_n(M(\mu_0)) \geqslant \frac{\lambda}{R^2}.$$

Since μ_0 is a measure satisfying that $\mu_0(\{0\}) = 0$, applying the second case we obtain that $L^2(\mu_0) \neq P^2(\mu_0)$. Again, this is not possible because $L^2(\mu) = P^2(\mu)$. This ends the proof of the theorem. $\square$

Remark 10. The converse of Theorem 9 is not true. We provide two examples, the second one involving a Toeplitz matrix.

Example 1. Consider the Pascal matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & 2 & 3 & 4 & \dots \\ 1 & 3 & 6 & 10 & \dots \\ 1 & 4 & 10 & 20 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

i.e., $c_{i,j} = \binom{i+j}{i}$. It is known (see e.g. [9]) that M is the moment matrix associated with the normalized Lebesgue measure μ in the circle with center 1 and radius 1. The sequence of orthonormal polynomials is given by $P_n(z) = (z-1)^n$ for every $n \in \mathbb{N}_0$. Thus $\sum_{n=0}^{\infty} |P_n(0)|^2 = \infty$ and, by Lemma 3, it follows that $\lim_{n\to\infty} \lambda_n = 0$. On the other hand, $P^2(\mu) \neq L^2(\mu)$ since $\frac{1}{z-1} \in L^2(\mu)$ and cannot be approximated by polynomials since, by Lemma 7, it follows that $dis^2(\frac{1}{z-1}, P^2(\mu)) = \frac{1}{\sum_{n=0}^{\infty} |P_n(1)|^2} = 1$.

Example 2. The Toeplitz matrix:

$$M = \begin{pmatrix} 2 & 1 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & \dots \\ 0 & 1 & 2 & 1 & \dots \\ 0 & 0 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

M is the moment matrix associated with the measure μ on the unit circle given by $d\mu(e^{i\theta}) = (w(\theta)/2\pi)\,d\theta$ with

$$w(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{-ik\theta} = e^{\theta i} + 2e^0 + e^{-i\theta} = 2 + 2\cos\theta.$$

Using the results in [11], it is easy to deduce that $\lambda_n = 2 + 2\cos\frac{(n+1)\pi}{n+2}$ and hence $\lim_{n\to\infty}\lambda_n = 0$. On the other hand, we show that $P^2(\mu)$ is not dense in $L^2(\mu)$. To prove it, we first show that for every n we have

$$|M_n| = n + 2.$$

We use induction on n. If $n = 1$, then $|M_1| = 3$. Assume that the result is true for $n \leqslant k$. Then, expanding the determinant we obtain

$$|M_{k+1}| = 2|M_k| - |M_{k-1}| = k + 3.$$

Since $\lim_{n\to\infty} e_0 M_n^{-1} e_0^* = \lim_{n\to\infty} |M_{n-1}|/|M_n| = 1$, then, by Corollary 2 and Corollary 8, we have that $\frac{1}{z} \notin P^2(\mu)$ and consequently $P^2(\mu)$ is not dense in $L^2(\mu)$. This result can also be proved using Szegö's theorem (see e.g. [8]).

Remark 11. Theorem 9 is true for measures with compact support on the real line. Moreover, measures compactly supported on the real line are always determinate (or equivalently $\lim_{n\to\infty}\lambda_n = 0$ by [2]) and $P^2(\mu) = L^2(\mu)$.

Remark 12. In Theorem 9 we cannot remove the assumption of boundedness of the support since there are measures with non-bounded support on the real line such that $P^2(\mu) = L^2(\mu)$ and nevertheless $\lim_{n\to\infty}\lambda_n > 0$: they are the N-extremal measures (see e.g. [13]).

Proposition 13. *Let μ be a positive measure with compact support on $\mathbb{C}$ such that $0 \notin \mathrm{Supp}(\mu)$. The following are equivalent:*

(1) $\sum_{n=0}^{\infty} |P_n(0)|^2 = \infty$.
(2) $P^2(\mu) = \overline{[1, z, \frac{1}{z}, z^2, \frac{1}{z^2}, \ldots]}$.

