Coupled Amplitude-Streaming Flow Equations for Nearly Inviscid Faraday Waves in Small Aspect Ratio Containers*

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Summary. We derive a set of asymptotically exact *coupled amplitude-streaming flow* (CASF) equations governing the evolution of weakly nonlinear nearly inviscid multi-mode Faraday waves and the associated streaming flow in finite geometries. The streaming flow is found to play a particularly important role near mode interactions. Such interactions come about either through a suitable choice of parameters or through breaking of degeneracy among modes related by symmetry. An example of the first case is provided by the interaction of two nonaxisymmetric modes in a circular container with different azimuthal wavenumbers. The second case arises when the shape of the container is changed from square to slightly rectangular, or from circular to slightly noncircular but with a plane of symmetry. The generation of streaming flow in each of these cases is discussed in detail and the properties of the resulting CASF equations are described. A preliminary analysis suggests that these equations can resolve discrepancies between existing theory and experimental results in the first two of the above cases.

Key words. vibrating flows, amplitude equations, streaming flow, mean flow, Faraday waves, mode interaction

1. Introduction and Formulation

The Faraday instability, that is the excitation of surface gravity-capillary waves by the vertical vibration of a container of fluid [1], has been of great interest from the point of view of pattern formation [2], [3]. This system has an additional appeal in the low viscosity limit because of its close connection with classical water wave theory. However, this limit is singular and must be treated with care. This is because viscous oscillatory boundary layers attached to the container and the free surface are capable of driving *streaming flows* that in turn interact with the waves responsible for them [4], [5], [6]. This interaction arises already at leading (i.e., cubic) order and as a result has a strong effect on the stability of the waves. Depending on circumstances the streaming flow can promote instability or stabilize the waves. As a result theories of the Faraday instability based on the potential formulation are fundamentally unreliable, even in the low viscosity limit. In a recent paper [5] we have discussed the origin of the streaming flow and derived equations describing the interaction of this flow with the Faraday waves in the case of an extended two-dimensional container. In such containers a mean flow is easily excited and consists of two contributions, the inviscid mean flow familiar from theories of inviscid water waves, and the streaming flow driven by nonzero time-averaged Reynolds stress in the oscillatory boundary layers along no-slip boundaries and the free surface. The mechanisms responsible for driving the streaming flow (also called "acoustic streaming" or "viscous mean flow") are well known and go back to the work of Schlichting [7] and Longuet-Higgins [8] (see [9] for a review). However, their importance for the dynamics of Faraday waves under experimentally relevant conditions has been recognized only relatively recently [5], [6]. In this paper we focus on three-dimensional containers of small aspect ratio, i.e., systems in which the frequency of the vibration selects a wavenumber of the instability that is comparable to the size of the container. In this case inviscid mean flows are much harder to excite (although as we shall see they are not entirely absent) and the viscous streaming flow provides the dominant interaction with the waves. We point out that mode interactions are very effective in generating such viscous streaming flows and hence that such flows must be included in any quantitative attempt to explain experiments on mode interactions in the nearly inviscid Faraday system. The case of a circular container is typical. If only one (axisymmetric) surface mode is excited, its evolution decouples from the streaming flow (see Section 2.3.1). In contrast, if two counter-rotating surface modes are involved, the system selects an equal amplitude superposition of these modes. The resulting oscillation is a standing wave and so is completely determined up to an overall phase; however, it is this phase that is coupled to the streaming flow and that can exhibit nontrivial dynamics (see Section 4.2). This special property of the system is a consequence of rotational invariance of the system, and is in direct contrast to (counter-rotating) waves excited directly by *lateral* vibration where the wave amplitudes couple to the streaming flow as well [4], [6]. In the present paper we show that even with vertical vibration the full coupling between the streaming flow and the wave amplitude and phase is restored when (i) the (circular) cross section of the Faraday container is slightly perturbed so that invariance under rotation is lost, or (ii) when a second pair a counter-rotating modes is present. In general, (iii) the presence of two surface modes suffices for full coupling if the cross section of the container is not circular or if it is circular but the two interacting modes are axisymmetric (for they then differ from one another in something besides the sign of their phase velocity). These three cases are considered explicitly below and used to illustrate the general theory presented in this paper.

Streaming flows are of interest in other areas of fluid mechanics as well, and have been studied theoretically and/or experimentally in connection with flows in blood vessels [10], generation of mean motions in the ear [11], interaction of sound waves with obstacles [12] as well as flows around vibrating bodies [13]. In these applications the interest is in steady flows generated by oscillations; such flows are sometimes called steady streaming. Analogous flows produced by a viscous boundary layer attached to a vibrating free surface are of interest in water wave theory ([14], [15], [16], [17], [18] and references therein) and play a fundamental role in the instability of the ocean to Langmuir circulations [19], [20]. They have also been studied in connection with capillary waves [21] and in conjunction with thermal effects in order to investigate the usability of the resulting streaming flow for controlling undesirable thermocapillary convection [22], [23], [24] that occurs in materials processing in microgravity [25], [26]. In all these cases the primary oscillating flow was given a priori. On the other hand, steady circulations are known to affect the dynamics of surface waves [27], [28], suggesting that the streaming flow generated by the waves themselves can also affect their dynamics. The techniques developed in this paper show that this is indeed the case. Similar coupling arises in vibrating liquid bridges [6], [29], and may well play a role in the dynamics of acoustically driven drops and bubbles. In particular the current description of self-propulsion of acoustically driven bubbles relies on the excitation of specific mode interactions but remains entirely inviscid [30], [31], [32].

To formulate the mathematical problem, we consider a cylindrical container of general cross-section Σ under vertical vibration. In order to avoid uncertainties associated with the modeling of contact line dynamics ([33], [34] and references therein) and additional difficulties due to the presence of a strong singularity in the velocity at a moving contact line when the contact angle differs from 0 or π [35], [36], we assume that the contact line is pinned to the upper edge of the vertical wall of the container, and that the liquid fills the container such that the unperturbed free surface is exactly horizontal. Faraday experiments on this configuration have been performed in an attempt to eliminate the lateral meniscus and the associated meniscus waves [3], [37], [38]. We nondimensionalize lengths using the unperturbed depth h and time using the *gravity-capillary time* $[g/h + T/(\rho h^3)]^{-1/2}$, where g is the gravitational acceleration, T is the coefficient of surface tension, and ρ is the density, all assumed to be constant. We use a Cartesian coordinate system attached to the vibrating container, with the $z = 0$ plane at the unperturbed free surface. The governing equations (continuity and momentum conservation) and boundary conditions (no-slip at solid boundaries, kinematic compatibility and tangential and normal stress balance at the free surface) are

$$\nabla \cdot \boldsymbol{v} = 0, \qquad \partial \boldsymbol{v}/\partial t - \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = -\nabla p + C_g \Delta \boldsymbol{v} \tag{1.1}$$

for $(x, y) \in \Sigma, \quad -1 < z < f$,

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{if } z = -1 \text{ or if } (x, y) \in \partial\Sigma, \qquad f = 0 \quad \text{if } (x, y) \in \partial\Sigma, \tag{1.2}$$

$$\boldsymbol{v} \cdot \boldsymbol{n} = (\partial f/\partial t)(\boldsymbol{e}_z \cdot \boldsymbol{n}), \qquad [(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\top}) \cdot \boldsymbol{n}] \times \boldsymbol{n} = \boldsymbol{0}, \quad \text{at } z = f, \tag{1.3}$$

$$p - |\boldsymbol{v}|^2/2 - (1-S)f + S\nabla \cdot [\nabla f/(1+|\nabla f|^2)^{1/2}]$$
$$= C_g[(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^\top) \cdot \boldsymbol{n}] \cdot \boldsymbol{n} - 4\mu\omega^2 f \cos 2\omega t, \quad \text{at } z = f, \quad (1.4)$$

together with appropriate initial conditions. Here p (= pressure + $|\boldsymbol{v}|^2/2 + (1-S)z - 4\mu\omega^2 z \cos 2\omega t$) is a modified (hydrostatic stagnation) pressure, $\boldsymbol{v}$ is the velocity, f is the vertical deflection of the free surface, $\boldsymbol{n}$ is the outward unit normal to the free surface, while $\partial\Sigma$ denotes the boundary of the cross-section Σ (i.e., the lateral walls) and $\boldsymbol{e}_z$ is the upward unit vector. The real parameters $\mu > 0$ and 2ω denote the amplitude and frequency of the forcing. The quantity $C_g = \nu/(gh^3 + Th/\rho)^{1/2}$ (with ν = kinematic viscosity) is a *capillary-gravity number* and $S = T/(T + \rho g h^2)$ is a *gravity-capillary balance parameter*; these are related to the usual *capillary number* $C \equiv \nu\sqrt{\rho/Th}$ and *Bond number* $B \equiv \rho g h^2/T$ by $C_g = C/(1+B)^{1/2}$ and $S = 1/(1+B)$. The parameter S is such that $0 \le S \le 1$, with the extreme cases $S = 0$ and $S = 1$ corresponding to the *purely gravitational* limit ($T = 0$) and the *purely capillary* limit ($g = 0$), respectively.

In this paper we consider the (nearly inviscid, nearly resonant, weakly nonlinear) limit

$$C_g \ll 1, \qquad |\omega - \Omega| \ll 1, \qquad \mu \ll 1, \quad (1.5)$$

where Ω is an inviscid eigenfrequency of the linearized problem around the flat state. In contrast to [5] we assume that Ω has (algebraic and geometric) multiplicity $N \ge 1$. Situations with $N > 1$ arise either due to the presence of symmetries or at mode interaction points that take place at particular values of μ and ω. This N-fold degeneracy of the linear inviscid problem can be lifted by forced symmetry breaking or by moving μ and ω slightly from the mode interaction point. The inclusion of viscosity also shifts the location of the mode interaction point. In either case these perturbations (generically) split the eigenfrequency Ω into N distinct frequencies, $\Omega_1, \ldots, \Omega_N$, assumed to be such that

$$|\Omega_k - \Omega| \ll 1 \quad \text{for } k = 1, \ldots, N. \quad (1.6)$$

As already mentioned, the streaming flow is expected to play a significant role in just these circumstances. This flow enters into the problem because the linearized problem admits *hydrodynamic* (or *viscous*) modes [39], [40], [5], in addition to the usual *surface modes*. In the nearly inviscid limit the former decay more slowly than the surface modes, and so are easily excited, forming the streaming flow. For small C_g, these modes take the form $(\boldsymbol{v}, p, f) = e^{C_g \lambda t}(\boldsymbol{U}, C_g P, C_g F) + \cdots$, with the (real) eigenvalue $\lambda < 0$ given by

$$\nabla \cdot \boldsymbol{U} = 0, \qquad \lambda \boldsymbol{U} = -\nabla P + \Delta \boldsymbol{U}, \quad \text{if } (x, y) \in \Sigma, \quad -1 < z < 0, \quad (1.7)$$

$$\boldsymbol{U} = \boldsymbol{0} \quad \text{if } z = -1 \text{ or if } (x, y) \in \partial\Sigma, \quad (1.8)$$

$$\boldsymbol{e}_z \cdot \boldsymbol{U} = 0, \qquad [\boldsymbol{e}_z \cdot (\nabla \boldsymbol{U} + \nabla \boldsymbol{U}^\top)] \times \boldsymbol{e}_z = \boldsymbol{0} \quad \text{at } z = 0. \quad (1.9)$$

The associated (scaled) free surface deflection F is calculated a posteriori from the normal stress balance and volume conservation:

$$S\Delta F - (1-S)F = (-P + [(\nabla \boldsymbol{U} + \nabla \boldsymbol{U}^\top) \cdot \boldsymbol{e}_z] \cdot \boldsymbol{e}_z)_{z=0} \quad \text{in } \Sigma, \quad (1.10)$$

$$F = 0 \quad \text{at } \partial\Sigma, \qquad \text{and} \quad \int_\Sigma F \, dx\, dy = 0. \quad (1.11)$$

Thus, in contrast to the surface modes, the hydrodynamic modes are nonoscillatory and exhibit $O(C_g)$ free surface deflection. Moreover these modes decay on an $O(C_g^{-1})$ timescale, in contrast to the $O(C_g{}^{1/2})$ timescale of the surface modes, and hence cannot be ignored a priori in a weakly nonlinear theory.

The remainder of the paper is organized as follows. In Section 2 we derive and discuss a system of *coupled amplitude-streaming flow* (CASF) equations that describe the slow evolution of the complex amplitudes of competing surface waves and the associated streaming flow. The streaming flow itself is incompressible and satisfies a Navier-Stokes–like equation in three dimensions. As a consequence only a limited description of the resulting system can be obtained analytically, and any study of the attractors must rely on costly numerical computations. Thus some effort has been made to simplify the CASF equations further, in order to obtain model problems which nonetheless capture the role played by the streaming flow. These models are constructed using Galerkin truncation, and as in a related problem [41] appear to perform well. In Sections 3 through 5 we focus on three particular cases, namely an interaction between two nearly degenerate modes in rectangular containers with almost square cross-section, as in the well-known experiments by Simonelli and Gollub [42] and Feng and Sethna [43], an analogous interaction in almost circular containers (to our knowledge, a situation not studied experimentally), and a mode-mode interaction in circular containers, as in the seminal experiment by Ciliberto and Gollub [44], [45]. Finally, in Section 6 we discuss in general terms the role of streaming flows in the nearly inviscid Faraday system.

2. Derivation of the Coupled Amplitude-Streaming Flow Equations

In this section we derive equations for the (complex) amplitudes A_k of modes with frequencies Ω_k created from the breakup of a N-fold degenerate inviscid mode by a small change in the system geometry or in the parameter values used. In the limit (1.5)–(1.6) these modes are nearly inviscid everywhere except in *viscous boundary layers*, of $O(C_g^{1/2})$ thickness, attached to the walls of the container and the free surface. Since all these modes oscillate with frequencies near ω, we write the velocity $\boldsymbol{v}$ and the modified pressure p in the bulk (i.e., outside of these boundary layers), and the free surface deflection f in the form

$$(\boldsymbol{v}, p, f) = \mathrm{e}^{\mathrm{i}\omega t}\Bigg[\sum_{k=1}^{N} A_k(\boldsymbol{V}_k, P_k, F_k) + (\boldsymbol{v}_{3s}, p_{3s}, f_{3s}) + \sum_{k,l,m=1}^{N} \bar{A}_k A_l A_m(\boldsymbol{V}_{klm}, P_{klm}, F_{klm}) + \cdots\Bigg] + \text{c.c.} + \sum_{k,l=1}^{N} \bar{A}_k A_l(\boldsymbol{h}_{kl}, P_{kl}, F_{kl}) + (\boldsymbol{u}^s, p^s, f^s) + NRT, \tag{2.1}$$

where NRT stands for *nonresonant terms* (depending on the short time variable $t \sim 1$ as $\exp(\mathrm{i}k\omega t)$, with the integer $k \neq \pm 1, 0$); the terms written out explicitly either resonate with the surface waves or with the streaming flow. The amplitudes $A_1, \ldots, A_N$, the *streaming flow velocity* $\boldsymbol{u}^s$ with the associated *modified pressure* p^s, and *free surface deflection* f^s are all small and depend weakly on time, namely,

$$|dA_k/dt| \ll |A_k| \ll 1, \quad \text{for } k = 1, \ldots, N, \qquad |\partial \boldsymbol{u}^s/\partial t| \ll |\boldsymbol{u}^s| \ll 1,$$
$$|\partial p^s/\partial t| \ll |p^s| \ll 1, \qquad |\partial f^s/\partial t| \ll |f^s| \ll 1. \tag{2.2}$$

In addition (2.1) also depends on powers of the small parameters C_g, μ, $\omega - \Omega$, $\Omega_1 - \Omega$, $\ldots$, $\Omega_N - \Omega$; the corresponding terms have not been written out explicitly because they will not be needed in what follows. All coefficients in (2.1) are $O(1)$ except for the quantities $\boldsymbol{v}_{3s}$, p_{3s}, and f_{3s}, which depend bilinearly on $(A_1, \ldots, A_N)$ and $\boldsymbol{u}^s$ (see below). The main objective of this section is to derive and discuss the following equations (hereafter the CASF equations) that describe the flow in the bulk, outside of the thin viscous boundary layers at the container walls and the fluid surface:

$$A_k'(t) = -[d_k + \mathrm{i}(\omega - \Omega)]A_k + \mathrm{i}\sum_{l,m=1}^{N} \beta_{klm}(\Omega_l - \Omega)A_m + \mathrm{i}\sum_{l,m,n=1}^{N} \alpha_{klmn}\bar{A}_l A_m A_n$$
$$- \mathrm{i}\Omega \sum_{l=1}^{N} \int_{-1}^{0}\int_{\Sigma} \boldsymbol{u}^s \cdot \boldsymbol{g}_{kl}\, d\boldsymbol{x} A_l + \mathrm{i}\mu \sum_{l=1}^{N} \alpha_{kl}\bar{A}_l, \quad \text{for } k = 1, \ldots, N, \tag{2.3}$$

$$\partial \boldsymbol{u}^s/\partial t - \left[\boldsymbol{u}^s + \sum_{k,l=1}^{N} \bar{A}_k A_l (\boldsymbol{h}_{kl} - \boldsymbol{g}_{kl})\right] \times (\boldsymbol{\nabla} \times \boldsymbol{u}^s) = -\boldsymbol{\nabla}\hat{p}^s + C_g \Delta \boldsymbol{u}^s,$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{u}^s = 0, \tag{2.4}$$

for $(x, y) \in \Sigma$, $-1 < z < 0$, subject to the boundary conditions

$$\boldsymbol{u}^s = \sum_{k,l=1}^{N} \bar{A}_k A_l \boldsymbol{\varphi}_{kl}^1 \quad \text{if } z = -1 \text{ or if } (x, y) \in \partial\Sigma, \tag{2.5}$$

$$\boldsymbol{u}^s \cdot \boldsymbol{e}_z = 0, \qquad \partial \tilde{\boldsymbol{u}}^s/\partial z = \sum_{k,l=1}^{N} \bar{A}_k A_l \boldsymbol{\varphi}_{kl}^2 \quad \text{at } z = 0, \tag{2.6}$$

where $\tilde{\boldsymbol{u}}^s$ is the horizontal projection of $\boldsymbol{u}^s$, and the modified pressure $\hat{p}^s$ and the various coefficients and vectors are determined below.

