On the pulsating instability of two-dimensional flames

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We consider a well-known thermo-diffusive model for the propagation of a premixed, adiabatic flame front in the large-activation-energy limit. That model depends only on one nondimensional parameter β, the reduced Lewis number. Near the pulsating instability limit, as $\beta \downarrow \beta_0 = 32/3$, we obtain an asymptotic model for the evolution of a quasi-planar flame front, via a multi-scale analysis. The asymptotic model consists of two complex Ginzburg–Landau equations and a real Burgers equation, coupled by non-local terms. The model is used to analyse the nonlinear stability of the flame front.

1 Introduction

This paper deals with the stability of a uniformly propagating premixed, adiabatic, plane flame front in two space dimensions. In the large-activation-energy limit, the chemical reaction is confined to an infinitely thin reaction sheet (whose location must be determined) and the governing mathematical model is a free boundary problem. We shall use a well-known thermo-diffusive model, first derived by Matkowsky & Sivashinsky (1979). After convenient nondimensionalization, the model is

$$\partial\theta/\partial t = \Delta\theta \quad \text{if} \quad x < \psi(y,t), \quad \theta = 1 \quad \text{otherwise}, \tag{1.1}$$

$$\partial S/\partial t = \Delta S + \beta\Delta\theta \quad \text{if} \quad x \neq \psi(y,t), \tag{1.2}$$

where for each $t > 0$, θ, S and $S_x + \beta\theta_x$ are continuous across the curve $x = \psi(y,t)$, while

$$\theta_x = -(1+\psi_y^2)^{-\frac{1}{2}} \exp(S/2) \quad \text{at} \quad x - \psi(y,t) = 0^-. \tag{1.3}$$

The boundary conditions are

$$\theta = S = 0 \quad \text{at} \quad x = -\infty, \quad |S| < \infty \quad \text{at} \quad x = \infty, \quad |\theta|, |S| < \infty \quad \text{at} \quad y = \pm\infty. \tag{1.4}$$

Here Δ is the Laplacian operator, t is the time variable, x and y are the space variables, θ is the temperature and S is an enthalpy (i.e. a linear combination of temperature and the concentration of the deficient reactant). The reaction sheet is located at $x = \psi(y,t)$ and, since $\theta_x = 0$ at $x - \psi(y,t) = 0^+$ (see (1.1)), according to (1.3) the slope of the temperature profile jumps across the reaction sheet to account for the effect of the exothermic chemical reaction. The model depends only on one parameter, the reduced Lewis number β.

In addition to the jump conditions and the boundary conditions above, we need some additional assumptions on the behaviour of the dependent variables as $y \to \pm\infty$, in order to ensure the existence of some spatial averages that will appear in the course of the analysis. For the sake of clarity, we do not anticipate these additional assumptions, and they will be imposed when needed.

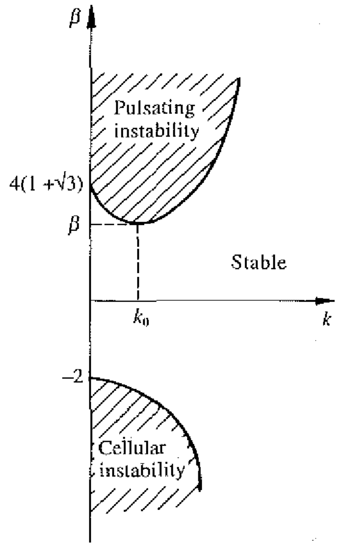

FIGURE 1.

We shall analyse the nonlinear stability of the following solution of (1.1)–(1.4)

$$\psi_s(y,t) \equiv -t; \quad \theta_s(x,y,t) \equiv \exp(x+t), \quad S_s(x,y,t) \equiv -\beta(x+t)\exp(x+t) \quad \text{if} \quad x+t<0,$$
$$\theta_s(x,y,t)-1 \equiv S_s(x,y,t) \equiv 0 \quad \text{if} \quad x+t \geqslant 0; \tag{1.5}$$

this solution is stationary in an appropriately moving reference frame and corresponds to a uniformly propagating plane flame front.

A word about nondimensionalization is now necessary. In (1.1)–(1.4) the space variables x and y are scaled with the characteristic length of the preheated zone, l_H. A typical value of l_H in practice is $l_H = 1$ mm. Since the cross-section of the burner is usually equal to, say, 10 cm, it makes sense to consider (1.1)–(1.4) in $-\infty < y < \infty$, as we shall do for the moment.

The linear stability of the plane solution (1.5) was first analysed by Sivashinsky (1977). When seeking solutions of (1.1)–(1.4) of the form

$$\left.\begin{aligned} \theta-\theta_s = Av_1(x)\exp(\omega t+iky), \quad S-S_s = Av_2(x)\exp(\omega t+iky), \\ \psi-\psi_s = A\exp(\omega t+iky), \end{aligned}\right\} \tag{1.6}$$

for small values of the constant A, the linear approximation is seen to have a non-trivial solution if

$$\beta(k^2-p_-^2) = 2p_-(p_+-p_-)^2, \tag{1.7}$$

where
$$p_\pm = [1 \pm (1+4k^2+4\omega)^{\frac{1}{2}}]/2.$$

By setting $\operatorname{Re}\omega = 0$ in (1.7) and eliminating $\operatorname{Im}\omega$, we obtain the neutral stability curves (Sivashinsky diagram) that are sketched in his fig. 1. In particular, the solution (1.5) is linearly stable if $-2 \leqslant \beta \leqslant \beta_0 = 32/3$, and it is linearly unstable otherwise (we are ignoring the continuous spectrum of the linearized problem; it is expected to be in the left-hand side of the complex plane).

A fairly complete and quite enlightening local nonlinear stability analysis near the lower instability limit, $\beta = -2$, was first made by Sivashinsky (1979) (see also Michelson & Sivashinsky), who showed that the flame exhibits a chaotic cellular behaviour at this limit. A similar analysis of the upper instability limit is lacking in the literature, although some works have given partial results. Matkowsky & Olagunju (1980) analysed the restricted one-dimensional problem near $\beta = 4(1+\sqrt{3})$ (see fig. 1), and showed that a Hopf bifurcation occurs at this limit. Matkowsky & Olagunju (1982) also made a first analysis near $\beta = \beta_0$, and found travelling and standing waves along the flame sheet, with wave number $k = k_0$; in addition, they performed a restricted linear stability analysis of those waves, and concluded that the travelling ones are stable while the standing ones are unstable. Travelling waves along the flame sheet had been found experimentally in, e.g., Markstein (1964), Ferguson & Keck (1979) and Sabathier *et al.* (1979). In a series of papers, Matkowsky *et al.* (1985, 1987, 1988) considered the restriction of (1.1)–(1.4) to strips (in two dimensions) and to tubes (in three dimensions), and found a large variety of secondary and tertiary bifurcations to quasiperiodic behaviour. These results suggest that as the transversal characteristic length of the tube grows the flame could exhibit chaotic behaviour. We shall come back to this point later on.

In the sequel, we shall analyse the local nonlinear stability of the solution (1.5) as $\beta \downarrow \beta_0$. In order to choose the appropriate scalings, let us point out that as $\beta \to \beta_0$, the marginally stable modes of the linearized problem are associated with a real eigenvalue ω_1 and with two pairs of complex conjugate eigenvalues, $\omega_2, \bar{\omega}_2$, ω_3 and $\bar{\omega}_3$, such that

$$\omega_1 = -(1+\beta_0/2)k^2+\ldots \quad \text{as} \quad k \to 0 \quad \text{and} \quad \beta \to \beta_0, \tag{1.8}$$

$$\omega_2 = i\Omega + ibk_0(k-k_0) - c(k-k_0)^2 + d(\beta-\beta_0)+\ldots \quad \text{as} \quad k \to k_0 \quad \text{and} \quad \beta \to \beta_0, \tag{1.9}$$

$$\omega_3 = i\Omega - ibk_0(k+k_0) - c(k+k_0)^2 + d(\beta-\beta_0)+\ldots \quad \text{as} \quad k \to -k_0 \quad \text{and} \quad \beta \to \beta_0, \tag{1.10}$$

where

$$\beta_0 = 32/3, \quad k_0 \simeq 0.204, \quad \Omega \simeq 0.781, \quad b \simeq 6.72, \quad c \simeq 1.68-i2.09, \quad d \simeq 0.106+i0.054. \tag{1.11}$$

Now, if we define the small parameter $\epsilon > 0$ as

$$\epsilon^2 = \beta - \beta_0,$$

from (1.9)–(1.10) we see that the significant modes in a nonlinear stability analysis are those corresponding to ω_2 with $k-k_0 \sim \epsilon$, and to ω_3, with $k+k_0 \sim \epsilon$. This means that we must consider an additional 'slow' variable in the y-direction

$$\eta = \epsilon y \tag{1.12}$$

and, consequently, two additional slow time variables,

$$T = \epsilon t \quad \text{and} \quad \tau = \epsilon^2 t. \tag{1.13}$$

Observe that the intermediate time scale T is associated with the purely oscillatory, non-dispersive terms $ibk_0(k-k_0)$ and $-ibk_0(k+k_0)$ of ω_2 and ω_3. In connection with the dependent variables, we assume that

$$\theta - \theta_s \sim S - S_s \sim \epsilon \tag{1.14}$$

because we expect cubic nonlinearities to appear in the equations giving the evolution of the complex amplitudes of the modes associated with ω_2 and ω_3, as occurs in Hopf bifurcation (see, e.g., Hassard *et al.* 1981). In fact, observe that if the mode associated with ω_1 were absent, then we would encounter an ordinary Hopf bifurcation as $\epsilon \to 0$. Finally, we assume that

$$\psi - \psi_s \sim 1. \tag{1.15}$$

This assumption is based on the fact that if the modes associated with ω_2 and ω_3 were absent, then the reaction sheet location ψ would evolve according to the phase-diffusion equation $\psi_\tau = (1+\beta_0/2)\psi_{\eta\eta} - \psi_\eta^2/2$ (see Kuramoto 1984), where the slow variables τ and η are as defined above (with ϵ small).