Proof. Let $\alpha > 0$ and $R > 0$ be such that $\mathrm{Supp}(\mu) \subset \{z \in \mathbb{C}\colon \alpha \leqslant |z| \leqslant R\}$. We prove (1) implies (2). Note that by Corollary 8 we have that (1) is equivalent to the fact that $\frac{1}{z} \in P^2(\mu)$ and, consequently, $P^2(\mu) = \overline{[\frac{1}{z}, 1, z, z^2, \ldots]}$. Then

$$\int \left|\frac{1}{z^2} - v_0\frac{1}{z} - v_1 - v_2 z - \cdots - v_n z^{n-1}\right|^2 d\mu = \int \frac{1}{|z|^2}\left|\frac{1}{z} - v_0 - v_1 z - \cdots - v_n z^n\right|^2 d\mu$$
$$\leqslant \frac{1}{\alpha^2}\, dis^2\left(\frac{1}{z}, \mathbb{P}_n[z]\right).$$

By taking the infimum over $v_0, \ldots, v_n$ and $n \in \mathbb{N}$ we have that

$$dis^2\left(\frac{1}{z^2}, P^2(\mu)\right) \leqslant \frac{1}{\alpha^2}\, dis^2\left(\frac{1}{z}, P^2(\mu)\right).$$

Proceeding in the same way we can prove that $\frac{1}{z^k} \in P^2(\mu)$ for all $k \in \mathbb{N}$.

(2) implies (1) is a consequence of Corollary 8. $\square$

Remark 14. Note that the condition of $0 \notin \mathrm{Supp}(\mu)$ cannot be removed in Proposition 13. Indeed, consider any measure with $0 \in \mathrm{Supp}(\mu)$ being a point mass with $\mu(\{0\}) = d > 0$, and such that $\mathrm{Supp}(\mu)$ is a compact set with empty interior and with K^c a connected set. By Mergelyan theorem (see e.g. [10]), the polynomials are dense in the space of the continuous functions on K with the uniform norm. Consequently, the polynomials are dense in $L^2(\mu)$, that is, $P^2(\mu) = L^2(\mu)$ and therefore (2) holds. However, since $M(\mu) \geqslant M(d\delta_0)$ by Lemma 1

$$\frac{1}{\sum_{k=0}^{n} |P_k(0)|^2} \geqslant \min_{v\in\mathbb{C}^n} (1 \quad v)\, M_n(d\delta_0) \begin{pmatrix} 1 \\ v^* \end{pmatrix} = d > 0,$$

and therefore

$$\sum_{k=0}^{\infty} \left|P_k(0)\right|^2 < \infty.$$

We finish this section with a result which relates the behaviour of the smallest eigenvalue λ_n with the norm of the monic polynomials, based in the results in [5]:

Lemma 15. *Let μ be a positive measure with infinite compact support on $\mathbb{C}$. For each $n \in \mathbb{N}$, let $P_n(z) = \sum_{j=0}^{n} v_{j,n} z^j$ be the orthonormal polynomial of degree n. If $k \leqslant n$, then*

$$\lambda_n \leqslant \frac{1}{\sum_{j=0}^{k} |v_{j,k}|^2}.$$

In particular, $\lambda_n \leqslant \|\Phi_n(z)\|^2$.

Proof. From the proof of Lemma 5,

$$\lambda_n \leqslant \frac{\int |p(z)|^2 \, d\mu}{\frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta},$$

for any polynomial not identically zero $p(z)$ of degree $\leqslant n$. In particular, if we apply the above inequality to $P_k(z)$ with $k \leqslant n$:

$$\lambda_n \leqslant \frac{1}{\sum_{j=0}^{k} |v_{j,k}|^2} \leqslant \frac{1}{v_{k,k}^2}.$$

Then, since $P_n(z) = v_{n,n} \Phi_n(z)$, $\lambda_n \leqslant \|\Phi_n(z)\|^2$. $\square$

Remark 16. If we define the $(n+1) \times (n+1)$-matrix $\mathcal{B}_n = (v_{j,k})$ with $v_{j,k}$ as in Lemma 15 if $j \leqslant k$ and $v_{j,k} = 0$ if $j > k$ and define $\mathcal{A}_n = \mathcal{B}_n \mathcal{B}_n^*$ then the entries $a_{j,k}^{(n)}$ of $\mathcal{A}_n$ are the coefficients in the kernel function

$$K_n(z, w) = \sum_{k=0}^{n} P_k(z) \overline{P_k(w)} = \sum_{j,k=1}^{n} a_{j,k}^{(n)} z^j \overline{w}^k,$$

and like in [5] $\mathcal{A}_n^{-1} = \overline{M_n}$.