The following remarks are in order.

(i) The frequency splitting arises as a result of an $O(\varepsilon) \sim d_k$ change in the shape of the container. This change has no effect at leading order on the damping d_k or on any of the remaining terms in (2.3).
(ii) The vectors $\boldsymbol{\varphi}_{kl}^j$ satisfy $\boldsymbol{\varphi}_{kl}^j = \bar{\boldsymbol{\varphi}}_{lk}^j$ so that the sums in (2.5) and (2.6) are real. Likewise $\boldsymbol{g}_{kl} = \bar{\boldsymbol{g}}_{lk}$ and $\boldsymbol{h}_{kl} = \bar{\boldsymbol{h}}_{lk}$ (see below).
(iii) The following estimates hold for $k, l, m, n = 1, \ldots, N$:

$$|d_k| \ll 1, \qquad |\beta_{klm}| \sim |\alpha_{klmn}| \sim |\alpha_{kl}| \sim |\boldsymbol{g}_{kl}| \sim |\boldsymbol{h}_{kl}| \sim |\boldsymbol{\varphi}_{kl}^1| \sim |\boldsymbol{\varphi}_{kl}^2| \sim 1, \tag{2.7}$$

and allow us to neglect higher order terms based on the fact

$$|\boldsymbol{u}^s| \sim |A_1|^2 \sim \cdots \sim |A_N|^2 \ll 1. \tag{2.8}$$

(iv) If $C_g \ll |A_k|^2$, the higher order viscous term retained in (2.4) plays no role in the bulk but remains responsible for the presence of secondary viscous boundary layers associated with the streaming flow.

2.1. The Amplitude Equations

The amplitude equations (2.3) can be derived (except for the singularity in the solution in the bulk near the contact line that must be handled with care [29]) by substituting expansion (2.1) into (1.1) and into the boundary conditions that result from *matching conditions* between the viscous boundary layers and the bulk, and imposing *solvability conditions* (i.e., eliminating secular terms on the fast timescale $t \sim 1$) at each order. The only unfamiliar term in the amplitude equations is that involving the streaming flow velocity; this term is derived below.

A. The *leading order terms* in (2.1) are linear combinations of the *surface modes* $(\boldsymbol{V}_k, P_k, F_k)$, for $k = 1, \ldots, N$, which are nontrivial solutions of the linearized, inviscid problem

$$\nabla \cdot \boldsymbol{V}_k = 0, \qquad \mathrm{i}\Omega \boldsymbol{V}_k = -\nabla P_k \quad \text{in } \Sigma\times]-1, 0[, \tag{2.9}$$

$$\boldsymbol{e}_z \cdot \boldsymbol{V}_k = 0 \quad \text{if } z = -1, \qquad \boldsymbol{n}_0 \cdot \boldsymbol{V}_k = 0, \qquad F_k = 0 \quad \text{if } (x, y) \in \partial\Sigma, \tag{2.10}$$

$$\boldsymbol{e}_z \cdot \boldsymbol{V}_k = \mathrm{i}\Omega F_k, \;\; P_k - (1 - S)F_k + S\Delta F_k = 0 \quad \text{on} \quad z = 0, \tag{2.11}$$

$$\int_\Sigma F_k \, dx \, dy = 0, \tag{2.12}$$

where $\boldsymbol{n}_0$ is the outward unit normal to the lateral wall. For convenience these modes are selected such that

$$\int_{-1}^{0} \int_\Sigma \boldsymbol{V}_k \cdot \boldsymbol{V}_l \, d\boldsymbol{x} + \int_\Sigma [(1 - S)\bar{F}_k F_l + S\nabla F_k \cdot \nabla F_l] \, dx \, dy = \delta_{kl}, \tag{2.13}$$

for $k, l = 1, \ldots, N$, where δ_{kl} is the Kronecker delta and the dot denotes the *inner product*

$$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \bar{u}_1 u_2 + \bar{v}_1 v_2 + \bar{w}_1 w_2 \quad \text{if} \quad \boldsymbol{v}_k = u_k \boldsymbol{e}_r + v_k \boldsymbol{e}_\theta + w_k \boldsymbol{e}_z \quad \text{for } k = 1 \text{ and } 2. \tag{2.14}$$

Here the overbar denotes the complex conjugate. Note that equations (2.9)–(2.12) are equivariant under the action

$$(\boldsymbol{V}, P, F) \to (-\bar{\boldsymbol{V}}, \bar{P}, \bar{F}). \tag{2.15}$$

B. The linear terms in the amplitude equations (2.3) account for damping, detuning, departure from degeneracy and forcing, and are obtained from the solvability conditions at orders $C_g^{1/2}|A_k|$, $C_g|A_k|$, $|\omega - \Omega||A_k|$, $|\Omega_k - \Omega||A_k|$, and $\mu|A_k|$.

B-1. *Viscous effects* result in both detuning and damping, yielding

$$d_k = \gamma_k^1 (1 + \mathrm{i})C_g^{1/2} + \gamma_k^2 C_g + O(C_g^{3/2}), \tag{2.16}$$

where γ_k^1 and γ_k^2 are real and strictly positive. The $O(C_g^{1/2})$ term comes from viscous dissipation and viscous detuning in the Stokes boundary layers attached to the boundary of the container while the $O(C_g)$ term results from dissipation in the bulk and a first correction to dissipation in the boundary layers; note that there is no viscous detuning at the latter order and that viscous dissipation in the boundary layer attached to the free surface is ignored (since it provides a $O(C_g^{3/2}|A_k|)$ contribution). The second term may be neglected *for sufficiently small* C_g. However, as first shown by [46] in the liquid bridge context and confirmed by [47] for brimful cylinders of circular cross section, the two-term approximation gives quantitatively much better results [47], [48], [49] for typical values of C_g (e.g., water in centimeter-deep containers) because of the relatively large value of the ratio γ_k^2/γ_k^1 for a fixed contact line [47]; this is expected to remain so for brimful containers of general cross section. Note that γ_k^2/γ_k^1 is necessarily large for high order modes for which

$$\gamma_k^1 \to 0, \qquad \gamma_k^2 \sim |\boldsymbol{k}|^2 \quad \text{as } \Omega \to \infty, \tag{2.17}$$

where $|\boldsymbol{k}|$ is the wavenumber of the mode and satisfies the inviscid dispersion relation

$$\Omega^2 \simeq (1-S)|\boldsymbol{k}| + S|\boldsymbol{k}|^3. \tag{2.18}$$

This is because for high frequencies the surface modes behave locally like (linear combinations of) plane waves, with well-defined wavevectors, except of course in the vicinity of the contact line. The estimates (2.17) follow from standard estimates of viscous dissipation taking into account that as Ω becomes large the inviscid eigenfunctions vanish exponentially with depth with a characteristic length scale $|\boldsymbol{k}|^{-1}$.

B-2. The term accounting for the departure from the N-fold degeneracy depends linearly on $\Omega - \Omega_k$, where Ω and $\Omega_1, \ldots, \Omega_N$ are the unperturbed and perturbed (inviscid) eigenfrequencies, respectively. Note that these are perturbations of a $O(1)$ quantity (namely Ω), while the corresponding effects in the remaining terms in the amplitude equations are neglected because they involve perturbations of terms that are already small. Generically, the perturbation splits the N-dimensional eigenspace in the degenerate problem into N *surviving eigenspaces* that need not coincide, even in first approximation, with those spanned by the N eigenvectors of (2.9)–(2.12) selected above although these eigenvectors can always be selected so that this is in fact so (in this case, $\mathrm{i}\sum \beta_{klm}(\Omega_l - \Omega)A_m$ would simplify to $\mathrm{i}(\Omega_k - \Omega)A_k$ in (2.3)). Since the term responsible for the frequency splitting is conservative and would be of the form $\mathrm{i}(\Omega_k - \Omega)A_k$ for suitably chosen eigenvectors, it follows that

$$\beta_{klm} = \bar{\beta}_{mlk} \quad \text{and} \quad \sum_{l=1}^{N} \beta_{klm} = \delta_{km} \quad \text{for} \quad k, l, m = 1, \ldots, N, \tag{2.19}$$

where we have taken into account that, according to (2.1) and (2.13), the *energy* of the system is

$$\begin{aligned} E &\equiv \int_{-1}^{0}\int_{\Sigma} |\boldsymbol{v}|^2 d\boldsymbol{x} + \int_{\Sigma} [(1-S)|f|^2 + S|\boldsymbol{\nabla} f|^2]dx\,dy \\ &= \sum_{k=1}^{N} |A_k|^2 + O\left(\sum_{k=1}^{N} |A_k|^4\right). \end{aligned} \tag{2.20}$$

B-3. The coefficients of the parametric forcing terms are

$$\alpha_{kl} = 2\omega^2 \int_\Sigma \bar{F}_k \bar{F}_l \, dx \, dy. \tag{2.21}$$

C. At second order in the complex amplitudes, equation (2.1) contains no resonant terms of the form $\exp(\pm i\omega t)$. At this order we obtain only nonresonant terms and the (slowly varying) resonant terms explicitly displayed in (2.1), namely $\sum \bar{A}_k A_l(\boldsymbol{h}_{kl}, P_{kl}, F_{kl}) + (\boldsymbol{u}^s, 0, 0)$; by definition, both $|p^s|$ and $|f^s|$ are $o(\sum |A_k|^2)$. The functions P_{kl}, F_{kl} in (2.1) can be calculated from the strictly inviscid problem but will not be needed below. The expressions $\sum \bar{A}_k A_l \boldsymbol{h}_{kl}$ and $\boldsymbol{u}^s$ are of the same order (see (2.8)), and both contribute to the *Eulerian mean flow velocity* at leading order. The associated mean flows will be called *inviscid mean flow* and *streaming* (or *viscous mean*) *flow*, respectively. The distinction between the two is made precise by requiring that

$$\boldsymbol{u}^s \cdot \boldsymbol{n}_0 = 0 \quad \text{if } (x, y) \in \partial\Sigma, \qquad \boldsymbol{u}^s \cdot \boldsymbol{e}_z = 0 \quad \text{if either } z = -1 \text{ or } z = 0, \tag{2.22}$$

where $\boldsymbol{n}_0$ is the outward unit normal to $\partial\Sigma$. Thus it is the *inviscid* mean flow that accounts for the normal component of the mean flow velocity at the unperturbed free surface. Note that the inviscid mean flow is slaved to the surface waves.

C-1. The velocity vectors $\boldsymbol{h}_{kl}$ appearing in the expression for the *inviscid mean flow velocity* can be written in the form

$$\boldsymbol{h}_{kl} = i\nabla H_{kl}, \tag{2.23}$$

with the velocity potential H_{kl} given by

$$\Delta H_{kl} = 0 \quad \text{if } (x, y) \in \Sigma \text{ and } -1 < z < 0, \tag{2.24}$$

$$H_{kl} = 0 \quad \text{if either } z = -1 \text{ or } (x, y) \in \partial\Sigma, \tag{2.25}$$

$$\partial H_{kl}/\partial z = -i\tilde{\nabla} \cdot (\bar{F}_k \tilde{\boldsymbol{V}}_l + F_l \bar{\tilde{\boldsymbol{V}}}_k) \quad \text{if } z = 0, \tag{2.26}$$

where $\tilde{\boldsymbol{V}}_{k,l}$ and $\tilde{\nabla}$ are the horizontal projections of $\boldsymbol{V}_{k,l}$ and ∇. The boundary condition (2.26) that forces this flow results from the short-time average of the left-hand side of equation (1.3a); note that $\boldsymbol{h}_{kl} = \bar{\boldsymbol{h}}_{lk}$. Moreover, according to (2.1), (2.9), (2.22), (2.23), and (2.26), the normal component of the mean flow velocity is

$$i\Omega^{-1} \sum_{k,l=1}^{N} \bar{A}_k A_l \tilde{\nabla} \cdot [\bar{F}_k \tilde{\nabla} P_l - F_l \tilde{\nabla} \bar{P}_k]_{z=0}, \tag{2.27}$$

and hence vanishes identically if F_k and $P_k(z = 0)$ are proportional for all k, with the proportionality constant independent of k. This situation in turn holds if either (a) capillary effects are absent or (b) the contact line is completely free (namely, if the dynamic contact angle is constant). In some cases this is true even for a fixed contact line (as in this paper), e.g., if (c) the surface wave is quasi-standing (see Section 2.3.1) or if (d) the cross section is circular and only two counter-rotating modes are present.

C-2. The *streaming flow velocity* $\boldsymbol{u}^s$ cannot be calculated from strictly inviscid theory, for which $\boldsymbol{u}^s = \mathbf{0}$. The mean flow described by Davey-Stewartson–like models [50] is strictly inviscid and has the same origin as the inviscid mean flow described above: It accounts for mean flow normal to the unperturbed free surface. In order to determine $\boldsymbol{u}^s$ (see Sections 2.2–2.3 below), we need to go to $O(\sum |A_k|^4)$ in the momentum equation and include viscous effects that allow vorticity creation in the oscillatory boundary layers. The resulting vorticity may then diffuse or be advected into the bulk.

D. *At third order in the complex amplitudes*, we obtain two kinds of *resonant terms*.

D-1. The terms in (2.1) that are explicitly cubic in the complex amplitudes are required for the calculation of the coefficients α_{klmn}. This calculation is omitted since despite the presence of viscous boundary layers, the resulting coefficients coincide with those obtained in the strictly inviscid limit. In this limit, the original problem (1.1)–(1.4) is invariant under the action $t \to -t, \boldsymbol{v} \to -\boldsymbol{v}$, and conservative, and we must therefore have

$$\bar{\alpha}_{klmn} = \alpha_{klmn}, \qquad \alpha_{klmn} + \alpha_{lkmn} = \alpha_{mnkl} + \alpha_{nmkl}, \qquad \alpha_{klmn} = \alpha_{klnm}. \quad (2.28)$$

Additional relations must hold if the system (2.3) is to be Hamiltonian [51], [52]. Explicit calculations of the coefficients in this limit for square [53], [54], rectangular [54], and circular [55] domains with a free contact line confirm these relations (see also [56]).

For use below, we note that the velocity vectors associated with the corresponding terms in (2.1) are potential, namely,

$$\nabla \times \boldsymbol{V}_{klm} = \mathbf{0} \quad \text{for } k, l, m = 1, \ldots, N, \qquad (2.29)$$

and that

$$|\alpha_{klmn}| \sim \Omega |\boldsymbol{k}|^2 |F|^2 \quad \text{as } \Omega \to \infty, \qquad (2.30)$$

where $|\boldsymbol{k}|$ and $|F|$ denote the order of magnitude of the wavevector and the free surface deflection of the surface modes, respectively. Both are of order unity when $\Omega \sim 1$. The estimate (2.30) follows from well-known explicit expressions for laterally unbounded waves [56]. Alternatively, it can be obtained by noting that if the period and wavelength of the waves are used for nondimensionalization (instead of the gravity-capillary time and h), these cubic coefficients are of order unity. For this new nondimensionalization, t is replaced by t/Ω and A_k by $A_k/(|F||\boldsymbol{k}|)$.

D-2. The resonant terms denoted by $(\boldsymbol{v}_{3s}, p_{3s}, f_{3s})e^{i\omega t}$ + c.c. in (2.1) generate the terms in the amplitude equations (2.3) that include the coupling to the streaming flow. These terms are distinguished from the remaining third-order resonant terms by the requirement that they depend bilinearly on both the streaming flow velocity and the complex amplitudes. Since these terms are new, we provide a detailed derivation here. To simplify notation we write the relevant part of equation (2.3) in the form

$$A_k' = \mathcal{H}_k. \qquad (2.31)$$

Thus we only need to show that

$$\mathcal{H}_k = -\mathrm{i}\Omega \sum_{l=1}^{N} \int_{-1}^{0} \int_{\Sigma} \boldsymbol{u}^s \cdot \boldsymbol{g}_{kl}\, d\boldsymbol{x} A_l, \tag{2.32}$$

for appropriate vector functions $\boldsymbol{g}_{kl}$, which are calculated below. To this end, we substitute equations (2.1) and (2.31) into (1.1)–(1.4) and retain terms that are either bilinear in $(\boldsymbol{u}^s, A_k)$ or linear in $\mathcal{H}_k$. We obtain

$$\nabla \cdot \boldsymbol{v}_{3s} = 0, \qquad \mathrm{i}\Omega \boldsymbol{v}_{3s} + \nabla p_{3s} = \sum_{l=1}^{N} A_l \boldsymbol{V}_l \times (\nabla \times \boldsymbol{u}^s) - \sum_{l=1}^{N} \boldsymbol{V}_l \mathcal{H}_l$$

$$\text{if } (x, y) \in \Sigma \text{ and } -1 < z < 0, \tag{2.33}$$

$$\boldsymbol{e}_z \cdot \boldsymbol{v}_{3s} = 0 \quad \text{if } z = -1, \qquad \boldsymbol{n}_0 \cdot \boldsymbol{v}_{3s} = 0 \quad \text{if } (x, y) \in \partial\Sigma,$$

$$f_{3s} = 0 \quad \text{if } (x, y) \in \partial\Sigma, \tag{2.34}$$

$$\boldsymbol{e}_z \cdot \boldsymbol{v}_{3s} - \mathrm{i}\Omega f_{3s} = \tilde{\nabla} \cdot \left[\left(\sum_{l=1}^{N} A_l F_l \right) \tilde{\boldsymbol{u}}^s \right] + \sum_{l=1}^{N} F_l \mathcal{H}_l \quad \text{if } z = 0, \tag{2.35}$$

$$p_{3s} - (1 - S) f_{3s} + S \Delta f_{3s} = \boldsymbol{u}^s \cdot \left(\sum_{l=1}^{N} A_l \boldsymbol{V}_l \right) \quad \text{if } z = 0, \tag{2.36}$$

where $\tilde{\boldsymbol{u}}^s$ and $\tilde{\nabla}$ are again the horizontal projections of $\boldsymbol{u}^s$ and ∇. In fact, equation (2.33) applies only in the bulk, outside (secondary) viscous boundary layers. Although these boundary layers should in principle be taken into account in the derivation of the boundary conditions, a straightforward calculation shows that the boundary layers do not contribute new terms to the boundary conditions at this order. Also, in order to obtain (2.35) we have taken into account that, according to (2.4a), $\partial(\boldsymbol{u}^s \cdot \boldsymbol{e}_z)/\partial z = -\tilde{\nabla} \cdot \tilde{\boldsymbol{u}}^s$ at $z = 0$. The coupling term $\mathcal{H}_k$ is now obtained by applying a *solvability condition* to equations (2.33)–(2.36). Using the inner product (2.14), we multiply $\boldsymbol{V}_k$ by (2.33b) and (2.9b) by $\boldsymbol{v}_{3s}$, add, and integrate the resulting equation over $(x, y) \in \Sigma$, $-1 < z < 0$. Repeated integration by parts using (2.13) together with the remaining equations and boundary conditions in (2.9)–(2.11) and (2.33)–(2.36) yields

$$\begin{aligned} \mathcal{H}_k &= \sum_{l=1}^{N} A_l \left[\int_{-1}^{0} \int_{\Sigma} \boldsymbol{V}_k \cdot [\boldsymbol{V}_l \times (\nabla \times \boldsymbol{u}^s)]\, d\boldsymbol{x} \right. \\ &\qquad \left. + \int_{\Sigma} [\mathrm{i}\Omega \bar{F}_k (\boldsymbol{u}^s \cdot \boldsymbol{V}_l) - \bar{P}_k \tilde{\nabla} \cdot (F_l \tilde{\boldsymbol{u}}^s)]_{z=0}\, dx\, dy \right] \\ &= -\mathrm{i}\Omega \sum_{l=1}^{N} A_l \int_{-1}^{0} \int_{\Sigma} \boldsymbol{u}^s \cdot \boldsymbol{g}_{kl}\, d\boldsymbol{x}, \end{aligned} \tag{2.37}$$

where $\boldsymbol{g}_{kl}$ is given by

$$\boldsymbol{g}_{kl} = \mathrm{i}\Omega^{-1} \nabla \times (\bar{\boldsymbol{V}}_k \times \boldsymbol{V}_l), \qquad \bar{\boldsymbol{g}}_{kl} = \boldsymbol{g}_{lk}. \tag{2.38}$$