In §2 we shall use these scalings to obtain, via a multiscale analysis, a real Burgers equation giving the leading order of the reaction sheet location and a pair of complex Ginzburg–Landau equations giving the leading order of the complex amplitudes of two wavetrains that travel along the reaction sheet in opposite directions; the three equations, to be referred to as the *amplitude equations* in the sequel, are coupled by nonlocal terms and give a first approximation of the nonlinear evolution of the flame at the slowest time scale $\tau \sim 1$. The amplitude equations are somewhat similar to those obtained at the onset of the so-called *oscillatory instability* that has been recently considered, in a systematic way, in the literature of pattern formation (see Fauve 1987; Hohenberg & Cross 1989; Newell 1989; and references given therein); the main differences are associated with the presence, in our case, of a third real amplitude (the reaction sheet location), and the fact that our analysis involves two time scales. We should mention also that a similar problem was considered by Booty *et al.* (1988, p. 529), who obtained also two complex Ginzburg–Landau equations, but without non-local terms; nevertheless, that model is not correct from an asymptotic point of view (it is intended to describe the slow evolution of the flame on the slowest time scale, but it involves the intermediate time scale explicitly).

In §3 we shall use the asymptotic model to describe the nonlinear evolution of the flame in transversely finite strips, of transverse length L, in two cases: (i) $L \gg \epsilon^{-1}$; and (ii) $L \sim \epsilon^{-1}$. The results of §3 indicate that the flame exhibits travelling waves in case (i), and time-oscillatory behaviour in case (ii), and that no chaotic behaviour is to be expected as $\beta \to \beta_0$ in two dimensions. It seems that, in order to encounter chaos by analytical means, one must consider three-dimensional effects, as will be commented on in §4.

After completing this work and sending it for publication, in September 1990, we heard about a manuscript by Knobloch & De Luca (1990), who used perturbation techniques to obtain averaged Ginzburg–Landau equations (similar to those obtained in this paper) as the amplitude equations in the context of pattern formation. In their manuscript the third (Burgers type of) amplitude equation is absent because the authors analyse the nonlinear stability of a steady state (while we consider a quasi-planar travelling wavefront). The formal derivation in Knobloch & De Luca (1990) has been justified rigorously in Vega (1991).

2 Asymptotic derivation of the amplitude equations

In this section we derive a coupled system of three PDEs giving the evolution, on the slowest time scale ($\epsilon^2 t = \tau \sim 1$), of those modes that are near-marginal at the onset of instability (i.e. the modes associated with the eigenvalues ω_1, ω_2 and ω_3, see (1.8)–(1.10)).

The derivation will be made according to the following multi-scale scheme. First, the solution of (1.1)–(1.4) will be expanded in powers of the small parameter

$$\epsilon = (\beta - \beta_0)^{\frac{1}{2}}.$$

Then, at each order ϵ^j, we shall apply the appropriate solvability conditions. As usual, these conditions are obtained by eliminating secular terms (that is, by requiring the dependent variables to be bounded) on the fast time scale, as $t \to \infty$, and also on the intermediate time scale, as $T \to \infty$. By imposing these conditions we shall obtain, at the order ϵ^3, a first approximation of the nonlinear evolution of the dependent variables on the slowest time scale $\tau \sim 1$.

For convenience, we first rewrite the problem in a reference frame attached to the reaction sheet, by using the new spatial variables $\xi = x - \psi(y, t)$ and y. In addition, we introduce the new slow time and space variables $T = \epsilon t$, $\tau = \epsilon^2 t$ and $\eta = \epsilon y$, as suggested by the scaling analysis in §1. Then (1.1)–(1.4) is written in the form

$$\begin{aligned} &-U_t + \psi_t U_\xi + D((1+\psi_y^2) U_{\xi\xi} - 2\psi_y U_{\xi y} + U_{yy} - \psi_{yy} U_\xi) \\ &= \epsilon[U_T - \psi_T U_\xi - 2D(\psi_y \psi_\eta U_{\xi\xi} - \psi_{y\eta} U_\xi - \psi_\eta U_{\xi y} - \psi_y U_{\xi\eta} + U_{y\eta})] \\ &+ \epsilon^2[U_\tau - \psi_\tau U_\xi - D(\psi_\eta^2 U_{\xi\xi} - \psi_{\eta\eta} U_\xi - 2\psi_\eta U_{\xi\eta} + U_{\eta\eta})] \quad \text{if} \quad \xi \neq 0 \end{aligned} \tag{2.1}$$

$$[U] = 0; \quad D[U_\xi] + (1 + \psi_y^2 + 2\epsilon\psi_y \psi_\eta + \epsilon^2 \psi_\eta^2)^{-\frac{1}{2}} \exp(u_2/2)\, \mathscr{E} = 0 \quad \text{at} \quad \xi = 0 \tag{2.2}$$

$$U = 0 \quad \text{at} \quad \xi = -\infty, \quad u_1 = 0 \quad \text{if} \quad \xi \geqslant 0, \quad u_2 < \infty \quad \text{at} \quad \xi = \infty, \tag{2.3}$$

where we use, as in the sequel, the vectorial notation

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \equiv \begin{pmatrix} \theta \\ S \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}, \quad \mathscr{E} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and the brackets denote the jump across the reaction sheet

$$[U] = (U)_{\xi=0^+} - (U)_{\xi=0^-}.$$

Also, as suggested by the scaling analysis in §1, we have $U - U_s = O(\epsilon)$ and $\psi - \psi_s = O(1)$, where U_s and ψ_s are given by

$$U_s = \begin{pmatrix} 1 \\ -\beta\xi \end{pmatrix} \exp(\xi) \quad \text{if} \quad \xi < 0, \quad U_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{if} \quad \xi \geqslant 0, \quad \psi_s = -t,$$

and correspond to the uniformly propagating plane flame front. Observe that since U_s and the diffusion matrix depend on $\beta = \beta_0 + \epsilon^2$, we have

$$U_s(\xi, \epsilon) = U_0 + \epsilon^2 U_2(\xi),$$

$$D = D_0 + \epsilon^2 D_2 \equiv \begin{pmatrix} 1 & 0 \\ \beta_0 & 1 \end{pmatrix} + \epsilon^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then we seek the expansions

$$U(\xi, y, \eta, t, T, \tau) = U_0(\xi) + \epsilon^2 U_2(\xi) + \sum_{j=1} \epsilon^j U^{(j)}(\xi, y, \eta, t, T, \tau), \tag{2.4}$$

$$\psi(y, \eta, t, T, \tau) = \psi_s(t) + \phi^{(0)}(\eta, T, \tau) + \sum_{j=1} \epsilon^j \psi^{(j)}(y, \eta, t, T, \tau), \tag{2.5}$$

where the fact that $\phi^{(0)}$ depends only on the slow variables may be seen as a consequence of eliminating secular terms at the order ϵ^0, on the time scale $t \sim 1$. When the expansions

(2.4)–(2.5) are inserted in (2.1)–(2.3) we get a sequence of recursive linear problems of the form

$$-U_t^{(j)} - U_\xi^{(j)} + D_0(U_{\xi\xi}^{(j)} + U_{yy}^{(j)}) + \psi_t^{(j)} U_0' - \psi_{yy}^{(j)} D_0 U_0' = F^{(j)} \quad \text{if} \quad \xi \neq 0, \tag{2.6}$$

$$[U^{(j)}] = 0, \quad D_0[U_\xi^{(j)}] + (1/2) u_2^{(j)} \mathscr{E} = f^{(j)}, \tag{2.7}$$

$$U^{(j)} = 0 \quad \text{at} \quad \xi = -\infty, \quad u_1^{(j)} = 0 \quad \text{if} \quad \xi \geqslant 0, \quad u_2^{(j)} < \infty \quad \text{at} \quad \xi = \infty, \tag{2.8}$$

where $U_0' = \mathrm{d}U_0/\mathrm{d}\xi$ and U_0 is given in the Appendix (equation (A 1)).

By eliminating secular terms in the time scale $t \sim 1$ at each order ϵ^j we readily obtain the following (three) solvability conditions

$$\int_0^{2\pi/\Omega} \int_0^{2\pi/k_0} l_0(F^{(j)}, f^{(j)}) \,\mathrm{d}t\,\mathrm{d}y = 0, \tag{2.9}$$

$$\int_0^{2\pi/\Omega} \int_0^{2\pi/k_0} l_1(F^{(j)}, f^{(j)}) \exp(-i\Omega t \mp ik_0 y) \,\mathrm{d}t\,\mathrm{d}y = 0, \tag{2.10}$$

where the linear operators l_0 and l_1 are defined in the Appendix (equations (A 2)–(A 3)).

At order ϵ, we have $F^{(1)} = \phi_T^{(0)} U_0', f^{(1)} = 0$. Then the solvability conditions (2.9)–(2.10) yield

$$\phi_T^{(0)} \equiv 0 \quad \text{i.e.} \quad \phi^{(0)} = \phi^{(0)}(\eta, \tau). \tag{2.11}$$

Consequently, the system (2.6)–(2.8) is homogeneous at order ϵ, and $U^{(1)}$ and $\psi^{(1)}$ may be written as

$$U^{(1)} = W^{(1)}(\eta, T, \tau) V(\xi) \exp(i\Omega t + ik_0 y) + X^{(1)}(\eta, T, \tau) V(\xi) \exp(i\Omega t - ik_0 y) + \text{c.c.}, \tag{2.12}$$

$$\psi^{(1)} = \phi^{(1)}(\eta, T, \tau) + [W^{(1)}(\eta, T, \tau) \exp(i\Omega t + ik_0 y) + X^{(1)}(\eta, T, \tau) \exp(i\Omega t - ik_0 y) + \text{c.c.}], \tag{2.13}$$

where c.c. stands for the complex conjugate, and the eigenfunction V is given in the Appendix (equation (A 5)). Observe that the eigenfunction U_0' (associated with the eigenvalue $\omega = 0$) does not appear in (2.12) because the problems (2.6)–(2.8) are not invariant under ξ-translations (due to the fact that we have attached the axis $\xi = 0$ to the reaction sheet). Also, notice that in (2.12–2.13), $U^{(1)}$ and $\psi^{(1)} - \phi^{(1)}$ are superpositions of two transverse wave-trains, of complex amplitudes $W^{(1)}$ and $X^{(1)}$, travelling along the flame front (in the time scale $t \sim 1$) in opposite directions. Our final purpose in this section is to obtain three equations giving the evolution on the slowest time scale $\tau \sim 1$ of the amplitudes $\phi^{(0)}$ (the leading order of the reaction sheet location), $W^{(1)}$ and $X^{(1)}$. These equations will be called the *amplitude equations*.