Remark 17. In general, even for moment matrices, it is not true that $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \|\Phi_n(z)\|^2$. To show this it is enough to consider Example 2 in Remark 10. For such a Toeplitz matrix M we have that $\lim_{n\to\infty} \lambda_n = 0$ and nevertheless,

$$\lim_{n\to\infty} \|\Phi_n(z)\|^2 = \lim_{n\to\infty} \frac{1}{e_n M_n^{-1} e_n^*} = \lim_{n\to\infty} \frac{|M_n|}{|M_{n-1}|} = 1 \neq 0.$$

Remark 18. Note that in the case of a Toeplitz positive definite matrix M, since $e_0 M_n^{-1} e_0^* = e_n M_n^{-1} e_n^*$, we always have

$$\|\Phi_n(z)\|^2 = \frac{1}{\sum_{k=0}^{n} |P_k(0)|^2},$$

and consequently:

$$\lim_{n\to\infty} \|\Phi_n(z)\|^2 = \lim_{n\to\infty} \frac{1}{\sum_{k=0}^{n} |P_k(0)|^2}.$$

3. Related results for positive measures with support on $\overline{\mathbb{D}}$

We now consider measures μ with infinite support on $\overline{\mathbb{D}}$. Note that in this case $\|\Phi_{n+1}(z)\| \leqslant \|\Phi_n(z)\|$ for every $n \in \mathbb{N}$. Indeed, by the extremal property of the monic polynomials we have

$$\|\Phi_{n+1}(z)\|^2 \leqslant \|z\Phi_n(z)\|^2 \leqslant \|\Phi_n(z)\|^2,$$

and therefore $\lim_{n\to\infty} \|\Phi_n(z)\|$ exists.

We may decompose such measures μ on $\overline{\mathbb{D}}$ as $\mu = \eta + \nu$ where $\eta = \mu/\mathbb{D}$ and $\nu = \mu/\mathbb{T}$. Since we are going to use two measures μ and ν, we denote the corresponding monic polynomials as $\Phi_n(z; \mu)$ and $\Phi_n(z; \nu)$ respectively. We prove that the n large asymptotic of the norm of monic polynomials has a harmonic behaviour in the following sense:

Proposition 19. *Let μ be a positive measure with infinite support on $\overline{\mathbb{D}}$ and suppose that $\nu = \mu/\mathbb{T}$ has infinite support on $\mathbb{T}$. Then*

$$\lim_{n\to\infty} \|\Phi_n(z;\mu)\| = \lim_{n\to\infty} \|\Phi_n(z;\nu)\|.$$

Proof. Since $M_n(\nu) \leqslant M_n(\mu)$, we have $vM_n(\nu)v^* \leqslant vM_n(\mu)v^*$. Then

$$\|\Phi_n(z;\nu)\|^2 = \inf_{v\in\mathbb{C}^n} \begin{pmatrix} v & 1 \end{pmatrix} M_n(\nu) \begin{pmatrix} v^* \\ 1 \end{pmatrix} \leqslant \inf_{v\in\mathbb{C}^n} \begin{pmatrix} v & 1 \end{pmatrix} M_n(\mu) \begin{pmatrix} v^* \\ 1 \end{pmatrix} = \|\Phi_n(z;\mu)\|^2.$$

As a consequence,

$$\lim_{n\to\infty} \|\Phi_n(z;\nu)\| \leqslant \lim_{n\to\infty} \|\Phi_n(z;\mu)\|.$$