The second equality in (2.37) follows on integrating by parts the surface term (and taking into account that, according to (2.10), $F_k = 0$ at the contact line) and using (2.11a) and the expression

$$
\begin{aligned}
&\int_{-1}^{0}\int_{\Sigma} \boldsymbol{V}_k \cdot [\boldsymbol{V}_l \times (\boldsymbol{\nabla} \times \boldsymbol{u}^s)]\, d\boldsymbol{x} \\
&= \int_{-1}^{0}\int_{\Sigma} (\boldsymbol{\nabla} \times \boldsymbol{u}^s) \cdot (\bar{\boldsymbol{V}}_k \times \boldsymbol{V}_l)\, d\boldsymbol{x} \\
&= \int_{-1}^{0}\int_{\Sigma} [\boldsymbol{u}^s \cdot (\boldsymbol{\nabla} \times (\bar{\boldsymbol{V}}_k \times \boldsymbol{V}_l)) + \boldsymbol{\nabla} \cdot ((\boldsymbol{u}^s \cdot \boldsymbol{V}_l)\bar{\boldsymbol{V}}_k - (\boldsymbol{u}^s \cdot \bar{\boldsymbol{V}}_k)\boldsymbol{V}_l)]\, d\boldsymbol{x} \\
&= -\mathrm{i}\Omega \int_{-1}^{0}\int_{\Sigma} \boldsymbol{u}^s \cdot \boldsymbol{g}_{kl}\, d\boldsymbol{x} \\
&\quad + \int_{\Sigma} [(\boldsymbol{u}^s \cdot \boldsymbol{V}_l)(\boldsymbol{e}_z \cdot \bar{\boldsymbol{V}}_k) - (\boldsymbol{u}^s \cdot \bar{\boldsymbol{V}}_k)(\boldsymbol{e}_z \cdot \boldsymbol{V}_l)]_{z=0}\, dx\, dy,
\end{aligned}
$$

obtained from standard vector identities and integration by parts. Equation (2.37) now yields the required expression (2.32) for $\mathcal{H}_k$.

2.2. *The Streaming Flow Equations and Boundary Conditions*

We now consider the slowly varying velocity associated with the streaming flow, $\boldsymbol{u}^s$, and show that it evolves according to equations (2.4)–(2.6).

2.2.1. The Continuity and Momentum Equations. Equations similar to (2.4) are well known [19], [20], [17], but for completeness they are obtained here by substitution of expansion (2.1) into the original continuity and momentum equations (1.1). Since (1.1a) is linear, the oscillatory flow introduces no new terms and (2.4a) follows. The momentum equation does, however, involve additional terms resulting from products (in the quadratic advection term) of oscillatory terms that are of first and third order in the complex amplitudes; these are of the same order as the usual $O(|\boldsymbol{u}^s|^2)$ advection terms (see (2.8)). In addition, due to the very nature of the streaming flow, we must also retain viscous terms, however small these may be. From equations (2.23) and (2.29) it follows that

$$
\begin{aligned}
&\partial \boldsymbol{u}^s/\partial t - \left(\sum_{k,l=1}^{N} \bar{A}_k A_l \boldsymbol{h}_{kl} + \boldsymbol{u}^s\right) \times \boldsymbol{\nabla} \times \boldsymbol{u}^s - \left[\left(\sum_{l=1}^{N} A_l \boldsymbol{V}_l\right) \times \boldsymbol{\nabla} \times \bar{\boldsymbol{v}}_{3s} + \text{c.c.}\right] \\
&\quad = -\boldsymbol{\nabla}[p^s + \sum_{k,l=1}^{N} (\bar{A}_k' A_l + \bar{A}_k A_l') H_{kl}] + C_g \Delta \boldsymbol{u}^s + \cdots, \qquad (2.39)
\end{aligned}
$$

where $\boldsymbol{v}_{3s}$, $\boldsymbol{h}_{kl}$, and H_{kl} are defined in (2.1) and (2.23), and given by (2.33)–(2.36) and (2.24)–(2.26). Equation (2.33b) yields

$$
\boldsymbol{\nabla} \times \boldsymbol{v}_{3s} = -\mathrm{i}\Omega^{-1} \sum_{k=1}^{N} A_k \boldsymbol{\nabla} \times (\boldsymbol{V}_k \times \boldsymbol{\nabla} \times \boldsymbol{u}^s), \qquad (2.40)
$$

and we only need to use the vector identity

$$\mathrm{i}\boldsymbol{u}\times\nabla\times(\bar{\boldsymbol{u}}\times\nabla\times\boldsymbol{w})+\text{c.c.}=\mathrm{i}[\nabla\times(\boldsymbol{u}\times\bar{\boldsymbol{u}})]\times\nabla\times\boldsymbol{w}+\mathrm{i}\nabla[(\nabla\times\boldsymbol{w})\cdot(\boldsymbol{u}\times\bar{\boldsymbol{u}})], \quad (2.41)$$

which holds for any real vector $\boldsymbol{w}$ and any complex vector $\boldsymbol{u}$ such that $\nabla\cdot\boldsymbol{u}=0$ and $\nabla\times\boldsymbol{u}=\mathbf{0}$ [6], to obtain equation (2.4), with $\boldsymbol{g}_{kl}$ as in (2.38) and $\hat{p}^s$ given by

$$\hat{p}^s=p^s+\sum_{k,l=1}^{N}(\bar{A}_k'A_l+\bar{A}_kA_l')H_{kl}+\mathrm{i}\Omega^{-1}\sum_{k,l=1}^{N}\bar{A}_kA_l[(\nabla\times\boldsymbol{u}^s)\cdot(\bar{\boldsymbol{V}}_k\times\boldsymbol{V}_l)]. \quad (2.42)$$

2.2.2. The Boundary Conditions. The form of the boundary conditions (2.5)–(2.6) readily follows from the following properties:

a. *The forcing terms depend bilinearly on* $(A_1,\dots,A_N)$ *and* $(\bar{A}_1,\dots,\bar{A}_N)$*, with* $O(1)$ *coefficients depending on position only.*
b. *The Stokes boundary layer near the solid walls provides a forcing tangential velocity, and the boundary layer near the free surface provides a forcing shear stress.* The component of $\boldsymbol{u}^s$ perpendicular to the boundary vanishes in both cases by the definition (2.22) of the streaming flow.
c. *The boundary conditions must be invariant under any symmetry that applies the original problem.*
d. *The forcing shear stress at the free surface vanishes at leading order if the associated surface wave is quasi-standing* (see Section 2.3.1 below), that is, if the phase of $\sum A_k\boldsymbol{V}_k$ is independent of position.

Property (a) is a direct consequence of the slowly varying nature of the streaming flow, and property (c) is obvious. Properties (b) and (d) are well known in two dimensions [7], [57], [8], [58] and have been checked [59] for general, not necessarily plane, solid and free boundaries in three dimensions. In fact, the formulae in [59] become simple for plane or cylindrical rigid boundaries (see Appendix) and for plane unperturbed free surfaces such as those in this paper, and allow a quick calculation of the vector functions $\boldsymbol{\varphi}_{kl}^1$ and $\boldsymbol{\varphi}_{kl}^2$ appearing in equations (2.5)–(2.6):

$$\boldsymbol{\varphi}_{kl}^1 = -(2\Omega)^{-1}[(2+3\mathrm{i})(\tilde{\nabla}\cdot\bar{\boldsymbol{V}}_k)\boldsymbol{V}_l+(\bar{\boldsymbol{V}}_k\cdot\tilde{\nabla})\boldsymbol{V}_l+\text{c.c.}]-\boldsymbol{h}_{kl}$$
$$\text{if either } z=-1 \text{ or } (x,y)\in\partial\Sigma, \quad (2.43)$$

$$\boldsymbol{\varphi}_{kl}^2 = \tilde{\nabla}(\tilde{\nabla}\cdot(\bar{F}_k\tilde{\boldsymbol{V}}_l))+2(\tilde{\nabla}\bar{F}_k\cdot\tilde{\nabla})\tilde{\boldsymbol{V}}_l+2(\tilde{\nabla}\cdot\tilde{\boldsymbol{V}}_l)\tilde{\nabla}\bar{F}_k+\text{c.c.}-(\partial\boldsymbol{h}_{kl}/\partial z)\cdot\boldsymbol{e}_z$$
$$\text{if } z=0. \quad (2.44)$$

Here, as above, $\tilde{\boldsymbol{V}}_k$ and $\tilde{\nabla}$ are the tangential projections of $\boldsymbol{V}_k$ and ∇ on either the solid boundary or the unperturbed free surface. Note that the inviscid oscillatory velocity $\boldsymbol{V}_k$ is tangential to the solid boundary, and thus $\tilde{\boldsymbol{V}}_k=\boldsymbol{V}_k$ in (2.43).

2.3. Some General Remarks on the CASF Equations

Before proceeding to particular cases, several remarks about the CASF equations are in order.

2.3.1. Single-Mode, Standing, and Quasi-Standing Surface Waves. In the generic case $N = 1$ (already considered in a related context in [29]) the eigenfrequency Ω is algebraically simple and the only eigenfunction $(\boldsymbol{V}_1, P_1, F_1)$ is necessarily invariant under (2.15); thus $\bar{\boldsymbol{V}}_1$ and $\boldsymbol{V}_1$ are collinear and (see (2.38)) $\boldsymbol{g}_{11} \equiv \mathbf{0}$. Consequently the integral term in the (only) amplitude equation (2.3) vanishes identically and *the evolution of A_1 decouples from the streaming flow*, as anticipated in Section 1. This conclusion does not require any additional conditions on the streaming flow.

In the context of this paper, we shall say that a wave is *standing* if the free surface exhibits stationary nodal lines. This condition holds for all single-mode waves, but is quite stringent in the multimode case. Specifically, if we rewrite (2.1) in the form

$$(\boldsymbol{v}, p, f) = \mathrm{e}^{\mathrm{i}\omega t}(\boldsymbol{V}, P, F) + \text{c.c.} + \cdots, \tag{2.45}$$

where $(\boldsymbol{V}, P, F) = \sum A_k(\boldsymbol{V}_k, P_k, F_k)$, this requirement holds if and only if $(\boldsymbol{V}, P, F)$ can be written as $(\boldsymbol{V}, P, F) = B(\tau)(\boldsymbol{V}_0(\boldsymbol{x}), P_0(\boldsymbol{x}), F_0(\boldsymbol{x}))$, with $(\boldsymbol{V}_0, P_0, F_0)$ invariant under (2.15). For instance, in square containers a wave is standing only if the integral term appearing in the amplitude equations (3.13) vanishes, a requirement generically satisfied only if the streaming flow is reflection-symmetric.

In general, *standing waves are independent of the streaming flow*. To see this we simply take $(\boldsymbol{V}_1, P_1, F_1) = (\boldsymbol{V}_0, P_0, F_0)$ in equation (2.1), with $A_1 \neq 0$, $A_2 = \cdots = A_N = 0$. The streaming flow contribution to the A_1 amplitude equation (2.3) then vanishes because $\boldsymbol{g}_{11} = 0$, while the remaining equations are satisfied identically. This does not mean, however, that the stability properties of such standing waves are independent of the streaming flow, as elaborated further below. In cases in which the nodal lines move but only on the slow timescale τ we shall say that the wave is *quasi-standing*. For such waves the phase of $(\boldsymbol{V}, P, F)$ is still independent of position (but will depend on τ). An example of such a wave is provided by an axisymmetric oscillation in which the radial nodes move (slowly) in and out. This example also shows that not all reflection-symmetric waves are standing.

2.3.2. Mass Transport Velocity, Stokes Drift, and Related Issues. The above analysis of the mean flow has been made for convenience in terms of the Eulerian velocity. This velocity is given by $\boldsymbol{u}^s + \boldsymbol{u}^i$, where the mean flows associated with $\boldsymbol{u}^s$ and

$$\boldsymbol{u}^i = \sum_{k,l=1}^{N} \bar{A}_k A_l \boldsymbol{h}_{kl} \tag{2.46}$$

are the *viscous* and *inviscid* mean flows, respectively; here, $\boldsymbol{h}_{kl}$ is given by (2.23)–(2.26). In contrast, the mass transport, or Lagrangian, velocity [8], [58],

$$\boldsymbol{u}^{mt} = \boldsymbol{u}^s + \boldsymbol{u}^i + \boldsymbol{u}^{Sd}, \tag{2.47}$$

is associated with the time-averaged (on the timescale $t \sim 1$) trajectories of material elements; the difference between them (the Stokes drift) is

$$\boldsymbol{u}^{Sd} = -\sum_{k,l=1}^{N} \bar{A}_k A_l \boldsymbol{g}_{kl}, \tag{2.48}$$

where $\boldsymbol{g}_{kl}$ is again given by (2.38); this expression for $\boldsymbol{u}^{Sd}$ is readily obtained from the standard one [58]. Note that the Stokes drift, like the inviscid mean flow, is slaved to the surface waves, in contrast to the streaming flow (see below), and that the normal component of the Eulerian mean flow velocity does not lead to any mass transport across the *unperturbed* free surface, i.e., $\boldsymbol{u}^{mt} \cdot \boldsymbol{e}_z = 0$ at $z = 0$. This result follows from equations (2.22) and (2.46)–(2.48) since equations (2.23)–(2.26), (2.38), and standard formulae from vector analysis imply that $\boldsymbol{h}_{kl} \cdot \boldsymbol{e}_z = \boldsymbol{g}_{kl} \cdot \boldsymbol{e}_z$ at $z = 0$.

The mass transport velocity is the relevant one for comparison with flow visualizations (with an exposure time long compared to the forcing period) and, more generally, for transport (and mixing) of passive scalars [60], [61]; unfortunately, both the streaming flow and the inviscid mean flows are often ignored, e.g. [61], presumably under the (mistaken) assumption that they are small compared to the Stokes drift. The mass transport velocity is also the appropriate one for calculating some global properties of the flow, such as the total momentum or angular momentum of the fluid, averaged over the short timescale $t \sim 1$. For an axisymmetric container, the angular momentum about the z-axis is $M^s + M$, where M^s is the *angular momentum of the streaming flow* $\boldsymbol{u}^s$ and

$$M = \sum_{k,l=1}^{N} \bar{A}_k A_l M_{kl} \tag{2.49}$$

is the angular momentum of the inviscid mean flow and the Stokes drift. Here M_{kl} is the angular momentum of $\boldsymbol{h}_{kl} - \boldsymbol{g}_{kl}$. In inviscid theories the conservation of angular momentum plays an important role, but this is no longer so once viscosity (and hence streaming flow) is included. Indeed, in such systems there is no reason why an initial condition with zero angular momentum cannot evolve into a final state that spins clockwise or counterclockwise [62]. In contrast neither the mean (inviscid + streaming) flow nor the Stokes drift affects the energy E of the system at leading order because the contribution of both is of order $\sum |A_k|^4$, while E is quadratic in the complex amplitudes (see (2.20)). This is consistent with the fact that the coupling to the streaming flow in the amplitude equations (2.3) is conservative. However, neither flow can be ignored at higher order in the energy equation, even though the dissipation in the streaming flow is in general small [63].

2.3.3. Neglected Higher Order Terms. The neglected higher order terms in the amplitude equations (2.3) are of order

$$C_g^{3/2}|A|, \qquad |A|^5, \qquad \mu|A|^3, \qquad C_g^{1/2}(|A|^3 + |A\boldsymbol{u}^s| + |\mu A|),$$
$$\varepsilon(C_g^{1/2} + |A|^3 + |A\boldsymbol{u}^s| + |\mu A|),$$

and account, respectively, for viscous dissipation in the boundary layer attached to the free surface, higher order nonlinearity, the effect of viscosity on the nonlinearity, coupling to the streaming flow and forcing, and the effects of departure from the N-fold eigenvalue degeneracy (as measured by $\varepsilon \sim |\Omega - \Omega_k| \ll 1$) on the linear damping, nonlinearity, coupling to the streaming flow and forcing. Some of these terms are sometimes retained in the literature [37], [64], [65], [66]. Equation (2.4a) is exact (recall that $\boldsymbol{u}^s$ includes the total streaming flow velocity, not just the first approximation) while the neglected terms in (2.4b) and (2.5)–(2.6) are, respectively, of order

$$(C_g^{1/2} + \varepsilon)\left(\sum |A_k|^2 + |\boldsymbol{u}^s|\right)|\boldsymbol{u}^s| + \left(\sum |A_k|^2 + |\boldsymbol{u}^s|\right)^3$$
$$\text{and} \quad (C_g^{1/2} + \varepsilon)\sum |A_k|^2 + \sum |A_k|^4, \tag{2.50}$$

and originate from higher order effects in the advection terms and in the oscillatory boundary layers. Finally, when the degeneracy is lifted by forced symmetry breaking (e.g., by perturbing the cross section Σ of the container) the resulting change in the domain also has an effect on the streaming flow. However, these corrections are of higher order and may also be ignored.