At order ϵ^2, the right-hand sides of the equation (2.6) and the jump condition (2.7), $F^{(2)}$ and $f^{(2)} = (f_1^{(2)}, f_2^{(2)})^{\mathrm{T}}$, are given in the Appendix (see equations (A 6)–(A 8)). Then, by applying the three solvability conditions (2.9)–(2.10), on the time scale $t \sim 1$, we get the following equations on the time scale $T \sim 1$

$$W_T^{(1)} - k_0 b W_\eta^{(1)} + ik_0 W^{(1)} \phi_\eta^{(0)} = 0, \tag{2.14}$$

$$X_T^{(1)} + k_0 b X_\eta^{(1)} - ik_0 X^{(1)} \phi_\eta^{(0)} = 0, \tag{2.15}$$

$$\phi_\tau^{(0)} + \phi_T^{(1)} = (1 + \beta_0/2) \phi_{\eta\eta}^{(0)} - |\phi_\eta^{(0)}|^2/2 + e_1(|W^{(1)}|^2 + |X^{(1)}|^2), \tag{2.16}$$

where the constants b and e_1 are given in (1.11) and in the Appendix (see (A 9)). Notice that the coefficient of $\phi^{(0)}_{\eta\eta}$ in (2.16) coincides with that of $-k^2$ in (1.8), as was to be expected.

Now the equations (2.14)–(2.15) are readily integrated on the intermediate time scale $T \sim 1$ to obtain (notice that τ is constant on this time scale)

$$W^{(1)}(\eta, T, \tau) = Y^{(1)}(\tilde{\eta}, \tau) \exp(i\phi^{(0)}(\eta, \tau)/b); \quad X^{(1)}(\eta, T, \tau) = Z^{(1)}(\hat{\eta}, \tau) \exp(i\phi^{(0)}(\eta, \tau)/b) \tag{2.17}$$

where the functions $Y^{(1)}$ and $Z^{(1)}$ are arbitrary functions (at this order) and

$$\tilde{\eta} = \eta + k_0 bT, \quad \text{and} \quad \hat{\eta} = \eta - k_0 bT. \tag{2.18}$$

Observe that the solutions (2.17) contain no secular terms in the intermediate time scale. Notice also that the moduli of the amplitudes $W^{(1)}$ and $X^{(1)}$ are travelling at constant speed (in the intermediate time scale) in opposite directions.

In order to integrate the equation (2.16) on the intermediate time scale observe that, according to (2.11) and (2.17), it may be written as

$$\phi^{(1)}_T = G_1(\eta, \tau) + G_2(\eta + k_0 bT, \tau) + G_3(\eta - k_0 bT, \tau) \tag{2.19}$$

where $\quad G_1 \equiv -\phi^{(0)}_\tau + (1+\beta_0/2)\phi^{(0)}_{\eta\eta} - |\phi^{(0)}_\eta|^2/2, \quad G_2 \equiv e_1 |Y^{(1)}|^2, \quad G_3 \equiv e_1 |Z^{(1)}|^2.$

Then (recall that τ is a constant in this time scale)

$$\phi^{(1)}(\eta, T, \tau) = \phi^{(1)}(\eta, 0, \tau) + TG_1(\eta, \tau) + \int_0^T [G_2(\eta + k_0 b\tilde{T}, \tau) + G_3(\eta - k_0 b\tilde{T}, \tau)]\, \mathrm{d}\tilde{T},$$

and by imposing that $\phi^{(1)}$ remains bounded as $T \to \infty$ we get

$$G_1(\eta, \tau) + \lim T^{-1} \int_0^T [G_2(\eta + k_0 b\tilde{T}, \tau) + G_3(\eta - k_0 b\tilde{T}, \tau)]\, \mathrm{d}\tilde{T} = 0 \quad \text{as} \quad T \to \infty.$$

Notice that the limit appearing above is independent of η (when it exists). This condition readily yields the following equation for the evolution of $\phi^{(0)}$ on the slowest time scale

$$\phi^{(0)}_\tau = (1+\beta_0/2)\phi^{(0)}_{\eta\eta} - |\phi^{(0)}_\eta|^2/2 + e_1(\langle |Y^{(1)}|^2 \rangle^+ + \langle |Z^{(1)}|^2 \rangle^-), \tag{2.20}$$

where the spatial averages $\langle \cdot \rangle^+$ and $\langle \cdot \rangle^-$ are defined by

$$\langle h \rangle^\pm = \lim \left(a^{-1} \int_0^a h(\pm z, \tau)\, \mathrm{d}z \right) \quad \text{as} \quad a \to \infty, \tag{2.21}$$

with h standing for $|Y^{(1)}|^2 = |Y^{(1)}|^2(\tilde{\eta}, \tau)$ or $|Z^{(1)}|^2 = |Z^{(1)}|^2(\hat{\eta}, \tau)$. In the sequel we shall assume that the initial conditions of (1.1)–(1.4) are such that the averages appearing in (2.20) exist (this is true, in particular, if $|Y^{(1)}|^2$ and $|Z^{(1)}|^2$ are either periodic or quasi-periodic in their dependence on the spatial variables, or if they converge as $\eta \to \pm\infty$). The appearance of the averaged terms in the evolution equation (2.20) was to be expected, since $|Y^{(1)}|^2$ and $|Z^{(1)}|^2$ are travelling at an infinite speed on the slowest time scale.

To conclude the analysis at this order, we calculate the solution of (2.6)–(2.8) for $j = 2$ (see equations (A 10)–(A 11) in the Appendix) that will be needed at the next order. In (A 10)–(A 11) $\phi^{(2)}$, $W^{(2)}$ and $X^{(2)}$ are arbitrary functions (at this order) that appear through the general solution of the homogeneous part of (2.6)–(2.8).

At order ϵ^3, the right-hand sides of the equation (2.6) and the jump condition (2.7), $F^{(3)}$ and $f^{(3)} = (f_1^{(3)}, f_2^{(3)})^{\mathrm{T}}$, are given in the Appendix (see equations (A 12)–(A 14)). Then, by applying the two solvability conditions (2.10), in the time scale $t \sim 1$, we get the following equations in the time scale $T \sim 1$

$$
\begin{aligned}
&W_T^{(2)} - k_0 b W_\eta^{(2)} + ik_0\, W^{(2)} \phi_\eta^{(0)} + ik_0\, W^{(1)} \phi_\eta^{(1)} \\
&\qquad = -W_\tau^{(1)} + cW_{\eta\eta}^{(1)} + AW_\eta^{(1)} \phi_\eta^{(0)} + W^{(1)} (d - B\phi_{\eta\eta}^{(0)} - C|\phi_\eta^{(0)}|^2 - D|W^{(1)}|^2 - E|X^{(1)}|^2), \quad (2.22) \\
&X_T^{(2)} + k_0 b X_\eta^{(2)} - ik_0\, X^{(2)} \phi_\eta^{(0)} - ik_0\, X^{(1)} \phi_\eta^{(1)} \\
&\qquad = -X_\tau^{(1)} + cX_{\eta\eta}^{(1)} + AX_\eta^{(1)} \phi_\eta^{(0)} + X^{(1)} (d - B\phi_{\eta\eta}^{(0)} - C|\phi_\eta^{(0)}|^2 - D|X^{(1)}|^2 - E|W^{(1)}|^2), \quad (2.23)
\end{aligned}
$$

where the coefficients c and d are those obtained in the linear stability analysis (see (1.11)), as was to be expected, while A, B, C, D and E are given in the Appendix (see (A 18)).

Since $\phi^{(1)}$ appears explicitly in (2.22)–(2.23), at first sight it seems that we need a third equation for $\phi^{(1)}$, which would be obtained by applying the as yet unused solvability condition (2.9). Nevertheless, by introducing the new dependent variables $Y^{(2)}$ and $Z^{(2)}$, given by

$$
\left.
\begin{aligned}
Y^{(2)}(\eta, T, \tau) &= W^{(2)}(\eta, T, \tau) \exp(-i\phi^{(0)}(\eta, \tau)/b) - (i\phi^{(1)}(\eta, T, \tau)/b)\, Y^{(1)}(\eta + k_0 bT, \tau), \\
Z^{(2)}(\eta, T, \tau) &= X^{(2)}(\eta, T, \tau) \exp(-i\phi^{(0)}(\eta, \tau)/b) - (i\phi^{(1)}(\eta, T, \tau)/b)\, Z^{(1)}(\eta - k_0 bT, \tau).
\end{aligned}
\right\} \quad (2.24)
$$

(where $Y^{(1)}$ and $Z^{(1)}$ are defined in (2.17)), and taking into account (2.16), we obtain the following equations for $Y^{(2)}, Z^{(2)}, Y^{(1)}, Z^{(1)}$ and $\phi^{(0)}$

$$
\begin{aligned}
Y_T^{(2)} - k_0 b Y_\eta^{(2)} &= -Y_\tau^{(1)} + cY_{\eta\eta}^{(1)} + A_1\, Y_\eta^{(1)} \phi_\eta^{(0)} \\
&\quad + Y^{(1)}(d - B_1 \phi_{\eta\eta}^{(0)} - C_1 |\phi_\eta^{(0)}|^2 - D_1 |Y^{(1)}|^2 - E_1 |Z^{(1)}|^2), \quad (2.25) \\
Z_T^{(2)} + k_0 b Z_\eta^{(2)} &= -Z_\tau^{(1)} + cZ_{\eta\eta}^{(1)} + A_1\, Z_\eta^{(1)} \phi_\eta^{(0)} \\
&\quad + Z^{(1)}(d - B_1 \phi_{\eta\eta}^{(0)} - C_1 |\phi_\eta^{(0)}|^2 - D_1 |Z^{(1)}|^2 - E_1 |Y^{(1)}|^2), \quad (2.26)
\end{aligned}
$$

where the coefficients A_1, B_1, C_1, D_1 and E_1 are given in the Appendix (see (A 19)).