Therefore, if $\lim_{n\to\infty} \|\Phi_n(z;\mu)\| = 0$, then $\lim_{n\to\infty} \|\Phi_n(z;\nu)\| = 0$. On the other hand, assume that $\lim_{n\to\infty} \|\Phi_n(z;\mu)\| = c > 0$. Since $\{\|\Phi_n(z;\mu)\|\}_{n=0}^{\infty}$ is a non-increasing sequence, then $\|\Phi_n(z;\mu)\| \geqslant c$ for every $n \in \mathbb{N}$. Let n be fixed and let $p(z) = z^n + a_1z^{n-1} + \cdots + a_n$ be any monic polynomial of degree n. If $Q_k(z) = z^k p(z)$, which is a monic polynomial of degree $n+k$, by the extremal property of monic polynomials we have

$$\int_{\mathbb{D}} |z^k p(z)|^2\, d\eta + \int_{\mathbb{T}} |p(z)|^2\, d\nu = \int_{\overline{\mathbb{D}}} |z|^{2k} |p(z)|^2\, d\mu \geqslant \|\Phi_{n+k}(z;\mu)\|^2 \geqslant c^2.$$

Taking limits when $k \to \infty$, note that since $z^k p(z)$ is pointwise convergent to 0 in $\mathbb{D}$, by Lebesgue dominated convergence theorem:

$$\int_{\mathbb{T}} |p(z)|^2\, d\nu \geqslant c^2.$$

Consequently $\|\Phi_n(z;\nu)\| \geqslant c$. When $n \to \infty$ it follows that $\lim_{n\to\infty} \|\Phi_n(z;\nu)\| \geqslant c$. Therefore,

$$\lim_{n\to\infty} \|\Phi_n(z;\nu)\| = \lim_{n\to\infty} \|\Phi_n(z;\mu)\|$$

as required. $\square$

In the following result we prove an analogous result for the smallest eigenvalue of the finite truncated moment matrices:

Proposition 20. *Let μ be a measure with infinite support on $\overline{\mathbb{D}}$ and let $\nu = \mu/\mathbb{T}$. If $M(\mu)$ and $M(\nu)$ are the moment matrices associated with μ and ν, respectively then*

$$\lim_{n\to\infty} \lambda_n\big(M(\mu)\big) = \lim_{n\to\infty} \lambda_n\big(M(\nu)\big).$$

Proof. As in Proposition 19 we have that $M_n(\nu) \leqslant M_n(\mu)$ for every $n \in \mathbb{N}$. Consequently:

$$\lim_{n\to\infty} \lambda_n\big(M(\nu)\big) \leqslant \lim_{n\to\infty} \lambda_n\big(M(\mu)\big).$$

Moreover, if $\lim_{n\to\infty} \lambda_n(M(\mu)) = 0$ then $\lim_{n\to\infty} \lambda_n(M(\nu)) = 0$. Suppose now that $\lim_{n\to\infty} \lambda_n(M(\mu)) = C > 0$ for some $C > 0$. Then, if $\eta = \mu/\mathbb{D}$, by Lemma 5, we have

$$\int_{\overline{\mathbb{D}}} |p(z)|^2\, d\mu = \int_{\mathbb{D}} |p(z)|^2\, d\eta + \int_{\mathbb{T}} |p(z)|^2\, d\nu \geqslant C \int_{\mathbb{T}} |p(z)|^2\, \mathbf{dm},$$

whenever $p(z)$ is a polynomial. Let $p(z)$ be fixed, and consider the polynomial $z^n p(z)$ then

$$\int_{\overline{\mathbb{D}}} |z^n p(z)|^2\, d\mu = \int_{\mathbb{D}} |z^n p(z)|^2\, d\eta + \int_{\mathbb{T}} |p(z)|^2\, d\nu \geqslant C \int_{\mathbb{T}} |p(z)|^2\, \mathbf{dm}.$$

Again, proceeding as in Proposition 19, by taking limits when $n \to \infty$ we have that

$$\lim_{n\to\infty} \int_{\mathbb{D}} |z^n p(z)|^2\, d\eta = 0,$$

hence,

$$\int_{\mathbb{T}} |p(z)|^2 \, d\nu \geqslant C \int_{\mathbb{T}} |p(z)|^2 \, d\mathbf{m}.$$

Then $\lim_{n\to\infty} \lambda_n(M(\nu)) \geqslant C$, and therefore

$$\lim_{n\to\infty} \lambda_n\big(M(\nu)\big) = \lim_{n\to\infty} \lambda_n\big(M(\mu)\big). \qquad \square$$