2.3.4. Surface Wave-Streaming Flow Coupling. The momentum equation (2.4b) is the usual Navier-Stokes equation with a volumetric force

$$(\boldsymbol{u}^i + \boldsymbol{u}^{Sd}) \times (\nabla \times \boldsymbol{u}^s), \tag{2.51}$$

called the *vortex force*. This force does not drive any flow by itself (it vanishes if $\boldsymbol{u}^s = \boldsymbol{0}$) but can enhance or inhibit the effect of the remaining forcing terms; in fact this term can destabilize shear flows produced by water waves. The vortex force depends on the *streaming flow vorticity*, $\boldsymbol{\Omega}^s \equiv \nabla \times \boldsymbol{u}^s$, which evolves according to

$$\partial\boldsymbol{\Omega}^s/\partial t + [(\boldsymbol{u}^s + \boldsymbol{u}^i + \boldsymbol{u}^{Sd}) \cdot \nabla]\boldsymbol{\Omega}^s - (\boldsymbol{\Omega}^s \cdot \nabla)(\boldsymbol{u}^s + \boldsymbol{u}^i + \boldsymbol{u}^{Sd}) = C_g \Delta \boldsymbol{\Omega}^s, \tag{2.52}$$

as in [17], p. 119. The streaming flow is directly forced by the boundary conditions (2.5)–(2.6). If, as implicitly assumed, $\Omega \sim 1$, then the functions φ_{kl}^j appearing in (2.5)–(2.6) are also of order unity and the streaming flow velocity $\boldsymbol{u}^s$ satisfies (2.8), perhaps after an initial transient (see Section 2.3.5 below). Since, in addition, $|\boldsymbol{g}_{kl}|$ and $|\alpha_{klmn}|$ are also of order unity, the streaming flow terms in (2.3) are of the same order as the cubic terms, and *it is inconsistent to include the latter and neglect the streaming flow*. This is true even more so when Ω is large since then the wavenumber $|\boldsymbol{k}|$ is also large (according to the inviscid dispersion relation (2.18)) and the inviscid eigenfunctions then vanish exponentially fast outside of a layer of thickness $|\boldsymbol{k}|^{-1}$ near the free surface (the surface wave layer). According to equations (2.13) and (2.18), in this layer

$$|\boldsymbol{V}_k| \sim \Omega|F_k| \sim |\boldsymbol{k}|^{1/2}. \tag{2.53}$$

It follows from equations (2.30), (2.38), (2.43), and (2.44) that

$$|\alpha_{klmn}| \sim |\boldsymbol{k}||\boldsymbol{g}_{kl}| \sim |\boldsymbol{k}||\varphi_{kl}^1| \sim |\varphi_{kl}^2| \sim \Omega^{-1}|\boldsymbol{k}|^3. \tag{2.54}$$

In this limit, the forcing term in (2.5) can be neglected in comparison to that in (2.6). We must consider two cases.

(a) If the streaming flow velocity vanishes outside the surface wave layer, then $|\boldsymbol{u}^s| \sim |\boldsymbol{k}|^{-1} \sum |\varphi_{kl}^2||A_k|^2$ in this layer and, according to (2.54), we have

$$|\sum \alpha_{klmn} \bar{A}_l A_m A_n| \sim \Omega \,|\sum \int_{-1}^{0} \int_{\Sigma} \boldsymbol{u}^s \cdot \boldsymbol{g}_{kl}\, d\boldsymbol{x}\; A_l| \sim \Omega^{-1} |\boldsymbol{k}|^3 \sum |A_k|^3. \quad (2.55)$$

Note that this case requires that the streaming flow be confined to the surface wave layer. In some geometries (e.g., in Section 4 below if the radial wavenumber remains bounded when $\Omega \gg 1$) the spatial derivative of the right-hand side of (2.6b) remains bounded. In this case, a standard order of magnitude estimate shows that viscous diffusion is large compared to advection and the streaming flow can only remain confined if its time average vanishes, for otherwise the streaming flow vorticity must grow linearly at the edge of the surface wave layer and confinement is not possible. See Section 4.5 below.

(b) If, instead, the streaming flow is not confined to the surface wave layer, then $|\boldsymbol{u}^s| \sim \sum |\varphi_{kl}^2||A_k|^2$ everywhere and

$$\left|\sum \alpha_{klmn} \bar{A}_l A_m A_n\right| \sim \Omega^{-1} |\boldsymbol{k}|^3 \sum |A_k|^3 \ll \Omega \left|\sum \int_{-1}^{0} \int_{\Sigma} \boldsymbol{u}^s \cdot \boldsymbol{g}_{kl}\, d\boldsymbol{x}\; A_l\right|$$
$$\sim \Omega^{-1} |\boldsymbol{k}|^4 \sum |A_k|^3. \quad (2.56)$$

In this case, the streaming flow terms in (2.3) dominate the cubic terms, and the streaming flow provides the nonlinearity that saturates the instability.

We conclude that the effect of the streaming flow on the dynamics of the surface waves cannot in general be neglected in comparison to the usual cubic nonlinear terms and that, roughly speaking, the importance of this effect is larger for higher order modes. For instance, streaming flow effects should be more important in [42] than in [43] because the former studies the interaction of (2, 3) and (3, 2) modes in an almost square container while the latter focuses on the modes (0, 1) and (1, 0).

2.3.5. The Role of Transients. In addition to the basic fast timescale $t \sim 1$, the Faraday system exhibits several slower timescales. The amplitude equations (2.3) exhibit a *surface wave dissipation timescale*, given by (see (2.16))

$$t \sim t_d = |d_k|^{-1} = |\gamma_k^1 C_g^{1/2} + \gamma_k^2 C_g|^{-1}, \quad (2.57)$$

and a shorter timescale associated with the forcing if $|\mu|$ is large compared to $|d_k|$. When $|d_k| \sim |\mu|$, only one timescale is present, and

$$t_d \sim |A_k|^{-2}. \quad (2.58)$$

Similarly, the streaming flow momentum equation (2.4) exhibits the *viscous timescale*

$$t \sim t_v = C_g^{-1}, \quad (2.59)$$

which is much longer than the timescale (2.57) if either $C_g \to 0$ for a fixed mode or (see (2.17)) if $\Omega \to \infty$ for fixed C_g ($\ll 1$). However, in practice, as explained after equation (2.16), both timescales can be comparable for low order modes provided C_g is not too small and the contact line is pinned, as assumed here. This viscous timescale is the relevant one for the diffusion of streaming flow momentum and vorticity (see (2.4b) and (2.52)) from the boundaries into the bulk. In addition, the streaming flow manifests the timescale (2.57) of the surface waves, which according to (2.8) and (2.58), is also the *convective timescale* of the streaming flow. But the ultimate, long time behavior of the system is approached only on the viscous timescale t_v. In dimensional terms this timescale is given by h^2/ν, and so varies from a few minutes to a few hours for water in centimeter-deep containers [42], [43], [45]. Such a timescale can exert an influence over the duration of a typical experiment [42], [45].

We now examine the implication of the above comments for the dynamics of surface waves in the generic case when the timescales (2.57) and (2.59) are well separated, i.e., $t_d \ll t_v$. For simplicity, we assume that these are the only relevant timescales. We distinguish two cases, depending on initial conditions:

A. If $\nabla \times \boldsymbol{u}^s \equiv \mathbf{0}$ at $t = 0$, then according to (2.4a) and (2.22) we must also have $\boldsymbol{u}^s \equiv \mathbf{0}$ at $t = 0$, and thus $\boldsymbol{u}^s$ remains small on the timescale t_d (see (2.4)), until such time as the momentum and vorticity diffuse from the boundary layer into the bulk. During this transient we have $|\int\int \boldsymbol{u}^s \cdot \boldsymbol{g}\, d\boldsymbol{x}| \ll |A_k|^2$, and thus the system approaches an attractor of the amplitude equations usually considered in the literature, namely those with the streaming flow omitted. However, after this transient, the streaming flow begins to manifest itself and the solution evolves towards the true attractor of the full CASF equations. Transients of this type may have been detected in laterally vibrated [67, Fig. 8] and Faraday [42, §7.1] systems.

B. If the streaming flow vorticity is nonzero to begin with, the streaming flow affects the dynamics of the surface waves from the very beginning. During the initial transient of duration t_d viscous diffusion in the momentum equation (2.4b) can be ignored; the resulting simplified CASF equations can exhibit attractors that need not be close to the true attractors of the system; the latter will be reached only on the viscous timescale t_v. Transients of this type might be responsible for the striking behavior reported in [42, Fig. 16].

We conclude that the behavior of the Faraday system during the long initial transient should depend strongly on initial conditions: If the initial streaming velocity is not controlled, the system can appear to be "structurally unstable", as reported in [42]. If the viscous timescale is longer than the duration of the experiment and the initial streaming velocity is appropriately small, the influence of the streaming flow will not be apparent. These conclusions could explain why the results of Feng and Sethna [43] largely agreed with the predictions of a weakly nonlinear theory without streaming flow, while those of Simonelli and Gollub [42] did not. In the former case the dimensions of the container were much larger and the kinematic viscosity somewhat smaller; the viscous timescale was therefore much longer (i.e., $t_v^* = h^2/\nu \simeq (25.4 \text{ cm})^2/(0.01 \text{ cm}^2\text{s}^{-1}) \simeq 18$ hours in [43], and $t_v^* \simeq (2.5 \text{ cm})^2/(0.036 \text{ cm}^2\text{s}^{-1}) \simeq 3$ min in [42]). To obtain these estimates we used [68], [69] for the physical parameters not given in [42], [43] even though Feng and Sethna [43] state only that their container is "similar" to that used by [69]. Despite this uncertainty it is clear that the viscous timescale in [43] is much longer than that in [42].

3. Mode-Mode Interaction in Almost Square Containers

Let us assume now that the cross section of the container is a rectangle that is close to the square

$$\Sigma\colon\ |x| < L/2, \qquad |y| < L/2. \tag{3.1}$$

We suppose that $i\Omega$ is a double eigenvalue of the inviscid problem (2.9)–(2.12) in Σ, i.e., $N = 2$ in the terminology of Sections 1 and 2. This assumption implies that the surface wave mode excited by the parametric forcing breaks the D_4 symmetry of the system; this mode and the corresponding one obtained by reflection in a diagonal are excited simultaneously and hence interact strongly in the nonlinear regime. In the following we refer to the nonlinear states that resemble these eigenmodes as pure modes. In addition to the pure modes, the system admits nonlinear states in the form of mixed modes, consisting of an equal amplitude "superposition" of the pure modes. Both the pure modes and the mixed modes are excited at the same value of μ and are standing waves; their relative stability depends on the nonlinear terms in the corresponding amplitude equations. This set-up was investigated both experimentally [42], [43] and theoretically [43], [53], [54], [70], [71], [72]. However, a number of discrepancies between experiment and theory remain. When the container is square, the predicted shape of several bifurcation curves in the (μ, ω) plane differs from that reported in the experiment [42]. These predictions are based either on the assumption that the primary bifurcation is generic [70] and hence that mean flows are slaved to the dynamics of the mode amplitudes, or on a velocity potential formulation with the a posteriori addition of small damping [43], [53], [54], [71]. Both approaches thus leave out the streaming flow, leaving open the possibility that it is this flow that is responsible for the observations. Simonelli and Gollub [42] also demonstrated that perturbing the container cross section to a rectangular one unfolds the mode interaction point and produces chaotic oscillations in its vicinity. In the theories put forward, this behavior depends on the cubic coefficients computed on the basis of inviscid theory. However, as already indicated, the streaming flow comes in at the same order and hence is expected to have a profound effect on the chaotic dynamics as well. In Section 3.1 we first write down a scaled form of the CASF equations that apply to this problem, and then analyze some of their properties in order to make in Section 3.2 a qualitative comparison with the experiments in [42], [43]. We also comment on the appropriateness of some additional simplifications, including a Galerkin truncation (§3.3).

3.1. The Scaled CASF Equations

We begin by considering the dynamics in a square container. Motivated by experiments [42], [43], we let (V_1, P_1, F_1) and (V_2, P_2, F_2) be two eigenfunctions of (2.9)–(2.12) related by reflection in the xy diagonal, with F_1 odd in the x-direction and even in the y-direction. The odd-odd and even-even cases are treated similarly but lead to somewhat different amplitude equations. The chosen eigenfunctions are linearly independent, and we denote their amplitudes by A_1 and A_2, respectively. Since they break the D_4 symmetry of the square, the group D_4 acts on these amplitudes and on the associated streaming flow in a nontrivial way:

$$x \to -x, \qquad A_1 \to -A_1, \qquad (u_1^s, u_2^s, u_3^s) \to (-u_1^s, u_2^s, u_3^s), \tag{3.2}$$

$$x \leftrightarrow y, \qquad A_1 \leftrightarrow A_2, \qquad (u_1^s, u_2^s, u_3^s) \leftrightarrow (u_2^s, u_1^s, u_3^s). \tag{3.3}$$

In view of the symmetry (2.15) of the inviscid eigenvalue problem (2.9)–(2.12) we can take F_1 and F_2 to be real, with $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ purely imaginary. From equations (2.21), (2.23)–(2.26), and (2.38) it now follows that

$$\bar{\alpha}_{kl} = \alpha_{kl}, \qquad \boldsymbol{h}_{11} \equiv \boldsymbol{h}_{22} \equiv \boldsymbol{0}, \qquad \boldsymbol{h}_{12} \equiv -\boldsymbol{h}_{21} \equiv \mathrm{i}\boldsymbol{h}, \qquad \boldsymbol{g}_{11} \equiv \boldsymbol{g}_{22} \equiv \boldsymbol{0},$$
$$\boldsymbol{g}_{12} \equiv -\boldsymbol{g}_{21} \equiv \mathrm{i}\boldsymbol{g}. \tag{3.4}$$

Here $\boldsymbol{h} \equiv \nabla H_{12}$, $\boldsymbol{g} \equiv \Omega^{-1}\nabla \times (\bar{\boldsymbol{V}}_1 \times \boldsymbol{V}_2)$ are both real and equivariant under

$$x \to -x, \qquad (u_1, u_2, u_3) \to (u_1, -u_2, -u_3), \tag{3.5}$$

$$x \leftrightarrow y, \qquad (u_1, u_2, u_3) \to (-u_2, -u_1, -u_3), \tag{3.6}$$

where $\boldsymbol{u} \equiv (u_1, u_2, u_3)$ stands for either $\boldsymbol{h}$ or $\boldsymbol{g}$, and the function H_{12} is given by (2.24)–(2.26). The reflection symmetries (3.2)–(3.3) also imply that in a square

$$\alpha_{klmn} = \alpha_{\hat{k}\hat{l}\hat{m}\hat{n}},$$
$$\alpha_{1lmn} = 0 \quad \text{if exactly two of the indices } l, m, n \text{ are equal to } 1, \tag{3.7}$$
$$\alpha_{11} = \alpha_{22}, \qquad \alpha_{12} = \alpha_{21} = 0, \quad d_1 = d_2, \tag{3.8}$$

for $k, l, m, n = 1, 2$, where the symbol $\hat{}$ means that the value of the index has been changed (from 1 to 2 or vice versa).

If the square container is now perturbed to a rectangle, the eigenfrequencies are split as discussed in Section 2, with the modes $(\boldsymbol{V}_1, P_1, F_1)$, $(\boldsymbol{V}_2, P_2, F_2)$ being precisely the *surviving eigenmodes* discussed in the comment above eq. (2.19). Thus

$$\beta_{klm} = 0 \quad \text{if} \quad (k, l, m) \neq (1, 1, 1), (2, 2, 2), \qquad \beta_{111} = \beta_{222} = 1. \tag{3.9}$$

Since the remaining coefficients in (2.3) are unaffected (at leading order) by this perturbation, they are constrained by (3.4)–(3.8), and hence take the following form:

- The damping-detuning coefficients are $d_1 = d_2 = \gamma_1 C_g^{1/2} + \gamma_2 C_g + \mathrm{i}(\gamma_1 C_g^{1/2} + \omega - \Omega)$, with $\gamma_1 > 0$ and $\gamma_2 > 0$.
- All the coefficients α_{klmn} and α_{kl} vanish, except for

$$\begin{gathered}\alpha_{1111} = \alpha_{2222} = \alpha_1, \qquad \alpha_{1212} = \alpha_{2112} = \alpha_{1221} = \alpha_{2121} \equiv \alpha_2/2, \\ \alpha_{1122} = \alpha_{2211} \equiv \alpha_3, \qquad \alpha_{11} = \alpha_{22} \equiv \alpha_4,\end{gathered} \tag{3.10}$$

where the $\alpha_1, \dots, \alpha_4$ are real. Explicit expressions for the coefficients $\alpha_1, \dots, \alpha_4$ can be found in [53], [54] for several different mode interactions and a free contact line.