Equations (2.25)–(2.26) allow us to obtain solvability conditions on the time scale $T \sim 1$, in a similar manner as we did at order ϵ^2 with the equation (2.16). Let us first consider equation (2.25). Since $Y^{(1)} = Y^{(1)}(\eta + k_0 bT, \tau)$, $Z^{(1)} = Z^{(1)}(\eta - k_0 bT, \tau)$ and $\phi^{(0)} = \phi^{(0)}(\eta, \tau)$, the equation (2.25) is readily integrated to yield (again, recall that τ is constant in the intermediate time scale)

$$
\begin{aligned}
Y^{(2)}(\eta, T, \tau) &= Y^{(2)}(\eta, 0, \tau) + T(-Y_\tau^{(1)} + cY_{\eta\eta}^{(1)} + dY^{(1)} - D_1\, Y^{(1)} |Y^{(1)}|^2) \\
&\quad + (A_1/k_0 b)\, Y_\eta^{(1)} \int_\eta^{\eta + k_0 bT} \phi_\eta^{(0)}(\eta, \tau)\, \mathrm{d}\eta \\
&\quad - (1/k_0 b)\, Y^{(1)} \int_\eta^{\eta + k_0 bT} (B_1 \phi_{\eta\eta}^{(0)}(\eta, \tau) + C_1 |\phi_\eta^{(0)}(\eta, \tau)|^2)\, \mathrm{d}\eta \\
&\quad - (E_1/2k_0 b)\, Y^{(1)} \int_{\tilde\eta - k_0 bT}^{\tilde\eta + k_0 bT} |Z^{(1)}(\tilde\eta, \tau)|^2\, \mathrm{d}\tilde\eta. \quad (2.27)
\end{aligned}
$$

Now, when using the space variable $\tilde\eta = \eta + k_0 bT$ (i.e. in a reference frame moving at a constant speed on the intermediate time scale, where $Y^{(1)}$ is stationary on this time scale), and imposing that $Y^{(2)}(\tilde{\eta}, T, \tau)$ remains bounded as $T \to \infty$ for each fixed value of $\tilde{\eta}$, we readily obtain the following solvability condition

$$Y_\tau^{(1)} - cY_{\tilde{\eta}\tilde{\eta}}^{(1)} - Y^{(1)}(d - D_1 |Y^{(1)}|^2) = A_1 Y_{\tilde{\eta}}^{(1)} \lim_{T\to\infty} (1/k_0 bT) \int_{\tilde{\eta}-k_0 bT}^{\tilde{\eta}} \phi_\eta^{(0)}(\eta, \tau)\, \mathrm{d}\eta$$

$$+ Y^{(1)} \lim_{T\to\infty} (1/k_0 bT) \int_{\tilde{\eta}-k_0 bT}^{\tilde{\eta}} (B_1 \phi_{\eta\eta}^{(0)}(\eta, \tau) + C_1 |\phi^{(0)}(\eta, \tau)|^2)\, \mathrm{d}\eta$$

$$- E_1 Y^{(1)} \lim_{T\to\infty} (1/2k_0 bT) \int_{\tilde{\eta}-2k_0 bT}^{\tilde{\eta}} |Z^{(1)}(\hat{\eta}, \tau)|^2\, \mathrm{d}\hat{\eta}. \tag{2.28}$$

We assume that the initial conditions of (1.1)–(1.4) are such that the limits appearing in (2.28) exist (this is true, in particular, if $\phi^{(0)}$ and $Z^{(1)}$ are periodic or quasi-periodic in the space variables, or if they converge as $\eta \to \pm\infty$). Notice that the limits are independent of $\tilde{\eta}$. In addition, if (as we assume in the sequel) $\phi^{(0)}$ and $\phi_\eta^{(0)}$ are bounded as $\eta \to \pm\infty$ (for each fixed τ) then

$$\lim_{T\to\infty} (1/k_0 bT) \int_{\tilde{\eta}-k_0 bT}^{\tilde{\eta}} \phi_\eta^{(0)}(\eta, \tau)\, \mathrm{d}\eta = \lim_{T\to\infty} (1/k_0 bT) \int_{\tilde{\eta}-k_0 bT}^{\tilde{\eta}} \phi_{\eta\eta}^{(0)}(\eta, \tau)\, \mathrm{d}\eta = 0.$$

Then, (2.28) is readily written as

$$Y_\tau^{(1)} = cY_{\tilde{\eta}\tilde{\eta}}^{(1)} + Y^{(1)}(d - D_1 |Y^{(1)}|^2 - C_1 \langle |\phi_\eta^{(0)}|^2 \rangle^- - E_1 \langle |Z^{(1)}|^2 \rangle^-), \tag{2.29}$$

with the spatial average $\langle \cdot \rangle^-$ as defined in (2.21), with h standing now for $|\phi^{(0)}|^2 = |\phi^{(0)}|^2(\eta, \tau)$ or $|Z^{(1)}|^2 = |Z^{(1)}|^2(\hat{\eta}, \tau)$.

By eliminating secular terms from the solution of (2.26), in a similar way, we obtain

$$Z_\tau^{(1)} = cZ_{\hat{\eta}\hat{\eta}}^{(1)} + Z^{(1)}(d - D_1 |Z^{(1)}|^2 - C_1 \langle |\phi_\eta^{(0)}|^2 \rangle^+ - E_1 \langle |Y^{(1)}|^2 \rangle^+), \tag{2.30}$$

again with the spatial average $\langle \cdot \rangle^+$ as defined in (2.21), with h standing for $|\phi^{(0)}|^2 = |\phi^{(0)}|^2(\eta, \tau)$ or $|Y^{(1)}|^2 = |Y^{(1)}|^2(\tilde{\eta}, \tau)$.

As happened with (2.20), the appearance of the spatial average terms in the evolution equations (2.29) and (2.30) was to be expected, since in the reference frame attached to $Y^{(1)}$ (resp. to $Z^{(1)}$), $\phi^{(0)}$ and $Z^{(1)}$ (resp. $\phi^{(0)}$ and $Y^{(1)}$) are travelling to the right (resp. to the left) at an infinite speed, on the slowest time scale, $\tau \sim 1$.

Equations (2.20), (2.29) and (2.30) are the evolution equations giving the amplitudes $\phi^{(0)}$, $Y^{(1)}$ and $Z^{(1)}$, in the slowest time scale $\tau \sim 1$, as we anticipated at the beginning of this section. Notice that they are coupled by the averaged terms $\langle \cdot \rangle^+$ and $\langle \cdot \rangle^-$, which depend only on time. Thus, the coupling is non-local, and fairly weak. The well-posed model (2.20, 2.29–2.30) will be analysed in next section. Now, observe that the amplitude equations yield the weakly nonlinear evolution of the flame front. In fact, the solution (2.4)–(2.5) of (2.1)–(2.3) may be written (in first approximation) in terms of the amplitudes $\phi^{(0)}$, $Y^{(1)}$ and $Z^{(1)}$, as, (see (2.12–2.13, 2.17))

$$\begin{aligned} U(x, y, t) &\simeq U_s(x + t - \phi^{(0)}(\eta, \tau)) \\ &+ \varepsilon[Y^{(1)}(\eta + k_0 bT, \tau)\, V(x + t - \phi^{(0)}(\eta, \tau)) \exp(i\Omega t + ik_0 y + i\phi^{(0)}(\eta, \tau)/b) \\ &+ Z^{(1)}(\eta - k_0 bT, \tau)\, V(x + t - \phi^{(0)}(\eta, \tau)) \exp(i\Omega t - ik_0 y + i\phi^{(0)}(\eta, \tau)/b) + \text{c.c.}]. \end{aligned} \tag{2.31}$$

3 Local nonlinear stability analysis of the flame through the amplitude equations

Here we shall use the model posed by the amplitude equations (2.20, 2.29–2.30) to analyse the nonlinear stability of flames in two-dimensional strips whose transverse size L is large. Thus we shall consider the problem (1.1)–(1.4) in $-\infty < x < \infty, 0 < y < L$, with Neumann boundary conditions at $y = 0, L$. Two cases will be considered.

If $1/\epsilon^2 L = O(1)$ as $\epsilon \to 0$, we shall ignore those points of the strip whose distance to the boundary is of order $O(1/\epsilon)$. Then, the flame appears as transversely infinite in the spatial variable η and no boundary conditions are necessary for the amplitude equations. We shall assume that the largest spatial scale in the transverse direction is $y \sim \epsilon^{-1}$ (i.e. $\eta \sim 1$); this assumption might fail on slower time scales ($\epsilon^2 t \equiv \tau \sim 1/\epsilon$) which we shall ignore (thus, the attractors we shall find might be slow transients on these slower time scales). In addition, we shall assume that the initial conditions are such that the spatial averages appearing in the amplitude equations are well-defined; this is true, in particular, if $\phi^{(0)}$, $Y^{(1)}$ and $Z^{(1)}$ are periodic in the first variable ($\eta, \tilde{\eta}$ and $\hat{\eta}$, respectively), or equivalently, if the initial conditions for (1.1)–(1.4) are periodic in the y-variable, with a period of the order of $1/\epsilon$. This limit will be referred to as the *transversely infinite* one.