Remark 21. There is no analogue of the above propositions for the case of the asymptotic behaviour of the n-reproducing kernels at 0. To see this, consider a positive measure ν on $\mathbb{T}$ which satisfies that $L^2(\nu) = P^2(\nu)$, for instance a positive measure with an infinite amount of atomic points $\{z_n\}_{n=1}^{\infty}$ with weights $p_n > 0$ for $n \geqslant 0$ such that $\sum_{n\geqslant 0} p_n < \infty$ and $\lim_{n\to\infty} z_n = z_0$. By Mergelyan's theorem (see e.g. [10]) the polynomials are dense in the space of continuous functions in $\mathrm{Supp}(\nu) = \{z_n \colon n \in \mathbb{N}\} \cup \{z_0\}$ with supremum norm, and consequently $P^2(\nu) = L^2(\nu)$. Therefore $\frac{1}{z} \in P^2(\nu)$ and by Corollary 8, it follows that $\sum_{k=0}^{\infty} |P_n(0;\nu)|^2 = \infty$. Consider now the measure $\mu = \nu + d\delta_0$ where $d\delta_0$ is the measure with mass d concentrated at the singleton set $\{0\}$. Consider the sequences $\{P_n(z;\nu)\}_{n=0}^{\infty}$ and $\{P_n(z;\mu)\}_{n=0}^{\infty}$. For every $n \in \mathbb{N}$ we have $M_n(\mu) \geqslant M_n(d\delta_0)$, and then, by Lemma 1 we have

$$\frac{1}{\sum_{k=0}^{n} |P_n(0;\mu)|^2} = \min\left\{ (1 \quad v)\, M_n(\mu) \begin{pmatrix} 1 \\ v^* \end{pmatrix} \colon v \in \mathbb{C}^n \right\} \geqslant \min\left\{ (1 \quad v)\, M_n(d\delta_0) \begin{pmatrix} 1 \\ v^* \end{pmatrix} \colon v \in \mathbb{C}^n \right\} = d.$$

Hence

$$\sum_{n=0}^{\infty} \left| P_n(0;\mu) \right|^2 \leqslant \frac{1}{d} < \infty.$$

We finish the paper with several results relating density of polynomials with the asymptotic behaviour of monic polynomials and of n-reproducing kernels:

Proposition 22. *Let μ be a positive measure with support on $\overline{\mathbb{D}}$ and with* $\mathrm{Supp}(\mu/\mathbb{T})$ *infinite, such that $P^2(\mu) = L^2(\mu)$. Then*

$$\lim_{n\to\infty} \left\| \Phi_n(z;\mu) \right\| = 0.$$

Proof. Assume that $L^2(\mu) = P^2(\mu)$. Then, if $\nu = \mu/\mathbb{T}$ we have that $L^2(\nu) = P^2(\nu)$. Consequently, combining Lemma 7 and Remark 18, it follows that

$$0 = \frac{1}{\sum_{n=0}^{\infty} |P_n(0;\nu)|^2} = \lim_{n\to\infty} \left\| \Phi_n(z;\nu) \right\|^2.$$

Finally, using Proposition 19, we have

$$\lim_{n\to\infty} \left\| \Phi_n(z;\mu) \right\|^2 = \lim_{n\to\infty} \left\| \Phi_n(z;\nu) \right\|^2 = 0. \qquad \square$$

Remark 23. The converse of this result is not true. It is enough to consider the Lebesgue measure η in the closed unit disk. Then,

$$\left\| \Phi_n(z) \right\|^2 = \min\left\{ (w \quad 1)\, M_n(\eta) \begin{pmatrix} w^* \\ 1 \end{pmatrix} \colon w \in \mathbb{C}^n \right\} = \frac{\pi}{n+1}$$

and $\lim_{n\to\infty} \|\Phi_n(z)\|^2 = 0$. Nevertheless $L^2(\eta) \neq P^2(\eta)$, since $P^2(\eta)$ consists of all analytic functions $\sum_{n=0}^{\infty} a_n z^n$ such that $\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty$.

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