If we now introduce the rescaling

$$
\begin{aligned}
t = \tau/\delta, \qquad \gamma_1 C_g^{1/2} + \omega - (\Omega_1 + \Omega_2)/2 = \delta\Gamma, \qquad (\Omega_2 - \Omega_1)/2 = \delta\Lambda, \\
A_{1,2} = \delta^{1/2} A_{\pm}, \qquad \mu = \delta\Upsilon/\alpha_4, \qquad \boldsymbol{u}^s = \delta\boldsymbol{u}, \qquad \hat{p}^s = \delta^2 p,
\end{aligned}
\tag{3.11}
$$

where

$$
\delta = \gamma_1 C_g^{1/2} + \gamma_2 C_g \tag{3.12}
$$

is the damping rate, we may use (3.4) to rewrite equations (2.3)–(2.4) in the form

$$
\begin{aligned}
A'_{\pm}(\tau) \;=\; & -[1 + \mathrm{i}(\Gamma \pm \Lambda)]A_{\pm} + \mathrm{i}(\alpha_1|A_{\pm}|^2 + \alpha_2|A_{\mp}|^2)A_{\pm} + \mathrm{i}\alpha_3\bar{A}_{\pm}A_{\mp}^2 + \mathrm{i}\Upsilon\bar{A}_{\pm} \\
& \pm\Omega\int_{-1}^{0}\int_{\Sigma}\boldsymbol{u}\cdot\boldsymbol{g}\,d\boldsymbol{x}\,A_{\mp},
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{u} \;=\; & 0, \\
\partial\boldsymbol{u}/\partial\tau - [\boldsymbol{u} + \boldsymbol{H}(A_+, A_-) - \boldsymbol{G}(A_+, A_-)] & \times (\boldsymbol{\nabla}\times\boldsymbol{u}) = -\boldsymbol{\nabla}p + Re^{-1}\Delta\boldsymbol{u},
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{aligned}
\boldsymbol{H} \;=\; & \mathrm{i}(\bar{A}_- A_- - A_+\bar{A}_-)\boldsymbol{h}, \qquad \boldsymbol{G} = \mathrm{i}(\bar{A}_+ A_- - A_-\bar{A}_-)\boldsymbol{g}, \\
Re \;=\; & (\gamma_1 C_g^{1/2} + \gamma_2 C_g)/C_g
\end{aligned}
\tag{3.15}
$$

are the inviscid mean flow velocity, the Stokes drift, and the effective Reynolds number of the streaming flow, respectively. In eqs. (3.13) the terms $\Lambda \neq 0$ describe the leading order effect of perturbing the cross section Σ to a rectangular one.

Equivariance under (3.2)–(3.3) and the properties *c* and *d* in Section 2.2.2 imply that the boundary conditions (2.5)–(2.6) take the form

$$
\begin{aligned}
\boldsymbol{u} \;=\; & (|A_+|^2 + |A_-|^2)\boldsymbol{\varphi}_1 + (|A_+|^2 - |A_-|^2)\boldsymbol{\varphi}_2 + (\bar{A}_+ A_- + A_+\bar{A}_-)\boldsymbol{\varphi}_3 \\
& + \mathrm{i}(\bar{A}_+ A_- - A_+\bar{A}_-)\boldsymbol{\varphi}_4 \quad \text{if either } z = -1 \text{ or } (x, y) \in \partial\Sigma,
\end{aligned}
\tag{3.16}
$$

$$
\boldsymbol{u}\cdot\boldsymbol{e}_z \;=\; 0, \qquad \partial\tilde{\boldsymbol{u}}/\partial z = \mathrm{i}(\bar{A}_+ A_- - A_+\bar{A}_-)\boldsymbol{\varphi}_5 \quad \text{if } z = 0, \tag{3.17}
$$

where $\tilde{\boldsymbol{u}}$ is again the horizontal projection of $\boldsymbol{u}$, and the (real) vector functions $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_5$ (which can be calculated in terms of the inviscid eigenmodes by means of (2.43)–(2.44)) are tangent to the boundary of the container and exhibit the following symmetry properties:

A. $\boldsymbol{\varphi}_1$ transforms like $\boldsymbol{u}^s$ under (3.2)–(3.3).
B. $\boldsymbol{\varphi}_2$ transforms like (3.2) and (3.6).
C. $\boldsymbol{\varphi}_3$ transforms like (3.5) and (3.3).
D. $\boldsymbol{\varphi}_4$ and $\boldsymbol{\varphi}_5$ transform like (3.5) and (3.6).

The form of the boundary condition (3.17) follows from (2.44) and shows that the velocity shear at $z = 0$ vanishes whenever $\bar{A}_+ A_- = A_+\bar{A}_-$. This requirement is equivalent to the requirement that the phase of $A_+\boldsymbol{V}_1 + A_-\boldsymbol{V}_2$ be independent of position, i.e., that the surface wave be quasi-standing. These boundary conditions hold at leading order in the rectangular container as well.

The final CASF equations for the nearly square container are thus (3.13)–(3.14), (3.16)–(3.17), with the vector functions $\boldsymbol{h}$, $\boldsymbol{g}$, $\varphi_1, \ldots, \varphi_5$ as calculated from (2.43)–(2.44) and (3.4). The latter functions satisfy the symmetry properties indicated in (3.4) and A–D above; these are relevant to the analysis that follows. The real coefficients α_1, α_2, and α_3 have been calculated independently in [43], [53], [54], [72] for a free contact line; in the present problem the contact line is assumed to be pinned, which makes the calculations of these coefficients substantially more involved [29]. However, once the cross section is fixed, with $\Lambda = 0$ and $\Lambda \neq 0$ for squares and rectangles, respectively, the only free parameters are the scaled *forcing amplitude* Υ and the scaled *detuning* Γ.

3.2. Qualitative Comparison with Experiments

A. In a square ($\Lambda = 0$) there are two types of primary branches that bifurcate from the flat state $(A_-, A_-) = (0, 0)$ simultaneously: (i) pure states, of the form $(A, 0)$ or $(0, A)$, and (ii) mixed states of the form $(A, \pm A)$. Both are standing waves. Equations (3.13) show immediately that in both cases $\int\int \boldsymbol{u} \cdot \boldsymbol{g}\, d\boldsymbol{x} = 0$. Thus neither state involves the streaming flow. These results can be traced to the equivariance of $\boldsymbol{g}$ under (3.5)–(3.6), and are a consequence of the reflection symmetry of the streaming flows associated with these states. However, their stability properties with respect to reflection symmetry-breaking perturbations do depend on the coupling to the streaming flow. There are two instabilities of this kind. The first results in steady states of the form (A_+, A_-), $A_-\bar{A}_- \neq \bar{A}_+A_-$. These states resemble the alternating roll states $(A, \pm iA)$ familiar from studies of the Hopf bifurcation with D_4 symmetry [73] that occurs when $\Upsilon = 0$; Feng and Sethna [43] call these states “rotational”. In addition there are instabilities that would not be present *without* the coupling to the streaming flow; here the excitation of the streaming flow destabilizes standing waves that would otherwise be stable (see below).

B. In rectangular domains ($\Lambda \neq 0$) the CASF equations are invariant under the group D_2 only. This group is generated by (3.2) and

$$y \to -y, \qquad A_2 \to -A_2, \qquad \boldsymbol{u}^s \equiv (u_1^s, u_2^s, u_3^s) \to (u_1^s, -u_2^s, u_3^s). \qquad (3.18)$$

As a result there are only two primary branches of pure steady states and these are excited at different thresholds, i.e., the multiple bifurcation when $\Lambda = 0$ is split apart. The mixed modes no longer form a primary branch and instead appear only through secondary bifurcations. Simonelli and Gollub [42] did not attempt a comparison with theory for this case. They did report, however, observation of several different types of time-dependence in the mode amplitudes near the original mode interaction point. It is likely that such oscillations are the consequence of the interaction of the two nearly degenerate modes much as discussed in [74]. Since most of the reported oscillations lack instantaneous D_2 symmetry, streaming flow is likely to couple to the amplitude dynamics in an essential way. A comparison by Feng and Sethna [43] of their theory for $\Lambda \neq 0$ with experiments worked quite well for D_2-symmetric states but failed completely [43, Figs. 5,6] for the rotational states that should be accompanied by streaming flow. It should be noted that in a rectangular domain these states differ qualitatively from those in a square domain, and correspond instead to waves that rotate back and forth. In fact, for the parameter values for which their theory predicts waves of this type, Feng and Sethna did observe experimentally extremely slow, amplitude modulated, and apparently chaotic back-and-forth rotations, but were unable to establish their properties with confidence because of the long timescales involved. In contrast, the remaining features of the bifurcation diagram involve only reflection-symmetric states and these were found to be in reasonable quantitative agreement with the theory.

These experiments are consistent with our suggestion that streaming flows are generated whenever the state of the system lacks reflection symmetry, and that these flows might very well be responsible for the existing discrepancies between theory and experiments involving these states. Quantitative comparison with these experiments can only be performed on the basis of a (numerical) solution of the full CASF equations. We do not report here the results of such computations since the contact line in both experiments was left free, while for our theory it must be pinned. However, with this modification the experiments would fall within the scope of our theory because (i) the aspect ratio is evidently small, and (ii) C_g is also sufficiently small, viz., $C_g = 2.5 \cdot 10^{-6}$ in Feng and Sethna [43] and $2.4 \cdot 10^{-4}$ in Simonelli and Gollub [42], using $h = 25.4$ cm, $\rho = 1$ g cm^{-3}, $\nu = 0.01$ cm^2 s^{-1}, $T = 30$ dyn cm^{-1} [43], and $h = 2.5$ cm, $\rho = 0.81$ g cm^{-3}, $\nu = 0.032$ cm^2 s^{-1}, $T = 24.8$ dyn cm^{-1} [42]. Note that in both cases surface tension can in fact be ignored since $S = 4.7 \cdot 10^{-5}$ and $5.0 \cdot 10^{-3}$, respectively. To reach these conclusions we have, once again, used [68], [69] for the physical parameters not given in [42], [43].

3.3. *Truncation of the CASF Equations*

Given the complexity of the CASF equations, we have constructed a hierarchy of simplified models based on Galerkin truncation of the Navier-Stokes equation for the streaming flow. The streaming flow will in general contain terms of different symmetries. In a square these are determined by the two reflections (3.2) and (3.3) generating the group D_4. Terms in the streaming flow can be odd/odd, odd/even, even/odd, and even/even under these symmetries. In the following we write down the leading terms of each type. Using the results of the preceding section we have

$$\begin{aligned} A'_\pm(\tau) &= -(1+\mathrm{i}\Gamma)A_\pm + \mathrm{i}(\alpha_1|A_\pm|^2 + \alpha_2|A_\mp|^2)A_\pm + \mathrm{i}\alpha_3\bar{A}_\pm A_\mp^2 \\ &\quad + \mathrm{i}\Upsilon\bar{A}_\pm \mp \gamma v_1 A_\mp, \end{aligned} \tag{3.19}$$

$$v_1'(\tau) = \varepsilon_1[-v_1 + \mathrm{i}(\bar{A}_+A_- - A_+\bar{A}_-)] + \gamma_0 v_1 v_4 + \gamma_1 v_2 v_3, \tag{3.20}$$

$$v_2'(\tau) = \varepsilon_2[-v_2 + \bar{A}_+A_- + A_+\bar{A}_-] + \gamma_2 v_1 v_3 + \gamma_3 v_2 v_4, \tag{3.21}$$

$$v_3'(\tau) = \varepsilon_3[-v_3 + |A_+|^2 - |A_-|^2] + \gamma_4 v_1 v_2 + \gamma_5 v_3 v_4, \tag{3.22}$$

$$v_4'(\tau) = \varepsilon_4[-v_4 + |A_+|^2 + |A_-|^2] + \gamma_6 v_1^2 + \gamma_7 v_2^2 + \gamma_8 v_3^2 + \gamma_9 v_4^2. \tag{3.23}$$

Here $\varepsilon_j = -\lambda_j Re^{-1} > 0$, $\lambda_j < 0$ are the corresponding hydrodynamic eigenvalues, and the v_j represent the (real) amplitudes of the four different contributions to the streaming flow. These equations can be constructed as in [75]: Each contribution must be independent of the fast timescale and hence be a product of an amplitude and a complex conjugate; each must be either odd or even under (3.2) and (3.3); each must couple to the amplitudes $A_\pm$ in a conservative fashion and only the first, odd/odd, mode can contribute to the amplitude equations because the remaining ones are reflection-symmetric (property §3.2A above). Thus γ is real and no term of the form $v_k A_\mp$ with $k > 1$ is present in (3.19); both facts can be checked by explicit computation. However, the reflection-symmetric modes do affect implicitly the surface wave dynamics through v_1 because of the nonlinear terms in (3.20). Note that in steady state the streaming flow associated with $(A, 0)$ takes the form $(v_1, v_2, v_3, v_4) = (0, 0, |A|^2, |A|^2)$ as $A \to 0$, while $(v_1, v_2, v_3, v_4) = (0, 2|A|^2, 0, 2|A|^2)$ for the mixed modes $(A, \pm A)$.

If we neglect the nonlinear terms in (3.20)–(3.23), equations (3.19) and (3.20) decouple from the rest; in an almost square container we therefore have

$$A'_\pm(\tau) = -[1 + \mathrm{i}(\Gamma \pm \Lambda)]A_\pm + \mathrm{i}(\alpha_1|A_\pm|^2 + \alpha_2|A_\mp|^2)A_\pm + \mathrm{i}\alpha_3\bar{A}_\pm A_\mp^2 + \mathrm{i}\Upsilon\bar{A}_\pm \mp \gamma v_1 A_\mp, \tag{3.24}$$

$$v'_1(\tau) = \varepsilon[-v_1 + \mathrm{i}(\bar{A}_+ A_- - A_+\bar{A}_-)]. \tag{3.25}$$

In the remainder of this subsection we discuss the consequences of this one mode approximation of the streaming flow, although the results apply to (3.19)–(3.23) as well.

Both the pure and the mixed modes can become unstable to perturbations involving the streaming flow. Let us first consider pure modes, $(A_+, A_-, v_1) = (A, 0, 0)$, with $A \neq 0$ such that

$$[1 + \mathrm{i}(\Gamma + \Lambda) - \mathrm{i}\alpha_1|A|^2]A = \mathrm{i}\Upsilon\bar{A};$$

the phase of A can be eliminated from this equation, to obtain

$$1 + [\Gamma + \Lambda - \alpha_1|A|^2]^2 = \Upsilon^2, \qquad \alpha_1 \neq 0.$$

Thus the instability sets in at

$$\Upsilon = \Upsilon_c \equiv [1 + (\Gamma + \Lambda)^2]^{1/2},$$

and the amplitude $|A|$ increases monotonically for $\Upsilon > \Upsilon_c$ provided $(\Gamma + \Lambda)/\alpha_1 \leq 0$; if $(\Gamma + \Lambda)/\alpha_1 > 0$ the branch bifurcates subcritically at $\Upsilon = \Upsilon_c$ before turning around towards larger Υ at a secondary saddle-node bifurcation.

To determine the linear stability of these states, we replace A_+, A_-, and v_1 by $A + X_+ e^{\lambda\tau} + \bar{Y}_+ e^{\bar{\lambda}\tau}$, $X_- e^{\lambda\tau} + \bar{Y}_- e^{\bar{\lambda}\tau}$, and $Ze^{\lambda\tau}$+c.c., respectively, and linearize. The resulting equations

$$[\lambda + 1 + \mathrm{i}(\Gamma + \Lambda) - 2\mathrm{i}\alpha_1|A|^2]X_+ - \mathrm{i}(\Upsilon + \alpha_1 A^2)Y_+ = 0, \tag{3.26}$$

$$[\lambda + 1 - \mathrm{i}(\Gamma + \Lambda) + 2\mathrm{i}\alpha_1|A|^2]Y_+ + \mathrm{i}(\Upsilon + \alpha_1\bar{A}^2)X_+ = 0, \tag{3.27}$$

$$[\lambda + 1 + \mathrm{i}(\Gamma - \Lambda) - \mathrm{i}\alpha_2|A|^2]X_- - \mathrm{i}(\Upsilon + \alpha_3 A^2)Y_- - \gamma A Z = 0, \tag{3.28}$$

$$[\lambda + 1 - \mathrm{i}(\Gamma - \Lambda) + \mathrm{i}\alpha_2|A|^2]Y_- + \mathrm{i}(\Upsilon + \alpha_3\bar{A}^2)X_- - \gamma\bar{A}Z = 0, \tag{3.29}$$

$$\mathrm{i}\varepsilon(\bar{A}X_- - AY_-) - (\lambda + \varepsilon)Z = 0, \tag{3.30}$$

have *pure* eigenmodes ($X_- = Y_- = Z = 0$) and *mixed* eigenmodes ($X_- = Y_- = 0$, $Z \neq 0$) with associated dispersion relations given by

$$(\lambda + 1)^2 + [\Gamma + \Lambda - 2\alpha_1 |A|^2]^2 = 1 + (\Gamma + \Lambda)^2, \tag{3.31}$$

$$\lambda^2 + 2\lambda + [2\Lambda + (\alpha_2 + \alpha_3 - \alpha_1)|A|^2][-2\Gamma + (\alpha_1 + \alpha_2 - \alpha_3)|A|^2 + 2\gamma\varepsilon|A|^2/(\lambda + \varepsilon)] = 0, \tag{3.32}$$

respectively. The former relation is quadratic and shows readily that pure-mode instabilities are always nonoscillatory at threshold (i.e., associated with $\lambda = 0$) and correspond either to the primary bifurcation at $\Upsilon = \Upsilon_c$ or to the secondary saddle-node bifurcation at

$$|A|^2 = (\Gamma + \Lambda)/\alpha_1. \tag{3.33}$$

In contrast the relation (3.32) is cubic, and shows that mixed-mode instabilities are either nonoscillatory, occurring when

$$|A|^2 = 2\Lambda/(\alpha_1 - \alpha_2 - \alpha_3) \quad \text{or} \quad |A|^2 = 2\Gamma/(\alpha_1 + \alpha_2 - \alpha_3 + 2\gamma), \tag{3.34}$$

or oscillatory, producing quasiperiodic oscillations, when

$$[2\Lambda + (\alpha_2 + \alpha_3 - \alpha_1)|A|^2][2\Gamma + (\varepsilon\gamma - \alpha_1 - \alpha_2 + \alpha_3)|A|^2] = \varepsilon(\varepsilon + 2), \tag{3.35}$$

provided in all cases that $|A|^2 > 0$. Since the corresponding eigenvalues $\lambda = \pm i\lambda_I$ are given by

$$\lambda_I^2 = -\varepsilon^2 + \varepsilon\gamma|A|^2[2\Lambda + (\alpha_2 + \alpha_3 - \alpha_1)|A|^2] > 0, \tag{3.36}$$

the presence of this bifurcation leading to quasi-periodic waves requires that $\gamma \neq 0$. Such bifurcation cannot therefore occur without the streaming flow. From equations (3.33)–(3.35) we also find conditions for codimension-two degeneracies: (i) a Takens-Bogdanov bifurcation, resulting from the coalescence of the symmetry-breaking and Hopf bifurcations, occurs if (3.34b) holds and

$$\varepsilon(\alpha_1 + \alpha_2 - \alpha_3 + 2\gamma)^2 = 4\gamma\Gamma[\Lambda(\alpha_1 + \alpha_2 - \alpha_3 + 2\gamma) + \Gamma(\alpha_2 + \alpha_3 - \alpha_1)]; \tag{3.37}$$

(ii) a saddle-node–symmetry-breaking bifurcation occurs when (3.33) holds and either

$$(\Gamma + \Lambda)(\alpha_1 - \alpha_2 - \alpha_3) = 2\Lambda\alpha_1 \quad \text{or} \quad (\Gamma + \Lambda)(\alpha_1 + \alpha_2 - \alpha_3 + 2\gamma) = 2\Gamma\alpha_1; \tag{3.38}$$

(iii) a saddle-node–Hopf bifurcation with one zero plus two nonzero imaginary eigenvalues occurs when (3.34b) holds and

$$[2\Lambda\alpha_1 + (\alpha_2 + \alpha_3 - \alpha_1)(\Gamma + \Lambda)][2\Gamma\alpha_1 + (\varepsilon\gamma - \alpha_1 - \alpha_2 + \alpha_3)(\Gamma + \Lambda)] = \varepsilon(\varepsilon + 2)\alpha_1^2. \tag{3.39}$$

The first two of these bifurcations contain within their unfolding periodic solutions that correspond to (different types of) asymmetric mixed-mode oscillations in the Faraday system. The last bifurcation contains symmetric quasi-periodic solutions [76], and these correspond to three-frequency states in the Faraday system. Chaotic dynamics are present near the global bifurcations with which the two-tori terminate [77], [78].