The second case to be considered, to be referred to as *transversely finite*, is that in which

$$\epsilon L \equiv \lambda = O(1) \quad \text{as} \quad \epsilon \to 0. \tag{3.1}$$

Now the Neumann boundary conditions at $y = 0$, L lead to reflection conditions for the transverse waves. This is readily seen by a reflection principle as follows. First, the problem (1.1)–(1.4) in the strip $-\infty < x < \infty$, $0 < y < L$, is extended to the whole plane by requiring θ, S and ψ to be symmetric with respect to the line $y = rL$, for each integer r and each $t > 0$, so that

$$\theta_y = S_y = 0, \quad \psi_y = 0 \quad \text{at} \quad y = 0, L, \tag{3.2}$$

as required. Now, by imposing (3.2) to order ϵ in (2.31), we readily obtain the following conditions on the amplitudes $\phi^{(0)}$, $Y^{(1)}$ and $Z^{(1)}$

$$\phi^{(0)}_\eta(r\lambda, \tau) = 0, \tag{3.3}$$

$$Y^{(1)}(r\lambda - T, \tau) \exp(irk_0 L/\epsilon) = Z^{(1)}(r\lambda + T, \tau) \exp(-irk_0 L/\epsilon), \tag{3.4}$$

for each integer r and each $T > 0$. Conditions (3.4) is equivalent to the following conditions

$$Y^{(1)}(z, \tau) = Z^{(1)}(-z, \tau), \quad Y^{(1)}(z + 2\lambda, \tau) = Y^{(1)}(z, \tau) \exp(-i\delta), \tag{3.5}$$

if $-\infty < z < \infty$ and $\tau > 0$, where

$$\delta \equiv \text{fractional part of} \quad k_0 L/\pi c. \tag{3.6}$$

Conditions (3.3) and (3.5) are to be imposed to the amplitude equations (2.20, 2.29–2.30) in this limiting case. Observe that as L varies by a $O(1)$ quantity, λ remains constant but δ varies.

Now let us see that in both cases

$$\phi^{(0)}_\eta \to 0, \quad \phi^{(0)}_{\eta\eta} \to 0 \quad \text{as} \quad \tau \to \infty.$$

This is a consequence of the fact that the averaged terms in the amplitude equation (2.20) depend only on time

$$e_1[\langle |Y^{(1)}|^2 \rangle^+ + \langle |Z^{(1)}|^2 \rangle^-] \equiv g(\tau). \tag{3.7}$$

Then (2.20) is reduced to the heat equation by a Hopf–Cole transformation. In fact, the solution of (2.20) may be written as

$$\phi^{(0)}(\eta,\tau) \cong \int_0^\tau g(\sigma)\,\mathrm{d}\sigma-(2+\beta_0)\log(h(\eta,\tau)), \tag{3.8}$$

where h satisfies the heat equation, $h_\tau = (1+\beta_0/2)\,h_{\eta\eta}$, and thus $h_\eta \to 0$ and $h_{\eta\eta} \to 0$ as $\tau \to \infty$.

Then, for sufficiently large τ, the amplitude equations (2.20, 2.29–2.30) reduce to

$$\mathrm{d}\phi^{(0)}/\mathrm{d}\tau = e_1[\langle|Y^{(1)}|^2\rangle^+ + \langle|Z^{(1)}|^2\rangle^-], \tag{3.9}$$

$$Y^{(1)}_\tau = cY^{(1)}_{\tilde{\eta}\tilde{\eta}} + Y^{(1)}[d-D_1|Y^{(1)}|^2-E_1\langle|Z^{(1)}|^2\rangle^-], \tag{3.10}$$

$$Z^{(1)}_\tau = cZ^{(1)}_{\dot{\eta}\dot{\eta}} + Z^{(1)}[d-D_1|Z^{(1)}|^2-E_1\langle|Y^{(1)}|^2\rangle^+]. \tag{3.11}$$

Observe that equation (3.9) yields a first correction to the burning rate, and that the system (3.10)–(3.11) is decoupled from (3.9).

Finally, we reduce (3.10)–(3.11) to a simpler form for convenience. If the following new variables are used

$$\sigma = \alpha_1\tau, \quad \tilde{\zeta} = \alpha_2\tilde{\eta}, \quad \dot{\zeta} = \alpha_2\dot{\eta}, \tag{3.12}$$

$$Y_1 = \alpha_3 Y^{(1)}\exp(i\alpha_4\tau), \quad Y_2 = \alpha_3 Z^{(1)}\exp(i\alpha_4\tau), \tag{3.13}$$

with the real constants $\alpha_1,\ldots,\alpha_4$, as defined by

$$\alpha_1 = \mathrm{Re}(d), \quad \alpha_2 = \surd(\mathrm{Re}(d)/\mathrm{Re}(c)), \tag{3.14}$$

$$\alpha_3 = \surd(\mathrm{Re}(D_1)/\mathrm{Re}(d)), \quad \alpha_4 = -\mathrm{Im}(d), \tag{3.15}$$

then $Y_1 = Y_1(\tilde{\zeta},\sigma)$ and $Y_2 = Y_2(\dot{\zeta},\sigma)$ satisfy

$$Y_{1\sigma} = (1+ia_1)\,Y_{1\tilde{\zeta}\tilde{\zeta}} + Y_1[1-(1+ia_2)|Y_1|^2-(a_3+ia_4)\langle|Y_2|^2\rangle^-], \tag{3.16}$$

$$Y_{2\sigma} = (1+ia_1)\,Y_{2\dot{\zeta}\dot{\zeta}} + Y_2[1-(1+ia_2)|Y_2|^2-(a_3+ia_4)\langle|Y_1|^2\rangle^+], \tag{3.17}$$

where the real constants $a_1,\ldots,a_4$ are given by

$$a_1 = \mathrm{Im}(c)/\mathrm{Re}(c) \simeq -1.242, \quad a_2 = \mathrm{Im}(D_1)/\mathrm{Re}(D_1) \simeq -4.507, \tag{3.18}$$

$$a_3 = \mathrm{Re}(E_1)/\mathrm{Re}(D_1) \simeq 7.328, \quad a_4 = \mathrm{Im}(E_1)/\mathrm{Re}(D_1) \simeq 1.664, \tag{3.19}$$

and the spatial averages $\langle\cdot\rangle^\pm$ are defined as above. Observe that for transversely finite flames, conditions (3.5) lead to

$$Y_1(z,\sigma) = Y_2(-z,\sigma), \quad Y_1(z+2\mu,\sigma) = Y_1(z,\sigma)\exp(-i\delta), \tag{3.20}$$

for all z and all $\sigma > 0$, where δ is defined in (3.6) and

$$\mu = \alpha_2\lambda. \tag{3.21}$$

3.1 Transversely infinite flames ($1/\epsilon^2L = O(1)$)

As explained above, for large τ the amplitudes of the transverse waves are given in this case by equations (3.16) and (3.17), which are now analysed. The simplest non-uniform solutions of (3.16) and (3.17) are given by the following pair of (one-parameter) families of solutions

$$Y_1 = \pm\surd(1-\alpha^2)\exp[i(-a_2+(a_2-a_1)\alpha^2)\sigma+i\alpha\tilde{\zeta}], \quad Y_2 = 0, \quad -1 \leqslant \alpha \leqslant 1, \tag{3.22}$$

$$Y_1 = 0, \quad Y_2 = \pm\surd(1-\alpha^2)\exp[i(-a_2+(a_2-a_1)\alpha^2)\sigma+i\alpha\dot{\zeta}], \quad -1 \leqslant \alpha \leqslant 1, \tag{3.23}$$

and by the following (two-parameter) family

$$Y_1 = R_1(\alpha,\beta)\exp(i\gamma_1(\alpha,\beta)\sigma + i\alpha\tilde{\zeta}), \quad Y_2 = R_2(\alpha,\beta)\exp(i\gamma_2(\alpha,\beta)\sigma + i\beta\tilde{\zeta}), \tag{3.24}$$

where the scalars R_1, R_2, γ_1 and γ_2 are defined as

$$R_1(\alpha,\beta)^2 = (1-a_3-\alpha^2+a_3\beta^2)/(1-a_3^2), \quad R_2(\alpha,\beta)^2 = (1-a_3-\beta^2+a_3\alpha^2)/(1-a_3^2),$$
$$\gamma_1(\alpha,\beta) = [(a_2+a_4)(a_3-1)+(a_2-a_3a_4-a_1(1-a_3^2))\alpha^2-(a_2a_3-a_4)\beta^2]/(1-a_3^2),$$
$$\gamma_2(\alpha,\beta) = [(a_2+a_4)(a_3-1)+(a_2-a_3a_4-a_1(1-a_3^2))\beta^2-(a_2a_3-a_4)\alpha^2]/(1-a_3^2),$$

for those values of the parameters α and β such that

$$1-a_3-\alpha^2+a_3\beta^2 < 0, \quad 1-a_3-\beta^2+a_3\alpha^2 < 0.$$

Observe that these inequalities define a bounded domain of the α-β plane.

The solutions of the families (3.22) and (3.23) (resp. of the family (3.24)) correspond to transverse periodic travelling waves (resp. to quasi-periodic waves) with a frequency and a wave-number close to Ω and k_0, respectively. In particular, if either (i) $\alpha = 0$ in (3.22), or (ii) $\alpha = 0$ in (3.23), or (iii) $\alpha = \beta = 0$ in (3.24), then the wavenumber of the corresponding wave of (1.1)–(1.4) is precisely equal to k_0 (to order ϵ). The solutions (i) and (ii) correspond to travelling waves of (1.1)–(1.4), while the solutions (iii) correspond to standing waves. These three particular members of the family (3.22)–(3.24) are the only ones found by Matkowsky & Olagunju (1982).

When taking into account the numerical values of the coefficients of (3.16)–(3.17) (see (3.18–3.19)), by a straightforward linear stability analysis, the following results are readily found. Since $a_3^2 > 1$, every member of the families (3.22) and (3.23) (resp. (3.24)) is stable (resp. unstable) under spatially uniform perturbations. Also, when considering spatially non-uniform perturbations, it is seen that the solutions (3.22) and (3.23) are stable if α^2 is sufficiently small, and unstable otherwise; at the threshold, a family of more complex solutions of (3.16)–(3.17) is expected to bifurcate from the families (3.22) and (3.23). Nevertheless, we have not analysed that bifurcation because we do not expect the new solutions to be of use in our search for chaotic solutions of (1.1)–(1.4).

Now, concerning the global dynamics of (3.16)–(3.17), observe that the coupling between both Ginzburg–Landau equations is fairly weak. If the averaged terms are ignored, then one obtains two uncoupled Ginzburg–Landau equations which are not expected to exhibit chaotic dynamics because $1+a_1a_2 > 0$ (see, e.g., Kuramoto 1984). Therefore, we do not expect the global dynamics of the model (3.16)–(3.17) to be chaotic.

3.2 Transversely finite flames ($\epsilon L = O(1)$)

Now we consider those solutions of (3.16)–(3.17) satisfying (3.20). By using (3.20), (3.16)–(3.17) may be reduced to the following problem on Y_1

$$Y_{1\sigma} = (1+ia_1)Y_{1\tilde{\zeta}\tilde{\zeta}} + Y_1[1-(1+ia_2)|Y_1|^2-(a_3+ia_4)\langle|Y_1|^2\rangle], \tag{3.25}$$
$$Y_1(\tilde{\zeta}+2\mu,\sigma) = Y_1(\tilde{\zeta},\sigma)\exp(-i\delta), \tag{3.26}$$

where, since $|Y_1|^2$ is spatially periodic, $\langle|Y_1|^2\rangle$ is the spatial average over a period, i.e.

$$\langle|Y_1|^2\rangle = (2\mu)^{-1}\int_0^{2\mu}|Y_1(\tilde{\zeta},\sigma)|^2\,d\tilde{\zeta}.$$

The simplest solutions of (3.25)–(3.26) are given by

$$Y_1 = \pm\sqrt{\frac{1-\alpha^2}{1+a_3}}\exp\left[i\gamma(\alpha)\,\sigma + i\alpha\tilde{\zeta}\right], \tag{3.27}$$

where $$\gamma(\alpha) = -[a_2 + a_4 + a_1(1+a_3)\,\alpha^2]/(1+a_3),$$

and the parameter α satisfies

$$\alpha^2 \leqslant 1; \quad 2\mu\alpha = -\delta + 2r\pi \quad \text{for some integer } r \quad \text{if} \quad \alpha \neq 0.$$

Then, for each pair of values of μ and δ, we have a finite number $N \geqslant 1$ of solutions of (3.16)–(3.17). These solutions are readily seen to correspond to time-periodic (or standing wave) solutions of (1.1)–(1.4). Observe that for fixed μ, N varies with δ. This fact yields to a somewhat surprising conclusion when we take into account the definition of the parameters μ and δ (see (3.1), (3.6), (3.21)). Namely, *a given periodic solution of* (1.1)–(1.4) *may fail to exist* as the transverse length of the flame $L(\sim 1/\epsilon)$ varies by a $O(1)$ quantity.

The solution (3.27) corresponding to $\alpha = 0$ is linearly stable as it is readily seen. The remaining solutions are stable to spatially uniform perturbations, but they are stable to spatially non-uniform perturbations only if α^2 is sufficiently small. Again, at the threshold, a family of more complex solutions of (3.25)–(3.26) is expected to bifurcate from the solutions (3.27). However, for the same reason as in §3.1, we do not pursue this matter further. As in §3.1, since $1 + a_1 a_2 > 0$ we do not expect the dynamics of (3.25)–(3.26) to be chaotic.

4 Concluding remarks

We have analysed the nonlinear stability of adiabatic two-dimensional plane flame fronts, near the so-called pulsating instability limit, by using a standard thermo-diffusive model which seems appropriate in the large activation energy limit. The well-known linear stability results, recalled in §1, predict the appearance of two transverse wave-trains of small amplitude along the flame sheet, travelling in opposite directions. In §2 we derived, via a multi-scale analysis, three amplitude equations (2.20, 2.29–2.30), giving the reaction sheet location to leading order $\phi^{(0)}$, and the complex amplitudes of the transverse travelling wavetrains $Y^{(1)}$ and $Z^{(1)}$ in terms of the slowest space and time variables. The three equations are coupled by non-local terms involving the spatial averages of the amplitudes.

In §3 we used the amplitude equations to analyse transversely finite flames, of large transverse characteristic length L. We considered two cases. For $L \sim \epsilon^{-2}$ or larger ($\epsilon^2 = \beta - \beta_0$ is the small parameter in the problem, the deviation of the reduced Lewis number β from its value at threshold) the flame is considered as transversely infinite. In this case we found three families of transverse waves; by a linear stability analysis, we concluded that only some of these waves, namely those that are periodic travelling ones, are stable. For $L \sim \epsilon^{-1}$ we derived the appropriate reflection conditions of the transverse waves at the sidewalls, which were added to the amplitude equations. By inspection of the resulting model we found a finite number $N \geqslant 1$ of periodic solutions. Of course, N depends on L and (surprisingly to some extent) N varies as L is varied by a $O(1)$ quantity. One of these branches of solutions is stable for all L, while the remaining ones may be stable or unstable, depending on L.

From these results we may get the following picture about the behaviour of the flame as ϵ varies, for a fixed, large, value of the transverse length L. For sufficiently small ϵ, $\epsilon L = O(1)$ and the flame is expected to exhibit cellular periodic behaviour. As ϵ increases, L becomes of order ϵ^{-2} or larger, and the flame is expected to exhibit transverse travelling waves in the bulk, except in the vicinity of the sidewalls.

Some remarks about these results are in order:

(A) For $L \sim \epsilon^2$ the behaviour of the flame near the side-walls (within a distance of order ϵ^{-1} from them) has not been analysed. Such an analysis should answer the question as to whether the walls affect the bulk, either on the slowest time scale considered in this paper ($t \sim \epsilon^{-2}$) or on still larger time scales. For a discussion of this question in a related problem, including an incomplete answer, see Knobloch & De Luca (1990).

(B) At the level of the analysis in this paper, i.e. weakly nonlinear, two space dimensional, no chaotic behaviour has been found. Nevertheless, spatially three-dimensional effects might lead to chaos. A first partial answer to this question will be obtained by considering the weakly three-dimensional case, in which the reaction sheet location and the complex amplitudes of the transversal waves are considered as slowly dependent on the third spatial coordinate (see Vega 1991).

(C) For simplicity in the calculation of the coefficients of the amplitude equations (which is fairly wearisome) we only considered the adiabatic case. If volumetric heat losses are taken into account, the analysis is similar, and leads again to the amplitude equations derived in §2, with the coefficients depending on the heat loss parameter m. If, for some value of m, the coefficients of the equations (3.16)–(3.17) were such that $1+a_1 a_2$ changes sign, then an analytical description of chaotic behaviour, via the Kuramoto–Sivashinsky equation, would readily follow (see, e.g., Kuramoto 1984).

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Appendix

The function U_0 appearing in (2.6) and thereafter is given by

$$U_0(\xi) = \begin{pmatrix} 1 \\ -\beta_0 \xi \end{pmatrix} \exp(\xi) \quad \text{if} \quad \xi \leqslant 0, \quad U_0(\xi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{if} \quad \xi > 0. \tag{A 1}$$

The linear operators l_0 and l_1, appearing in the solvability conditions (2.9)–(2.10) are defined as

$$\begin{aligned} l_0(F,f) = & \int_{-\infty}^{0} [2F_1(\xi, y, \eta, t, T, \tau) + F_2(\xi, y, \eta, t, T, \tau)]\,\mathrm{d}\xi \\ & + \int_0^{\infty} F_2(\xi, y, \eta, t, T, \tau) \exp(-\xi)\,\mathrm{d}\xi + 2f_1(y, \eta, t, T, \tau) + f_2(y, \eta, t, T, \tau), \end{aligned} \tag{A 2}$$

$$\begin{aligned} l_1(F,f) = & \int_{-\infty}^{0} \left\{\left[2(p-q) - \beta_0 \frac{q+i\Omega}{p-q}\xi\right] F_1(\xi, y, \eta, t, T, \tau) + F_2(\xi, y, \eta, t, T, \tau)\right\} \exp(-q\xi)\,\mathrm{d}\xi \\ & + \int_0^{\infty} F_2(\xi, y, \eta, t, T, \tau) \exp(-p\xi)\,\mathrm{d}\xi + 2(p-q) f_1(y, \eta, t, T, \tau) + f_2(y, \eta, t, T, \tau), \end{aligned} \tag{A 3}$$

where $F = (F_1, F_2)^{\mathrm{T}}$ and $f = (f_1, f_2)^{\mathrm{T}}$ stand for the right-hand sides of the equation (2.6) and the jump condition (2.7), respectively, and the (complex) constants p and q are given by

$$2p - 1 = 1 - 2q = (1 + 4k_0^2 + 4i\Omega)^{\frac{1}{2}}. \tag{A 4}$$

The eigenfunction V appearing in (2.12) and thereafter is associated with the eigenvalue ω_2 (see (1.9)) at $k = k_0$ and $\beta = \beta_0$. It is given by

$$\left.\begin{aligned} V(\xi) &= \begin{pmatrix} -1 \\ 2q+\beta_0+\beta_0 \dfrac{p^2-k_0^2}{p-q}\xi \end{pmatrix} \exp\,(p\xi) + \begin{pmatrix} 1 \\ -\beta_0(1+\xi) \end{pmatrix} \exp\,(\xi) \quad \text{if} \quad \xi \leqslant 0, \\ V(\xi) &= \begin{pmatrix} 0 \\ 2q \end{pmatrix} \exp\,(q\xi) \quad \text{if} \quad \xi > 0. \end{aligned}\right\} \tag{A 5}$$