The corresponding results for the other pure mode, $(A_+, A_-, v_1) = (A, 0, 0)$, can be obtained from the above results using the substitution $\Lambda \to -\Lambda$. Likewise, we can use these results to deduce the stability properties of the mixed modes $(A_+, A_-, v_1) = (A, \pm A, 0)$ in a square container ($\Lambda = 0$). This is because equations (3.19) and (3.25) are invariant under the transformation

$$A_\pm \to (A_+ \pm A_-)/2, \qquad v_1 \to -v_1/2, \qquad \alpha_1 \to \alpha_1 + \alpha_2 + \alpha_3,$$
$$\alpha_2 \to 2(\alpha_1 - \alpha_3), \qquad \alpha_3 \to (\alpha_1 - \alpha_2 + \alpha_3), \qquad \gamma \to -2\gamma, \tag{3.40}$$

while the mixed modes become pure modes. It follows that for the mixed modes

$$1 + [\Gamma - (\alpha_1 + \alpha_2 + \alpha_3)|A|^2]^2 = \Upsilon^2, \qquad \alpha_1 + \alpha_2 + \alpha_3 \neq 0,$$

and hence that these modes set in at

$$\Upsilon = \Upsilon_c \equiv (1 + \Gamma^2)^{1/2},$$

i.e., simultaneously with the pure modes. The stability results of these states follow immediately from the substitution (3.40) into (3.33)–(3.39) and setting $\Lambda = 0$; note, in particular, that the two dispersion relations are now associated with reflection-symmetric and symmetry-breaking perturbations, respectively. Once again all the same bifurcations and degeneracies are still present, and streaming flow is crucial for the presence of a symmetry-breaking Hopf bifurcation.

4. Mode-Mode Interaction in Almost Circular Containers

In this section we discuss the corresponding results for circular containers. This system has the symmetry group O(2) of rotations and reflection of a circle. We consider nonaxisymmetric modes so that the primary instability breaks the symmetry, and hence corresponds to a zero eigenvalue of double multiplicity. We can think of this instability as generating clockwise and counterclockwise rotating waves. When these waves are coupled via the parametric forcing, the primary state is a standing wave with reflection symmetry. In this case only the phase of this standing wave couples to the streaming flow. However, as soon as the shape of the container is perturbed from circular, both the phase and the amplitudes couple to the streaming flow. In these cases the presence of the streaming flow has a much more dramatic impact on the dynamics. This interesting case has, unfortunately, not been investigated in experiments.

4.1. The Scaled CASF Equations

We use cylindrical coordinates (r, θ, z) with associated unit vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, and $\boldsymbol{e}_z$, and take the unperturbed cross section of the container to be

$$\Sigma\colon\ 0 \le r < R. \tag{4.1}$$

The two ($N = 2$) surface eigenmodes appearing in (2.1) are taken to be

$$(\boldsymbol{V}_1, P_1, F_1) = (\mathrm{i}U\boldsymbol{e}_r + V\boldsymbol{e}_\theta + \mathrm{i}W\boldsymbol{e}_z, P, F)\mathrm{e}^{\mathrm{i}m\theta}, \qquad (\boldsymbol{V}_2, P_2, F_2) = (-\bar{\boldsymbol{V}}_1, \bar{P}_1, \bar{F}_1), \tag{4.2}$$

where $m \geq 1$ and the functions U, V, W, P, and F are real and independent of θ. From equations (2.23)–(2.26) and (2.38), we now have

$$\begin{aligned} \boldsymbol{h}_{11} \equiv \boldsymbol{h}_{22} \equiv \boldsymbol{h}_{12} \equiv \boldsymbol{h}_{21} \equiv \boldsymbol{g}_{12} \equiv \boldsymbol{g}_{21} \equiv 0, \\ \boldsymbol{g}_{11} \equiv -\boldsymbol{g}_{22} \equiv \boldsymbol{g} = -2\Omega^{-1}\nabla \times (VW\boldsymbol{e}_r - UV\boldsymbol{e}_z) = g(r, z)\boldsymbol{e}_\theta. \end{aligned} \tag{4.3}$$

When the cross section of the cylinder is perturbed while *preserving the reflection symmetry* in the plane $\theta = 0, \pi$, the amplitude equations must remain invariant under the action

$$A_1 \leftrightarrow A_2, \qquad \boldsymbol{u}^s \cdot \boldsymbol{e}_\theta \to -\boldsymbol{u}^s \cdot \boldsymbol{e}_\theta. \tag{4.4}$$

In view of (2.19) this fact implies that

$$\beta_{111} = \beta_{212} = 1 - \beta_{121} = 1 - \beta_{222}, \qquad \beta_{112} = \beta_{211} = -\beta_{122} = -\beta_{221}. \tag{4.5}$$

This fact is used below to construct the linear terms in the amplitude equations. The remaining terms commute with the symmetry O(2) of the unperturbed container, generated by (4.4) and the rotations

$$\theta \to \theta + \phi, \qquad A_1 \to A_1\mathrm{e}^{\mathrm{i}m\phi}, \qquad A_2 \to A_2\mathrm{e}^{-\mathrm{i}m\phi}. \tag{4.6}$$

Thus (i) the viscous damping-detuning terms must be such that

$$d_1 = d_2 \equiv \gamma_1^1(1 + \mathrm{i})C_g^{1/2} + \gamma_2^1 C_g, \tag{4.7}$$

and (ii) the coefficients of the nonlinear terms and of the forcing all vanish except for

$$\alpha_{1111} = \alpha_{2222} \equiv \alpha_1, \qquad \alpha_{1221} = \alpha_{2112} \equiv \alpha_2, \qquad \alpha_{12} = \alpha_{21} \equiv \alpha_3. \tag{4.8}$$

Here $\gamma_1^1 > 0$, $\gamma_2^1 > 0$, α_1, α_2, and α_3 are real, and we have taken into account (2.16) and (2.28). The coefficients α_1, α_2, and α_3 have been computed by Miles [55] for a particular case with a free contact line.

With these results and the rescaling

$$\begin{aligned} t = \tau/\delta, \qquad \gamma_1^1 C_g^{1/2} + \omega - \Omega_2 + \beta_{111}(\Omega_2 - \Omega_1) = \delta\Gamma, \qquad \beta_{112}(\Omega_1 - \Omega_2) = \delta\Lambda, \\ A_{1,2} = \delta^{1/2}A_\pm, \qquad \mu = \delta\Upsilon/\alpha_3, \qquad \boldsymbol{u}^s = \delta\boldsymbol{u}, \qquad \hat{p}^s = \delta^2 p, \end{aligned} \tag{4.9}$$

where

$$\delta = \gamma_1^1 C_g^{1/2} + \gamma_2^1 C_g \tag{4.10}$$

is the damping rate, the amplitude equations (2.3)–(2.4) become

$$\begin{aligned} A'_\pm(\tau) \;=\; & -(1 + \mathrm{i}\Gamma)A_\pm + \mathrm{i}\Lambda A_\mp + \mathrm{i}(\alpha_1|A_\pm|^2 + \alpha_2|A_\mp|^2)A_\pm + \mathrm{i}\Upsilon\bar{A}_\mp \\ & \mp \mathrm{i}\Omega\int_{-1}^{0}\int_0^{2\pi}\int_0^R g(r, z)\boldsymbol{u}\cdot\boldsymbol{e}_\theta r\,dr\,d\theta\,dz A_\pm, \end{aligned} \tag{4.11}$$

$$\begin{aligned} &\nabla \cdot \boldsymbol{u} = 0, \\ &\partial\boldsymbol{u}/\partial\tau - [\boldsymbol{u} + \boldsymbol{G}(A_+, A_-)] \times (\nabla \times \boldsymbol{u}) = -\nabla p + Re^{-1}\Delta\boldsymbol{u}, \end{aligned} \tag{4.12}$$

where the Stokes drift $\boldsymbol{G}$ and the effective Reynolds number Re are given by

$$\boldsymbol{G} = (|A_-|^2 - |A_+|^2) g \boldsymbol{e}_\theta \quad \text{and} \quad Re = (\gamma_1^1 C_g^{1/2} + \gamma_2^1 C_g)/C_g. \tag{4.13}$$

Note that the inviscid mean flow is absent from these equations, as expected from the fact that the Stokes drift is horizontal at the unperturbed free surface (see the discussion in §2.1C). If the streaming flow is ignored in (4.11) the resulting equations are a special case of those considered in [79], [80].

The presence of the O(2) symmetry (4.4,4.6) and the properties c and d in Section 2.2.2 together imply that the boundary conditions (2.5)–(2.6) take the form (in terms of the rescaled variables (4.9))

$$\begin{aligned} \boldsymbol{u} \;=\; & [\varphi_1 A_+ \bar{A}_- \mathrm{e}^{2\mathrm{i}m\theta} + \text{c.c.} + \varphi_2(|A_+|^2 + |A_-|^2)]\boldsymbol{n}_0 \times \boldsymbol{e}_\theta \\ & + [i\varphi_3 A_+ \bar{A}_- \mathrm{e}^{2\mathrm{i}m\theta} + \text{c.c.} + \varphi_4(|A_+|^2 - |A_-|^2)]\boldsymbol{e}_\theta \\ & \text{if either } r = R \text{ or } z = -1, \end{aligned} \tag{4.14}$$

$$\boldsymbol{u} \cdot \boldsymbol{e}_z = (\partial \boldsymbol{u}/\partial z) \cdot \boldsymbol{e}_r = 0, \quad (\partial \boldsymbol{u}/\partial z) \cdot \boldsymbol{e}_\theta = \varphi_5(|A_-|^2 - |A_+|^2), \quad \text{on } z = 0, \tag{4.15}$$

where $\boldsymbol{n}_0$ is again the outward unit normal. As before, the boundary condition (4.15c) follows from the requirement that the surface shear vanish for quasi-standing surface waves, i.e., waves for which the phase of $A_+ \boldsymbol{V}_1 + A_- \boldsymbol{V}_2$ is independent of position.

Equations (4.11)–(4.12), (4.14)–(4.15) constitute the rescaled CASF for the present problem. In these equations the (real) scalar functions $\varphi_1, \ldots, \varphi_5$ and g are independent of θ, and given by (2.43)–(2.44) and (4.3) in terms of the components (4.2) of the excited linear modes. For a pinned contact line the coefficients γ_1^1 and γ_2^1 in (4.10) and the corresponding inviscid eigenfunctions have been calculated in [47].

When $\Lambda = 0$ the CASF equations (4.11)–(4.12) are equivariant with respect to the full group O(2). As soon as $\Lambda \neq 0$ the symmetry of the problem is reduced to the group D_2 generated by

$$A_\pm \to -A_\pm, \qquad \theta \to \theta + \pi, \quad \text{and} \quad A_+ \leftrightarrow A_-, \qquad \theta \to -\theta, \qquad \boldsymbol{u} \cdot \boldsymbol{e}_\theta \to -\boldsymbol{u} \cdot \boldsymbol{e}_\theta. \tag{4.16}$$

The former arises from evolution in time by $2\pi/\omega$ while the latter is a consequence of the remaining spatial reflection symmetry. Once again the coupling to the streaming flow in the amplitude equations (4.11) vanishes identically when the surface wave is reflection-symmetric for all τ.

4.2. *The Circular Container*

When $\Lambda = 0$ the surface wave becomes quasi-standing after a transient, which means that it is determined up to a spatial phase θ_0. If we write

$$A_\pm = B_\pm \mathrm{e}^{-\mathrm{i}m\theta_0(\tau)}, \tag{4.17}$$

where

$$\theta_0'(\tau) = (\Omega/m) \int_{-1}^{0} \int_0^{2\pi} \int_0^R g(r, z) \boldsymbol{u} \cdot \boldsymbol{e}_\theta r \, dr \, d\theta \, dz, \tag{4.18}$$

then equations (4.11) reduce to

$$B'_{\pm}(\tau) = -(1 + i\Gamma)B_{\pm} + i(\alpha_1|B_{\pm}|^2 + \alpha_2|B_{\mp}|^2)B_{\pm} + i\Upsilon\bar{B}_{\mp}. \tag{4.19}$$

These equations provide the simplest description of nearly inviscid Faraday waves in O(2)-symmetric systems [81] and all their solutions converge to reflection-symmetric steady states of the form

$$B_{\pm} = R_0 e^{im\theta_0^0}, \tag{4.20}$$

i.e., to standing waves. Equations (2.1), (4.2), (4.9), and (4.20) imply that the corresponding free surface deflection is given by

$$f = 2\delta^{1/2} R_0 F_1 \cos(m[\theta - \theta_0(\tau) + \theta_0^0]), \tag{4.21}$$

and hence that only the spatial phase θ_0 couples to the streaming flow, as described by equation (4.18) and

$$\nabla \cdot \boldsymbol{u} = 0, \qquad \partial \boldsymbol{u}/\partial\tau - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) = -\nabla p + Re^{-1}\Delta\boldsymbol{u}, \tag{4.22}$$

$$\boldsymbol{u} = 2R_0^2[\varphi_1 \cos[2m(\theta - \theta_0)] + \varphi_2]\boldsymbol{n}_0 \times \boldsymbol{e}_\theta - 2\varphi_3 \sin[2m(\theta - \theta_0)]\boldsymbol{e}_\theta,$$

$$\text{if either } r = R \text{ or } z = -1, \tag{4.23}$$

$$\boldsymbol{u} \cdot \boldsymbol{e}_z = (\partial\boldsymbol{u}/\partial z) \cdot \boldsymbol{e}_r = (\partial\boldsymbol{u}/\partial z) \cdot \boldsymbol{e}_\theta = 0, \qquad \text{on } z = 0, \tag{4.24}$$

as obtained upon substitution of (4.17)–(4.20) into (4.12), (4.14)–(4.15). The (constant) arbitrary phase θ_0^0 appearing in (4.20) has been eliminated by an appropriate rotation. Eqs. (4.18), (4.22)–(4.24) possess, for all R_0^2, reflection-symmetric steady states of the form $\boldsymbol{u} = \boldsymbol{u}^s(r, \theta - \theta_0, z)$, $\theta_0 =$ constant, with $\boldsymbol{u}^s(r, \theta, z) \cdot \boldsymbol{e}_\theta = -\boldsymbol{u}^s(r, -\theta, z) \cdot \boldsymbol{e}_\theta$; note that there is a whole family of such states, obtained by an arbitrary rotation [82]. For small R_0^2 the existence and (orbital) asymptotic stability of these states can be ascertained analytically. It turns out that these states can lose stability at finite R_0 either through a parity-breaking bifurcation giving rise to uniformly drifting spatially uniform standing waves (such as those observed in Faraday experiments in annular containers [83]), or via a symmetry-breaking Hopf bifurcation that produces the so-called direction-reversing waves [84]. In the latter case the standing waves drift alternately clockwise and counterclockwise but their mean location remains fixed. Solutions of this type have been found in a two-dimensional Cartesian geometry with periodic boundary conditions, and represent the instability that sets in at smallest amplitude [85]. The corresponding three-dimensional results in cylindrical or annular domains remain unavailable.