When taking into account (2.12) and (2.13), the right-hand sides of equations (2.6) and (2.7) at order ϵ^2, $F^{(2)}$ and $f^{(2)} = (f_1^{(2)}, f_2^{(2)})^{\mathrm{T}}$ are seen to be given by

$$\begin{aligned} F^{(2)} \equiv{} & (|W^{(1)}|^2+|X^{(1)}|^2)\,[(k_0^2 D_0 + i\Omega I)\,V' - k_0^2 D_0\, U_0''] \\ & + [\phi_{\eta\eta}^{(0)} D_0\, U_0' - |\phi_\eta^{(0)}|^2 D_0\, U_0'' - (\phi_\tau^{(0)} + \phi_T^{(1)})\, U_0']/2 \\ & + [(W_T^{(1)} I - 2ik_0\, W_\eta^{(1)} D_0)(V - U_0') + 2ik_0\, W^{(1)} \phi_\eta^{(0)} D_0(V' - U_0'')] \exp\,(i\Omega t + ik_0 y) \\ & + [(X_T^{(1)} I + 2ik_0\, X_\eta^{(1)} D_0)(V - U_0') - 2ik_0\, X^{(1)} \phi_\eta^{(0)} D_0(V' - U_0'')] \exp\,(i\Omega t - ik_0 y) \\ & + [k_0^2 D_0(U_0'' - 3V') - i\Omega V']\,[(W^{(1)})^2 \exp\,(2i\Omega t + 2ik_0 y) + (X^{(1)})^2 \exp\,(2i\Omega t - 2ik_0 y)] \\ & + W^{(1)}\,\overline{X^{(1)}}\,[k_0^2 D_0(2U_0'' - 3V' - 3\overline{V}') + i\Omega(V' - \overline{V}')] \exp\,(2ik_0 y) \\ & + 2W^{(1)} X^{(1)}\,[k_0^2 D_0(V' - U_0'') - i\Omega V'] \exp\,(2i\Omega t) + \text{c.c.}, \end{aligned} \tag{A 6}$$

$$\begin{aligned} f_1^{(2)} \equiv{} & (k_0^2 - q^2)(|W^{(1)}|^2 + |X^{(1)}|^2)/2 + |\phi_\eta^{(0)}|^2/4 \\ & + ik_0\, \phi_\eta^{(0)}\,[W^{(1)} \exp\,(i\Omega t + ik_0 y) - X^{(1)} \exp\,(i\Omega t - ik_0 y)] \\ & - [(k_0^2 + q^2)/2]\,[(W^{(1)})^2 \exp\,(2ik_0 y) + (X^{(1)})^2 \exp\,(-2ik_0 y)] \exp\,(2i\Omega t) \\ & + W^{(1)} X^{(1)} (k_0^2 - q^2) \exp\,(2i\Omega t) - W^{(1)}\,\overline{X^{(1)}}\,(k_0^2 + |q|^2) \exp\,(2ik_0 y) + \text{c.c.}, \end{aligned} \tag{A 7}$$

$$f_2^{(2)} = 0, \tag{A 8}$$

where I is the 2×2 unit matrix and, as above, overbars and c.c. stand for the complex conjugate, and primes denote differentiation with respect to ξ.

The constant e_1 appearing in equation (2.16) and thereafter is found to be

$$e_1 = |q|^2 - k_0^2 - (q + \bar{q})^2 \simeq 0.0418. \tag{A 9}$$

When using the solvability conditions (2.14)–(2.16), the solution of (2.6)–(2.8) for $j = 2$, $(U^{(2)}, \psi^{(2)})$ is found to be

$$\begin{aligned} U^{(2)} \equiv{} & [\phi_{\eta\eta}^{(0)}(U^{(20)} - (1 + \beta_0/2)\,\xi U_0') - |\phi_\eta^{(0)}|^2 (\xi U_0')/2 + (|W^{(1)}|^2 + |X^{(1)}|^2)(U^{(21)} - e_1 \xi U_0')]/2 \\ & + [W_\eta^{(1)} (k_0 b U^{(22)} + U^{(23)}) + W^{(1)} \phi_\eta^{(0)} (U^{(24)} - ik_0\, U^{(22)}) \\ & + W^{(2)}(\eta, T, \tau)\, V] \exp\,(i\Omega t + ik_0 y) \\ & - [X_\eta^{(1)} (k_0 b U^{(22)} + U^{(23)}) + X^{(1)} \phi_\eta^{(0)} (U^{(24)} + ik_0\, U^{(22)}) - X^{(2)}(\eta, T, \tau)\, V] \exp\,(i\Omega t - ik_0 y) \\ & + [(W^{(1)})^2 \exp\,(2ik_0 y) + (X^{(1)})^2 \exp\,(-2ik_0 y)]\, U^{(25)} \exp\,(2i\Omega t) \\ & + W^{(1)} X^{(1)}\, U^{(26)} \exp\,(2i\Omega t) \\ & + W^{(1)}\,\overline{X^{(1)}}\, U^{(27)} \exp\,(2ik_0 y) + \text{c.c.}, \end{aligned} \tag{A 10}$$

$$\begin{aligned} \psi^{(2)} \equiv{} & \phi^{(2)}(\eta, T, \tau) + W^{(2)}(\eta, T, \tau) \exp\,(i\Omega t + ik_0 y) + X^{(2)}(\eta, T, \tau) \exp\,(i\Omega t - ik_0 y) \\ & + [(W^{(1)})^2 \exp\,(2ik_0 y) + (X^{(1)})^2 \exp\,(-2ik_0 y)]\, \alpha^{(25)} \exp\,(2i\Omega t) \\ & + W^{(1)} X^{(1)}\, \alpha^{(26)} \exp\,(2i\Omega t) \\ & + W^{(1)}\,\overline{X^{(1)}}\, \alpha^{(27)} \exp\,(2ik_0 y) + \text{c.c.} \end{aligned} \tag{A 11}$$

Here, the functions U_0 and V are given by (A 1) and (A 5), while the constants $\alpha^{(25)}, \ldots, \alpha^{(27)}$ and the functions $U^{(20)}, \ldots, U^{(27)}$ are given by

$$\alpha^{(25)} \simeq -(0.287 + i\,1.058), \quad \alpha^{(26)} \simeq -(0.615 + i\,0.906), \quad \alpha^{(27)} \simeq 0.448,$$

$$U^{(2j)} \equiv \begin{pmatrix} g_{j1}(\xi) \\ g_{j2}(\xi) \end{pmatrix} \quad \text{if } \xi \leqslant 0, \quad U^{(2j)} \equiv \begin{pmatrix} 0 \\ g_{j3}(\xi) \end{pmatrix} \quad \text{if } \xi > 0, \quad \text{for} \quad j = 0, \ldots, 7,$$

where
$$g_{01} \equiv \xi \exp(\xi), \quad g_{02} \equiv -\beta_0(1+\xi^2)\exp(\xi), \quad g_{03} \equiv -\beta_0,$$

$$g_{11} \equiv h_1(\xi) + p\exp(\xi) + \text{c.c.},$$

$$g_{12} \equiv h_2(\xi) + (A_{12} - \beta_0 p\xi)\exp(\xi) + \text{c.c.}, \quad g_{13} \equiv h_3(\xi) + \text{c.c.},$$

$$g_{21}(\xi) \equiv -(p-q)^{-1}\xi\exp(p\xi),$$

$$g_{22} \equiv \left\{A_{22} + \left[\frac{2q}{p-q} + \frac{\beta_0}{p-q}\left(1 + 2\frac{i\Omega + 2k_0^2}{(p-q)^2}\right)\right]\xi + \beta_0\frac{p^2-k_0^2}{(p-q)^2}\xi^2\right\}\exp(p\xi),$$

$$g_{23} \equiv \left(A_{22} - \frac{2q}{p-q}\xi\right)\exp(q\xi),$$

$$g_{31} \equiv -2ik_0\, g_{21}(\xi),$$

$$g_{32} \equiv -2ik_0\left[g_{22}(\xi) + \frac{\beta_0}{p-q}\left(\frac{1}{p-q} - \xi\right)\exp(p\xi)\right], \quad g_{33} \equiv -2ik_0\left[g_{23}(\xi) + \beta_0\frac{\exp(q\xi)}{(p-q)^2}\right],$$

$$g_{41} \equiv \frac{-2ik_0 p\xi}{p-q}\exp(p\xi),$$

$$g_{42} \equiv 2ik_0\left[A_{42} + \frac{\beta_0 p}{(p-q)^2} + \left(\frac{2pq}{p-q} + \frac{2p^2\beta_0}{(p-q)^2} - \beta_0\frac{p^2-k_0^2}{(p-q)^3}\right)\xi + p\beta_0\frac{p^2-k_0^2}{(p-q)^2}\xi^2\right]\exp(p\xi),$$

$$g_{43} \equiv 2ik_0\left[A_{42} + \frac{\beta_0 p}{(p-q)^2} - \frac{2q^2\xi}{p-q}\right]\exp(q\xi),$$

$$g_{51} \equiv h_1(\xi) + (\alpha^{(25)} + 1/2)\,h_4(\xi) + A_{51}\exp(p_1\xi),$$

$$g_{52} \equiv h_2(\xi) - \beta_0 h_4(\xi)/2 + (\alpha^{(25)} + 1/2)\,h_5(\xi) + \left(A_{52} + \beta_0 A_{51}\frac{4k_0^2 - p_1^2}{2p_1 - 1}\xi\right)\exp(p_1\xi),$$

$$g_{53} \equiv h_3(\xi) + A_{53}\exp[(1-p_1)\xi],$$

$$g_{61} \equiv 2h_1(\xi) + (\alpha^{(26)} + 1)\,h_4(\xi) + A_{61}\exp(p_2\xi),$$

$$g_{62} \equiv 2h_2(\xi) - \beta_0 h_4(\xi) + (\alpha^{(26)} + 1)\,h_5(\xi) + \left(A_{62} - \beta_0 A_{61}\frac{p_2^2}{2p_2 - 1}\xi\right)\exp(p_2\xi),$$

$$g_{63} \equiv 2h_3(\xi) + A_{63}\exp[(1-p_2)\xi],$$

$$g_{71} \equiv h_1(\xi) + (\alpha^{(27)} + 1)\,h_4(\xi)/2 + A_{71}\exp(p_3\xi) + \text{c.c.},$$

$$g_{72} \equiv h_2(\xi) - \beta_0 h_4(\xi)/2 + (\alpha^{(27)} + 1)\,h_5(\xi)/2 + \left(A_{72} + \beta_0 A_{71}\frac{4k_0^2 - p_3^2}{2p_3 - 1}\xi\right)\exp(p_3\xi) + \text{c.c.},$$

$$g_{73} \equiv h_3(\xi) + A_{73}\exp[(1-p_3)\xi] + \text{c.c.},$$

with the constants $A_{12}, \ldots, A_{73}, p_1, \ldots, p_3$, and the functions $h_1, \ldots, h_5$, as given by

$$A_{12} \simeq -24.00, \quad A_{22} \simeq 0.364+i0.275, \quad A_{42} \simeq -3.044+i3.75, \quad A_{51} \simeq 1.037+i1.579,$$
$$A_{52} \simeq -(17.29+i15.97), \quad A_{53} \simeq -0.891+i0.868, \quad A_{61} \simeq 2.115+i1.947,$$
$$A_{62} \simeq -(33.31+i20.22), \quad A_{63} \simeq -0.0750+i0.546,$$
$$A_{71} \simeq 0.526, \quad A_{72} \simeq -10.83, \quad A_{73} \simeq 0.112,$$
$$2p_1 - 1 = (1+16k_0^2+8i\Omega)^{\frac{1}{2}}, \quad 2p_2 - 1 = (1+8i\Omega)^{\frac{1}{2}}, \quad 2p_3 - 1 = (1+16k_0^2)^{\frac{1}{2}},$$
$$h_1(\xi) \equiv -p \exp(p\xi), \quad h_2(\xi) \equiv \left[p(2q+\beta_0)+\beta_0 \frac{p^2-k_0^2}{p-q}(1+p\xi)\right] \exp(p\xi),$$
$$h_3(\xi) \equiv 2q^2 \exp(q\xi), \quad h_4(\xi) \equiv \exp(\xi), \quad h_5(\xi) \equiv -\beta_0(1+\xi) \exp(\xi).$$

When taking into account (2.12) and (2.13) and (A 10) and (A 11), the right-hand sides of equations (2.6) and (2.7) at order ϵ^3, $F^{(3)}$ and $f^{(3)} = (f_1^{(3)}, f_2^{(3)})^{\mathrm{T}}$ are seen to be of the form

$$F^{(3)} \equiv \sum_{r,s} M_{(r,s)}(\xi, \eta, T, \tau) \exp(ir\Omega t + isk_0 y) + \text{c.c.}, \tag{A 12}$$

$$f_1^{(3)} \equiv \sum_{r,s} m_{1(r,s)}(\eta, T, \tau) \exp(ir\Omega t + isk_0 y) + \text{c.c.}, \tag{A 13}$$

$$f_2^{(3)} \equiv \sum_{r,s} m_{2(r,s)}(\eta, T, \tau) \exp(ir\Omega t + isk_0 y) + \text{c.c.}, \tag{A 14}$$

where the integers r and s are such that $0 \leqslant r \leqslant 3$, $-3 \leqslant s \leqslant 3$. In order to apply the (two) solvability conditions (2.10), we only need to calculate the coefficients of (A 12)–(A 14) for $(r,s) = (1,1)$ and for $(r,s) = (1,-1)$. For $r = s = 1$, those coefficients are given by

$$\begin{aligned}
M_{(1,1)} \equiv{} & -2ik_0 W_\eta^{(2)} D_0 V_1 + 2ik_0(\phi_\eta^{(0)} W^{(2)} + \phi_\eta^{(1)} W^{(1)}) D_0 V_1' + (W_T^{(2)} + W_\tau^{(1)}) V_1 \\
& - W_{\eta\eta}^{(1)}(D_0 V_1 + k_0 D_3 U_3) - W^{(1)}[D_2(V'' - k_0^2 V_1) + (k_0^2 + i\Omega) U_2'] \\
& + \phi_\eta^{(0)} W_\eta^{(1)}[2D_0(V_1' + ik_0 U_3') - ik_0 U_3 - k_0 D_3 U_4] \\
& + \phi_{\eta\eta}^{(0)} W^{(1)}[(D_0 - (1+\beta_0/2)I) V' - D_4 U_5' - ik_0 U_3 - 2ik_0 D_0 U_4'] \\
& + |\phi_\eta^{(0)}|^2 W^{(1)}[V'/2 - D_0 V'' + D_4(\xi U_0')'/2 - ik_0 U_4 + 2ik_0 D_0 U_4'] \\
& + W^{(1)}|W^{(1)}|^2[k_0^2 D_0(\bar{V}'' - 2V'') - e_1 V' - \alpha^{(25)}(4k_0^2 D_0 U_0'' + 2i\Omega \bar{V}') - D_4 U_6' + D_5 U_7'] \\
& - W^{(1)}|X^{(1)}|^2[2k_0^2 D_0 \bar{V}'' + e_1 V' + 2i\Omega\alpha^{(26)} \bar{V}' + 4k_0^2 \alpha^{(27)} D_0 U_0'' \\
& + D_4 U_6' + \bar{D}_4 U_8' - \bar{D}_5 U_9']
\end{aligned} \tag{A 15}$$

where I is the unit matrix, D_2 is defined in §2, $D_3 \equiv 2iD_0 - bI$, $D_4 = k_0^2 D_0 + i\Omega I$, $D_5 = 3k_0^2 D_0 + i\Omega I$, U_2 is defined in §2, $V_1 \equiv V - U_0'$, $U_3 \equiv U^{(23)} + k_0 b U^{(22)}$, $U_4 \equiv U^{(24)} - ik_0 U^{(22)}$, $U_5 \equiv U^{(20)} - (1+\beta_0/2)(\xi U_0')$, $U_6 \equiv U^{(21)} - e_1 \xi U_0'$, $U_7 \equiv U^{(25)}$, $U_8 \equiv U^{(26)}$, $U_9 \equiv U^{(27)}$, and the remaining constants are as defined above.

$$\begin{aligned}
m_{1(1,1)} \equiv{} & ik_0(\phi_\eta^{(0)} W^{(2)} + \phi_\eta^{(1)} W^{(1)}) + \phi_\eta^{(0)} W_\eta^{(1)} - \phi_{\eta\eta}^{(0)} W^{(1)} v_2(0) u_2^{(20)}(0)/4 + |\phi_\eta^{(0)}|^2 W^{(1)} v_2(0)/4 \\
& - W^{(1)}|W^{(1)}|^2[v_2(0)(|v_2(0)|^2/4 + u_2^{(21)}(0) - 2k_0^2) + \bar{v}_2(0)(u_2^{(25)}(0) + k_0^2) - 8k_0^2 \alpha^{(25)}]/4 \\
& - W^{(1)}|X^{(1)}|^2[v_2(0)(|v_2(0)|^2/2 + u_2^{(21)}(0) + u_2^{(27)}(0)) + \bar{v}_2(0)(u_2^{(26)}(0) - 2k_0^2) - 8k_0^2 \alpha^{(27)}]/4,
\end{aligned} \tag{A 16}$$

where v_2 is the second component of the function V, and, for $j = 0, \ldots, 7$, $u_2^{(2j)}$ is the second component of the function $U_2^{(2j)}$ defined above.

$$m_{2(1,1)} \equiv qW^{(1)}, \quad m_{2(1,-1)} \equiv qX^{(1)}. \tag{A 17}$$

To obtain $M_{(1,-1)}$ (resp. $m_{1(1,-1)}$) we only need to change the sign of the first three terms (resp. of the first term) in the right hand side of (A 15) (resp. of (A 16)) and interchange $W^{(1)}$ with $X^{(1)}$ and $W^{(2)}$ with $X^{(2)}$ everywhere.

When applying the solvability conditions (2.10) at order ϵ^3, the coefficients c and d of (2.22) and (2.23) are seen to coincide with those obtained in the linear stability analysis (see (1.11)), as it was to be expected, while the coefficients A, B, C, D and E are found to be

$$A = -1, \quad B \simeq -1.48 + i\,8.51, \quad C \simeq i\,0.139, \quad D \simeq 0.089 - i\,0.409, \quad E \simeq 0.655 + i\,0.143. \tag{A 18}$$

The coefficients A_1, B_1, C_1, D_1 and E_1 of equations (2.25) and (2.26) are

$$\left.\begin{array}{l} A_1 = A + 2ic/b \simeq -0.378 + i\,0.501, \quad B_1 = B + i(1 + \beta_0/2 - c)/b \simeq -1.79 + i\,9.20, \\ C_1 = C - iA/b + (c - ib/2)/b^2 \simeq 0.037 + i\,0.167, \quad D_1 = D + ie_1/b \simeq 0.089 - i\,0.403, \\ E_1 = E + ie_1/b \simeq 0.655 + i\,0.149. \end{array}\right\} \tag{A 19}$$