4.3. Low Reynolds Number Streaming Flow

Once $\Lambda \neq 0$ equations (4.11), can no longer be reduced to (4.19), and the streaming flow couples to the amplitudes as well. The description of this coupling becomes simpler when the Reynolds number of the streaming flow is small, for then the nonpotential term $-\boldsymbol{u} \times (\nabla \times \boldsymbol{u})$ in (4.12b) is negligible. In fact this approximation remains qualitatively useful even for larger Reynolds numbers; see, in particular, Section 4.4 below. The absence of nonlinear terms allows us to isolate the part of the streaming flow velocity that contributes to the nonlocal term in (4.11), by decomposing the streaming flow variables as

$$(\boldsymbol{u}, p) = (v(r, z, \tau)\boldsymbol{e}_\theta, 0) + (\hat{\boldsymbol{u}}, p), \qquad \text{where} \int_0^{2\pi} \hat{\boldsymbol{u}} \cdot \boldsymbol{e}_\theta \, d\theta \equiv 0. \tag{4.25}$$

Thus

$$\begin{aligned} A'_\pm(\tau) &= -(1 + \mathrm{i}\Gamma)A_\pm + \mathrm{i}\Lambda A_\mp + \mathrm{i}(\alpha_1|A_\pm|^2 + \alpha_2|A_\mp|^2)A_\pm + \mathrm{i}\Upsilon \bar{A}_\mp \\ &\quad \mp 2\pi\mathrm{i}\Omega \int_{-1}^{0}\int_0^R g(r, z)v(r, z, \tau)\, r\, dr\, dz A_\pm, \end{aligned} \tag{4.26}$$

$$v_\tau = Re^{-1}(v_{rr} + r^{-1}v_r - r^{-2}v + v_{zz}) \quad \text{if } 0 < r < R,\ -1 < z < 0, \tag{4.27}$$

$$v = 0 \quad \text{as } r \to 0, \tag{4.28}$$

$$\begin{aligned} v &= \varphi_4(|A_-|^2 - |A_+|^2) \quad \text{if either } r = R \text{ or } z = -1, \\ v_z &= \varphi_5(|A_-|^2 - |A_+|^2) \quad \text{if } z = 0. \end{aligned} \tag{4.29}$$

The resulting model can be integrated numerically by relatively inexpensive methods and facilitates further analytical progress as well. In fact, the linear stability analysis in Section 4.5 below for the even simpler model (4.40) is readily extended to the present case. Specifically, the stability properties of the symmetric steady states of (4.26)–(4.29), i.e., of $A_+ = A_- = A$, $v = 0$, with A satisfying (4.41), are governed by the dispersion relations (4.48)–(4.49) with $\gamma\varepsilon/(\lambda + \varepsilon)$ replaced by

$$2\pi\Omega \int_{-1}^{0}\int_0^R g(r, z)V(r, z, \lambda)\, r\, dr\, dz, \tag{4.30}$$

where V solves

$$\lambda V = Re^{-1}(V_{rr} + r^{-1}V_r - r^{-2}V + V_{zz}) \quad \text{if } 0 < r < R,\ -1 < z < 0, \tag{4.31}$$

$$\begin{aligned} V &= 0 \quad \text{as } r \to 0, \qquad V = \varphi_4 \quad \text{if either } r = R \text{ or } z = -1, \\ V_z &= \varphi_5 \quad \text{if } z = 0. \end{aligned} \tag{4.32}$$

Note that V depends analytically on λ except at the eigenvalues of the homogeneous version of (4.31)–(4.32), which are poles of V. These latter eigenvalues are real and negative, and correspond to the purely azimuthal, hydrodynamic eigenmodes of (1.7)–(1.9).

4.4. The High Frequency Limit

In the limit of high forcing frequency, the CASF equations simplify dramatically and are replaced by a system in one spatial dimension together with a linearized equation for the streaming flow. In this limit the azimuthal wavenumber m of the excited surface mode becomes large, and $g, \varphi_1, \ldots, \varphi_4$ vanish exponentially rapidly outside of the surface-wave layer of thickness m^{-1}. In this layer we have the estimates (see §2.3.4, case (a))

$$\alpha_1 \sim \alpha_2 \sim g \sim m(|\varphi_1| + |\varphi_2| + |\varphi_3| + |\varphi_4|) \sim |\varphi_5| \sim \Omega^{-1}m^3, \tag{4.33}$$

provided the radial wavenumber remains bounded. In addition in this layer

$$g(r, z) \simeq g_0(r)e^{\eta}, \tag{4.34}$$

where

$$\eta = mz \tag{4.35}$$

is a stretched variable. If the vorticity is confined to the surface-wave layer (see below), the velocity components decay exponentially outside of the layer. Inside the layer, the azimuthal component v satisfies $|v| \sim \Omega^{-1} m^3 |A_\pm|^2 \sim C_g m^2$, an estimate that follows from a balance between damping and nonlinearity in the amplitude equations (for appropriate forcing amplitude) and the estimate (2.17); the radial and vertical velocity components are then m^{-1} times smaller. Altogether, after a suitable rescaling of the equations, we obtain

$$\begin{aligned} B'_\pm(\tau) &= -(1 + i\Gamma)B_\pm + i\Lambda B_\mp + i(\tilde{\alpha}_1|B_\pm|^2 + \tilde{\alpha}_2|B_\mp|^2)B_\pm \\ &\quad + i\Upsilon\bar{B}_\mp \mp i\tilde{\gamma}\int_{-\infty}^{0} V(\eta, \tau)e^{\eta}\, d\eta B_\pm, \end{aligned} \tag{4.36}$$

$$\begin{aligned} V_\tau &= V_{\eta\eta} \quad \text{if } -\infty < \eta < 0, \qquad V = 0 \quad \text{as } \eta \to -\infty, \\ V_\eta &= |B_-|^2 - |B_+|^2 \quad \text{at } \eta = 0, \end{aligned} \tag{4.37}$$

where V is the following weighted average of the azimuthal velocity:

$$V = \left(\int_0^R \int_0^{2\pi} g_0(r)\varphi_5(r) r\, d\theta\, dr\right)^{-1} \int_0^R \int_0^{2\pi} g_0(r) v(r, \theta, z, \tau) r\, d\theta\, dr.$$

As an evolution problem, equations (4.36)–(4.37) possess a unique solution, whose L_2 norm is uniformly bounded as $\tau \to \infty$ provided

$$\frac{1}{\tau}\int_0^{\tau} (|B_-|^2 - |B_+|^2)\, d\tau \to 0 \quad \text{as } \tau \to \infty. \tag{4.38}$$

This condition follows from the exact relation

$$\frac{d}{d\tau}\int_{-\infty}^{0} V(\eta, \tau)\, d\eta = |B_-|^2 - |B_+|^2,$$

readily obtained from equations (4.37). Condition (4.38) is equivalent to the requirement that the attractor of the system be reflection-symmetric on average. When this is not the case, vorticity cannot be confined in the surface-wave layer, as assumed above, and will spread into the bulk, producing the much more involved regime (b) in Section 2.3.4. In this case equations (4.36)–(4.37) will have solutions such that $|\int V\, d\eta|$ is unbounded as $\tau \to \infty$.

The linear model (4.36)–(4.37) is even simpler than that derived in the preceding section. The reflection-symmetric steady states take the form $B_+ = B_- = A$, $V = 0$, with A satisfying (4.41), and there are no nonsymmetric steady states. The stability of these states is given by the dispersion relations (4.48)–(4.49) with $\gamma\varepsilon/(\lambda + \varepsilon)$ replaced by $\tilde{\gamma}/(\lambda + \lambda^{1/2})$, with a nonzero λ in (4.49).

4.5. Single Mode Approximation for the Streaming Flow

The one-dimensional problem (4.27)–(4.29) can be solved by expressing the azimuthal component of the streaming flow velocity, v, as a Fourier expansion in the purely azimuthal hydrodynamic modes. If only the first such mode is retained, the following counterpart of (3.19)–(3.23) is obtained:

$$\begin{aligned} A'_\pm(\tau) &= -(1+\mathrm{i}\Gamma)A_\pm + \mathrm{i}\Lambda A_\mp + \mathrm{i}(\alpha_1|A_\pm|^2+\alpha_2|A_\mp|^2)A_\pm \\ &\quad + \mathrm{i}\Upsilon\bar{A}_\mp \mp \mathrm{i}\gamma v_1 A_\pm, \qquad (4.39) \\ v'_1(\tau) &= \varepsilon(-v_1+|A_+|^2-|A_-|^2), \qquad (4.40) \end{aligned}$$

where $\varepsilon = -\lambda Re^{-1} > 0$, and $\lambda < 0$ is the first purely azimuthal hydrodynamic eigenvalue, cf. Section 3.3.

These equations possess reflection-symmetric steady states (corresponding to a pure standing wave) of the form $(A_+, A_-, v_1) = (A, A, 0)$, where A satisfies

$$[1+\mathrm{i}(\Gamma-\Lambda)-\mathrm{i}(\alpha_1+\alpha_2)|A|^2]A = \mathrm{i}\Upsilon\bar{A}, \qquad \alpha_1+\alpha_2 \neq 0, \qquad (4.41)$$

as well as nonsymmetric steady states. The stability properties of both types of steady states can be obtained in closed form, although the analysis of the latter is somewhat tedious. The phase of A can be eliminated in (4.41) to obtain

$$1+[\Gamma-\Lambda-(\alpha_1+\alpha_2)|A|^2]^2 = \Upsilon^2; \qquad (4.42)$$

thus the instability threshold for the standing waves is given by

$$\Upsilon = \Upsilon_c \equiv [1+(\Gamma-\Lambda)^2]^{1/2}. \qquad (4.43)$$

The amplitude $|A|$ increases monotonically for $\Upsilon > \Upsilon_c$ provided $(\Gamma-\Lambda)/(\alpha_1+\alpha_2) \leq 0$; if $(\Gamma-\Lambda)/(\alpha_1+\alpha_2) > 0$, the branch bifurcates subcritically at $\Upsilon = \Upsilon_c$ before turning around towards larger Υ at a secondary saddle-node bifurcation. The linear stability properties of these states can be deduced immediately from Section 3.3 on noticing that, in terms of the new variables $\hat{A}_\pm$ defined by

$$\hat{A}_+ = \mathrm{i}(A_+ - A_-)/2, \qquad \hat{A}_- = (A_+ + A_-)/2, \qquad \hat{v}_1 = -v_1/2, \qquad (4.44)$$

equations (4.39)–(4.40) become

$$\begin{aligned} \hat{A}'_\pm(\tau) &= -[1+\mathrm{i}(\Gamma\pm\Lambda)]\hat{A}_\pm + \mathrm{i}[(\alpha_1+\alpha_2)|\hat{A}_\pm|^2 + 2\alpha_1|\hat{A}_\mp|^2]\hat{A}_\pm \\ &\quad - \mathrm{i}(\alpha_1-\alpha_2)\bar{\hat{A}}_\pm\hat{A}_\mp^2 + \mathrm{i}\Upsilon\bar{\hat{A}}_\pm \mp 2\gamma\hat{v}_1\hat{A}_\mp, \qquad (4.45) \\ \hat{v}'_1(\tau) &= \varepsilon[-\hat{v}_1 + \mathrm{i}(\bar{\hat{A}}_+\hat{A}_- - \hat{A}_+\bar{\hat{A}}_-)], \qquad (4.46) \end{aligned}$$

which coincide with equations (3.24)–(3.25). This is a consequence of the fact that the chosen domain perturbation preserves a plane of reflection symmetry (see comment at the end of §4.1). Under this change of variables, the symmetric standing wave $(A_+, A_-) = (A, A)$ transforms into a pure mode $(\hat{A}_+, \hat{A}_-) = (0, A)$. It follows that the dispersion relations for the standing waves (A, A) are given by (3.31)–(3.32) using the transformation

$$\alpha_1 \to \alpha_1 + \alpha_2, \qquad \alpha_2 \to 2\alpha_1, \qquad \alpha_3 \to \alpha_2 - \alpha_1, \qquad \Lambda \to -\Lambda, \qquad \gamma \to 2\gamma. \tag{4.47}$$

Thus

$$(\lambda + 1)^2 + [\Gamma - \Lambda - 2(\alpha_1 + \alpha_2)|A|^2]^2 = 1 + (\Gamma - \Lambda)^2, \tag{4.48}$$

$$\lambda^2 + 2\lambda + 4\Lambda\Gamma - 8\Lambda[\gamma\varepsilon/(\lambda + \varepsilon) + \alpha_1]|A|^2 = 0. \tag{4.49}$$

Once again, the former dispersion relation is associated with reflection-symmetric (i.e., standing wave) perturbations, and the latter with symmetry-breaking perturbations. We summarize here the results obtained from (3.33)–(3.39) using (4.47).

The two steady state bifurcations, the saddle-node bifurcation involving reflection-symmetric perturbations and the symmetry-breaking bifurcation in (4.49), occur at

$$|A|^2 = (\Gamma - \Lambda)/(\alpha_1 + \alpha_2), \tag{4.50}$$

$$|A|^2 = \Gamma/[2(\alpha_1 + \gamma)], \quad \text{if } \Gamma\Lambda \neq 0, \tag{4.51}$$

respectively. Note that the symmetry-breaking bifurcation does *not* occur in a perfectly circular domain. This is so also for the symmetry-breaking Hopf bifurcation. This bifurcation produces a kind of blinking wave [79], [80], and occurs at

$$|A|^2 = (4\Gamma\Lambda + 2\varepsilon + \varepsilon^2)/[4\Lambda(2\alpha_1 - \varepsilon\gamma)] > 0. \tag{4.52}$$

The corresponding eigenvalues $\lambda = \pm i\lambda_I$ are given by

$$\lambda_I^2 = -\varepsilon^2 - 4\varepsilon\gamma\Lambda|A|^2 > 0, \tag{4.53}$$

implying that the presence of this Hopf bifurcation requires that

$$\gamma\Lambda < 0. \tag{4.54}$$

Such a symmetry-breaking Hopf bifurcation cannot therefore occur without the coupling to the streaming flow. The various codimension-two degeneracies identified in Section 3.3 are still present: The Takens-Bogdanov bifurcation occurs when (4.54) holds and

$$(\gamma + \alpha_1)\varepsilon + 2\gamma\Gamma\Lambda = 0; \tag{4.55}$$

and the saddle-node–symmetry-breaking and the saddle-node–Hopf bifurcations occur, respectively, at

$$(\alpha_1 - \alpha_2 + 2\gamma)\Gamma = 2(\gamma + \alpha_1)\Lambda, \tag{4.56}$$

$$(4\Gamma\Lambda + 2\varepsilon + \varepsilon^2)/[4\Lambda(2\alpha_1 - \varepsilon\gamma)] - (\Gamma - \Lambda)/(\alpha_1 + \alpha_2) = 0. \tag{4.57}$$

The first two of these bifurcations contain within their unfolding periodic orbits that correspond to quasi-periodic Faraday waves, both of which will be asymmetric. The third case contains symmetric quasi-periodic solutions in its unfolding that once again may lead to chaos.

5. Mode-Mode Interaction in Circular Containers

We now consider the interaction between two pairs of nonaxisymmetric surface modes in a circular container, as in Ciliberto and Gollub's experiment [45]. To obtain such an interaction we select appropriately the driving frequency and amplitude. Theoretical studies of such mode interactions include those based on amplitude equations for nearly inviscid flows but without the inclusion of streaming flow [86], [87], [88] and generic studies based on the O(2) symmetry of the system [62]; for a comparison and critique of these approaches, see [62] and the comment by Miles [51]. In this section we retain the exact O(2) symmetry of the system and focus on the role of the streaming flow generated by the mode interaction. We derive first (in §5.1) the rescaled CASF equations, and then analyze the surface wave-streaming flow coupling (§5.2), comparing the results with previous approaches in Section 5.3. In Section 5.4 we comment on the dynamics near the bicritical point and in Section 5.5 we present a simplified model based on a Galerkin truncation of the streaming flow.

5.1. The Scaled CASF Equations

We formulate the problem as in Section 4, and consider the linearly independent modes

$$\begin{aligned}
(\boldsymbol{V}_1, P_1, F_1) &= (\mathrm{i}U_1\boldsymbol{e}_r + V_1\boldsymbol{e}_\theta + \mathrm{i}W_1\boldsymbol{e}_z, Q_1, \Psi_1)\mathrm{e}^{\mathrm{i}n\theta},\\
(\boldsymbol{V}_2, P_2, F_2) &= (-\bar{\boldsymbol{V}}_1, \bar{P}_1, \bar{F}_1),\\
(\boldsymbol{V}_3, P_3, F_3) &= (\mathrm{i}U_3\boldsymbol{e}_r + V_3\boldsymbol{e}_\theta + \mathrm{i}W_3\boldsymbol{e}_z, Q_3, \Psi_3)\mathrm{e}^{\mathrm{i}m\theta},\\
(\boldsymbol{V}_4, P_4, F_4) &= (-\bar{\boldsymbol{V}}_3, \bar{P}_3, \bar{F}_3),
\end{aligned} \tag{5.1}$$

where, for $j = 1$ and 3, the functions U_j, V_j, W_j, Q_j, and Ψ_j are real and independent of θ, and the azimuthal wavenumbers are such that $1 \le m < n$. Thus these modes correspond to two pairs of counter-rotating surface waves of the system. With this selection, according to (2.23)–(2.26) and (2.38), we have

$$\begin{aligned}
\boldsymbol{g}_{12} &\equiv \boldsymbol{g}_{21} \equiv \boldsymbol{g}_{34} \equiv \boldsymbol{g}_{43} \equiv 0,\\
\boldsymbol{g}_{11} &\equiv -\boldsymbol{g}_{22} \equiv \mathrm{i}\Omega^{-1}\boldsymbol{\nabla} \times (\bar{\boldsymbol{V}}_1 \times \boldsymbol{V}_1) \equiv \boldsymbol{g}_1,\\
\boldsymbol{g}_{33} &\equiv -\boldsymbol{g}_{44} \equiv \mathrm{i}\Omega^{-1}\boldsymbol{\nabla} \times (\bar{\boldsymbol{V}}_3 \times \boldsymbol{V}_3) \equiv \boldsymbol{g}_2,\\
\boldsymbol{g}_{13} &\equiv -\boldsymbol{g}_{42} \equiv \mathrm{i}\Omega^{-1}\boldsymbol{\nabla} \times (\bar{\boldsymbol{V}}_1 \times \boldsymbol{V}_3) \equiv \boldsymbol{g}_{3+}\mathrm{e}^{\mathrm{i}(n-m)\theta},\\
\boldsymbol{g}_{31} &\equiv -\boldsymbol{g}_{24} \equiv \bar{\boldsymbol{g}}_{13} \equiv \boldsymbol{g}_{3-}\mathrm{e}^{-\mathrm{i}(n-m)\theta},\\
\boldsymbol{g}_{41} &\equiv -\boldsymbol{g}_{23} \equiv \mathrm{i}\Omega^{-1}\boldsymbol{\nabla} \times (\bar{\boldsymbol{V}}_4 \times \boldsymbol{V}_1) \equiv \boldsymbol{g}_{4+}\mathrm{e}^{\mathrm{i}(m+n)\theta},\\
\boldsymbol{g}_{14} &\equiv -\boldsymbol{g}_{32} \equiv \bar{\boldsymbol{g}}_{41} \equiv \boldsymbol{g}_{4-}\mathrm{e}^{-\mathrm{i}(m-n)\theta},\\
\boldsymbol{h}_{12} &\equiv \boldsymbol{h}_{21} \equiv \boldsymbol{h}_{34} \equiv \boldsymbol{h}_{43} \equiv \boldsymbol{h}_{11} \equiv \boldsymbol{h}_{22} \equiv \boldsymbol{h}_{33} \equiv \boldsymbol{h}_{44} \equiv 0,\\
\boldsymbol{h}_{13} &\equiv -\boldsymbol{h}_{42} \equiv \mathrm{i}\boldsymbol{\nabla} H_{13} \equiv \boldsymbol{h}_{3+}\mathrm{e}^{\mathrm{i}(n-m)\theta}, \qquad \boldsymbol{h}_{31} \equiv -\boldsymbol{h}_{24} \equiv \bar{\boldsymbol{h}}_{13} \equiv \boldsymbol{h}_{3-}\mathrm{e}^{-\mathrm{i}(n-m)\theta},\\
\boldsymbol{h}_{41} &\equiv -\boldsymbol{h}_{23} \equiv \mathrm{i}\boldsymbol{\nabla} H_{41} \equiv \boldsymbol{h}_{4+}\mathrm{e}^{\mathrm{i}(m+n)\theta}, \qquad \boldsymbol{h}_{14} \equiv -\boldsymbol{h}_{32} \equiv \bar{\boldsymbol{h}}_{41} \equiv \boldsymbol{h}_{4-}\mathrm{e}^{-\mathrm{i}(m+n)\theta},
\end{aligned} \tag{5.2}$$

where H_{13} and H_{41} are given by (2.24)–(2.26). The vector functions $\boldsymbol{g}_1$, $\boldsymbol{g}_2$, $\boldsymbol{g}_{3\pm}$, $\boldsymbol{g}_{4\pm}$, $\boldsymbol{h}_{3\pm}$, and $\boldsymbol{h}_{4\pm}$ are independent of θ and take the form

$$\begin{aligned}
\boldsymbol{g}_1 &= g_1\boldsymbol{e}_\theta, \qquad \boldsymbol{g}_2 = g_2\boldsymbol{e}_\theta, \qquad \boldsymbol{g}_{3\pm} = \pm \mathrm{i} g_3^1\boldsymbol{e}_r + g_3^2\boldsymbol{e}_\theta \pm \mathrm{i} g_3^3\boldsymbol{e}_z,\\
\boldsymbol{g}_{4\pm} &= \pm \mathrm{i} g_4^1\boldsymbol{e}_r + g_4^2\boldsymbol{e}_\theta \pm \mathrm{i} g_4^3\boldsymbol{e}_z,\\
\boldsymbol{h}_{3\pm} &= \pm \mathrm{i} h_3^1\boldsymbol{e}_r + h_3^2\boldsymbol{e}_\theta \pm \mathrm{i} h_3^3\boldsymbol{e}_z, \qquad \boldsymbol{h}_{4\pm} = \pm \mathrm{i} h_4^1\boldsymbol{e}_r + h_4^2\boldsymbol{e}_\theta \pm \mathrm{i} h_4^3\boldsymbol{e}_z,
\end{aligned} \tag{5.3}$$

for some real scalar functions g_1, g_2, g_3^j, g_4^j, h_3^j, and h_4^j.

Proceeding as in Section 4.1, we require that the amplitude equations be invariant under rotations and reflection and conclude that the viscous damping-detuning coefficients must be such that

$$d_1 = d_2 = \gamma_1^1(1+\mathrm{i})C_g^{1/2} + \gamma_1^2 C_g, \qquad d_3 = d_4 = \gamma_2^1(1+\mathrm{i})C_g^{1/2} + \gamma_2^2 C_g, \tag{5.4}$$

and that all the coefficients accounting for cubic nonlinearity, forcing, and departure from the mode interaction are zero, except for

$$\begin{aligned}
&\alpha_{1111} = \alpha_{2222} \equiv \alpha_0, \qquad \alpha_{1221} = \alpha_{2112} \equiv \alpha_1,\\
&\alpha_{1331} = \alpha_{2442} \equiv \alpha_2, \qquad \alpha_{1441} = \alpha_{2332} \equiv \alpha_3,\\
&\alpha_{1234} = \alpha_{2143} \equiv \alpha_4, \qquad \alpha_{3333} = \alpha_{4444} \equiv \alpha_5,\\
&\alpha_{3443} = \alpha_{4334} \equiv \alpha_6, \qquad \alpha_{3113} = \alpha_{4224} \equiv \alpha_7,\\
&\alpha_{3223} = \alpha_{4114} \equiv \alpha_8, \qquad \alpha_{3412} = \alpha_{4321} \equiv \alpha_9,\\
&\alpha_{12} = \alpha_{21}, \qquad \alpha_{34} = \alpha_{43}, \qquad \beta_{111} = \beta_{222} = 1.
\end{aligned} \tag{5.5}$$

In addition, we introduce the rescaling

$$\begin{aligned}
&t = \tau/\delta_1, \qquad [(\gamma_1^1 + \gamma_2^1)C_g^{1/2} - (\Omega_1 + \Omega_2)]/2 + \omega - \Omega = \delta_1\Gamma,\\
&[(\gamma_1^1 - \gamma_2^1)C_g^{1/2} + \Omega_2 - \Omega_1]/2 = \delta_1\Lambda,\\
&\mu = \delta_1\Upsilon/\alpha_{12}, \qquad \beta_1 = \delta_2/\delta_1, \qquad \beta_2 = \alpha_{34}/\alpha_{12}, \qquad A_{1,2} = \delta_1^{1/2}A_\pm,\\
&A_{3,4} = \delta_1^{1/2}B_\pm, \qquad \boldsymbol{u}^s = \delta_1\boldsymbol{u}, \qquad \hat{p}^s = \delta_1^2 p,
\end{aligned} \tag{5.6}$$

where

$$\delta_1 = \gamma_1^1 C_g^{1/2} + \gamma_1^2 C_g, \qquad \delta_2 = \gamma_2^1 C_g^{1/2} + \gamma_2^2 C_g, \tag{5.7}$$

and rewrite equations (2.3)–(2.4) as

$$\begin{aligned}
A'_\pm(\tau) &= -(1 + \mathrm{i}\Gamma + \mathrm{i}\Lambda)A_\pm + \mathrm{i}(\alpha_0|A_\pm|^2 + \alpha_1|A_\mp|^2 + \alpha_2|B_\pm|^2 + \alpha_3|B_\mp|^2)A_\pm\\
&\quad + \mathrm{i}\alpha_4\bar{A}_\mp B_\mp B_\pm + \mathrm{i}\Upsilon\bar{A}_\mp\\
&\quad \mp \mathrm{i}\Omega\int_{-1}^{0}\int_0^{2\pi}\int_0^R [A_\pm\boldsymbol{g}_1 + B_\pm \mathrm{e}^{\pm\mathrm{i}(n-m)\theta}\boldsymbol{g}_{3\pm} + B_\mp \mathrm{e}^{\mp\mathrm{i}(m+n)\theta}\boldsymbol{g}_{4\mp}]\\
&\quad \cdot \boldsymbol{u}\, r\, dr\, d\theta\, dz,
\end{aligned} \tag{5.8}$$

$$
\begin{aligned}
B'_\pm(\tau) \;=\; & -(\beta_1 + \mathrm{i}\Gamma - \mathrm{i}\Lambda)B_\pm + \mathrm{i}(\alpha_5|B_\pm|^2 + \alpha_6|B_\mp|^2 + \alpha_7|A_\pm|^2 + \alpha_8|A_\mp|^2)B_\pm \\
& + \mathrm{i}\alpha_9 \bar{B}_\mp A_\pm A_\pm + \mathrm{i}\beta_2 \Upsilon \bar{B}_\mp \\
& \mp \mathrm{i}\Omega \int_{-1}^{0}\int_{0}^{2\pi}\int_{0}^{R} [B_\pm \boldsymbol{g}_2 + A_\pm \mathrm{e}^{\mp\mathrm{i}(n-m)\theta}\boldsymbol{g}_{3\mp} - A_\mp \mathrm{e}^{\mp\mathrm{i}(m-n)\theta}\boldsymbol{g}_{4\mp}] \\
& \cdot \boldsymbol{u}\, r\,dr\,d\theta\,dz, \qquad (5.9)
\end{aligned}
$$

$$
\begin{aligned}
\nabla \cdot \boldsymbol{u} \;=\; & 0, \\
\partial \boldsymbol{u}/\partial\tau - [\boldsymbol{u} - \boldsymbol{H}(A_\pm, B_\pm) + \boldsymbol{G}(A_\pm, B_\pm)] \times (\nabla \times \boldsymbol{u}) = & -\nabla p + Re^{-1}\Delta\boldsymbol{u}, \qquad (5.10)
\end{aligned}
$$

where the inviscid mean flow velocity $\boldsymbol{H}$, the Stokes drift $\boldsymbol{G}$, and the effective Reynolds number Re are given by

$$
\boldsymbol{H} \;=\; (A_-\bar{B}_- - \bar{A}_-B_+)\mathrm{e}^{\mathrm{i}(n-m)\theta}\boldsymbol{h}_{3-} + (\bar{A}_-B_- - A_+\bar{B}_-)\mathrm{e}^{\mathrm{i}(m+n)\theta}\boldsymbol{h}_{4+} + \text{c.c.}, \qquad (5.11)
$$

$$
\begin{aligned}
\boldsymbol{G} \;=\; & (|A_-|^2 - |A_+|^2)\boldsymbol{g}_1 + (|B_-|^2 - |B_+|^2)\boldsymbol{g}_2 \\
& + [(A_-\bar{B}_- - \bar{A}_+B_-)\mathrm{e}^{\mathrm{i}(n-m)\theta}\boldsymbol{g}_{3+} + (\bar{A}_-B_- - A_+\bar{B}_-)\mathrm{e}^{\mathrm{i}(m+n)\theta}\boldsymbol{g}_{4+} \\
& + \text{c.c.}], \qquad (5.12)
\end{aligned}
$$

and

$$
Re = (\gamma_1^1 C_g^{1/2} + \gamma_1^2 C_g)/C_g. \qquad (5.13)
$$

Moreover, in view of the O(2) symmetry of the problem and the properties c and d in Section 2.2.2, the boundary conditions (2.5)–(2.6) may be written as

$$
\begin{aligned}
\boldsymbol{u} \;=\; & (\bar{A}_+ B_- \phi_1^1 + A_- \bar{B}_+ \phi_1^2)\mathrm{e}^{\mathrm{i}(n-m)\theta} \\
& + (\bar{A}_-B_-\phi_2^1 + A_+\bar{B}_-\phi_2^2)\mathrm{e}^{\mathrm{i}(n-m)\theta} \\
& + \phi_3^1 A_+\bar{A}_-\mathrm{e}^{2\mathrm{i}m\theta} + \phi_4^1 B_+\bar{B}_-\mathrm{e}^{2\mathrm{i}n\theta} + \text{c.c.} + (|A_+|^2 + |A_-|^2)\boldsymbol{\varphi}_1^1 \\
& + (|A_-|^2 - |A_-|^2)\boldsymbol{\varphi}_1^2 \\
& + (|B_+|^2 + |B_-|^2)\boldsymbol{\varphi}_2^1 + (|B_-|^2 - |B_-|^2)\boldsymbol{\varphi}_2^2, \\
& \text{if either } r = R \text{ or } z = -1, \qquad (5.14)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{u}\cdot\boldsymbol{e}_z \;=\; & 0, \\
\partial\tilde{\boldsymbol{u}}/\partial z \;=\; & (A_-\bar{B}_- - \bar{A}_+B_-)\phi_3^2\mathrm{e}^{\mathrm{i}(n-m)\theta} + (\bar{A}_-B_+ - A_+\bar{B}_-)\phi_4^2\mathrm{e}^{\mathrm{i}(n-m)\theta} + \text{c.c.} \\
& + (|A_-|^2 - |A_+|^2)\boldsymbol{\varphi}_3^2 + (|B_-|^2 - |B_+|^2)\boldsymbol{\varphi}_4^2, \quad \text{on } z = 0. \qquad (5.15)
\end{aligned}
$$

Here $\tilde{\boldsymbol{u}}$ is again the horizontal projection of $\boldsymbol{u}$, and

$$
\begin{aligned}
\boldsymbol{\varphi}_j^1 \;=\; & \varphi_j^1 \boldsymbol{n}\times\boldsymbol{e}_\theta, \qquad \boldsymbol{\varphi}_j^2 = \varphi_j^2\boldsymbol{e}_\theta, \qquad \boldsymbol{\phi}_j^1 = \phi_{j1}^1\boldsymbol{n}_0\times\boldsymbol{e}_\theta + \mathrm{i}\phi_{j2}^1\boldsymbol{e}_\theta, \\
\boldsymbol{\phi}_j^2 \;=\; & \mathrm{i}\phi_{j1}^2\boldsymbol{n}_0\times\boldsymbol{e}_\theta + \phi_{j2}^2\boldsymbol{e}_\theta, \qquad (5.16)
\end{aligned}
$$

where the functions φ_j^k and ϕ_{jl}^k are real and independent of θ, and $\boldsymbol{n}_0$ is the outward unit normal (to the solid boundary or the unperturbed free boundary). Once again (5.15) follows from the requirement that the surface shear vanishes for quasi-standing waves. Here such waves take the form $A_+V_1 + A_-V_2 + B_+V_3 + B_-V_4$ provided their phase is independent of position for all τ, i.e., provided

$$|A_+| - |A_-| = |B_+| - |B_-| = \bar{A}_+ B_+ - A_- \bar{B}_- = 0. \tag{5.17}$$

These conditions also imply that $\bar{A}_- B_+ - A_+ \bar{B}_- = 0$.

5.2. *The Influence of O(2) Symmetry on the Coupling to the Streaming Flow*

Motivated by the experiment of Ciliberto and Gollub [44], [45] with $m = 4$, $n = 7$, we suppose in the following that m and n are relatively prime. The symmetry O(2) acts on the CASF equations (5.8)–(5.10), (5.14)–(5.15) by

$$\theta \to \theta + \phi: \quad A_\pm \to e^{\mp \mathrm{i} m\phi} A_\pm, \qquad B_\pm \to e^{\mp \mathrm{i} n\phi} B_\pm, \tag{5.18}$$

$$\theta \to -\theta: \quad A_+ \leftrightarrow A_-, \qquad B_+ \leftrightarrow B_-, \qquad \boldsymbol{u} \cdot \boldsymbol{e}_\theta \to -\boldsymbol{u} \cdot \boldsymbol{e}_\theta. \tag{5.19}$$

The properties of being reflection-symmetric and being quasi-standing are now two independent properties of the solutions of the CASF equations, and on its own neither implies that the amplitude decouples from the streaming flow. However, the two together imply both (5.17) and

$$A_+^n B_-^m = A_-^n A_-^m. \tag{5.20}$$

Since more conditions are required for decoupling than in the previous cases (treated in §3 and §4), we expect the influence of the streaming flow on the surface wave dynamics to be more visible.

5.3. *Comparison with Previous Theoretical Approaches*

If the streaming flow is ignored, as is usually done in the literature, the resulting equations

$$\begin{aligned} A'_\pm(\tau) &= -[1 + \mathrm{i}(\Gamma + \Lambda - \alpha_0|A_\pm|^2 - \alpha_1|A_\mp|^2 - \alpha_2|B_\pm|^2 - \alpha_3|B_\mp|^2)]A_\pm \\ &\quad + \mathrm{i}(\alpha_4 B_\pm B_\mp + \Upsilon)\bar{A}_\mp, \end{aligned} \tag{5.21}$$

$$\begin{aligned} B'_\pm(\tau) &= -[\beta_1 + \mathrm{i}(\Gamma - \Lambda - \alpha_5|B_\pm|^2 - \alpha_6|B_\mp|^2 - \alpha_7|A_\pm|^2 - \alpha_8|A_\mp|^2)]B_\pm \\ &\quad + \mathrm{i}(\alpha_9 A_\pm A_\mp + \beta_2\Upsilon)\bar{B}_\mp, \end{aligned} \tag{5.22}$$

are degenerate because they admit two new symmetries (in addition to the O(2) actions (5.18)–(5.19)) that are not present in the original equations. Namely, equations (5.21)–(5.22) are invariant under the following *four* independent actions:

$$\begin{aligned} &A_\pm \to e^{\mp \mathrm{i} m\phi_1} A_\pm; \qquad B_\pm \to e^{\mp \mathrm{i} n\phi_2} B_\pm; \\ &A_+ \leftrightarrow A_-, \quad \boldsymbol{u} \cdot \boldsymbol{e}_\theta \to -\boldsymbol{u} \cdot \boldsymbol{e}_\theta; \qquad B_+ \leftrightarrow B_-, \quad \boldsymbol{u} \cdot \boldsymbol{e}_\theta \to -\boldsymbol{u} \cdot \boldsymbol{e}_\theta, \end{aligned} \tag{5.23}$$

which generate the larger group O(2)×O(2). This additional symmetry is an artifact of the truncation of the amplitude equations at third order. In [62] it is shown that if the amplitude equations are computed to a sufficiently high order ($m + n - 1 = 10$ in the Ciliberto-Gollub experiment), the O(2) symmetry of the original system is restored. However, if this is not done, equations (5.21)–(5.22) predict that

$$dM_A/d\tau = -2M_A, \qquad dM_B/d\tau = -2\beta_1 M_B, \tag{5.24}$$

where $M_A = |A_+|^2 - |A_-|^2$ and $M_B = |B_+|^2 - |B_-|^2$ are (proportional to) the angular momenta of the Stokes drifts associated with each pair of modes separately (see the comment in §2.3.2), and both M_A and M_B vanish exponentially as $\tau \to \infty$. Thus at large times $M_A = M_B = 0$, and from (5.21)–(5.22) we obtain that

$$\text{phase of } A_-\bar{A}_- = \text{constant}, \qquad \text{phase of } B_-\bar{B}_- = \text{constant}. \tag{5.25}$$

In this case the spatial phases of the two pairs of modes, see (2.1) and (5.1), can be fixed arbitrarily and we may write $A_+ = e^{2i\phi_1}A_-$ and $B_+ = e^{2i\phi_2}B_-$ for some constants ϕ_1 and ϕ_2; i.e., the system (5.21)–(5.22) reduces to two complex amplitude equations, as noted in [45], [87], [88]. However, these equations fail to reproduce essential features of the experimental bifurcation diagram. For instance, center manifold reduction at the bicritical point (see §5.4 below) yields a two-dimensional system, suggesting that chaos is not possible in the vicinity of this point, contrary to observations [45]. In fact, as shown by Crawford et al. [62], higher order terms in the amplitude equations reinstate the coupling between the mode amplitudes and a certain phase difference, leading to a center manifold description of the dynamics that is of third order. However, the analysis in [62] was based only on the symmetry properties of the system, and thus implicitly assumed that viscosity is large enough that any mean flows are slaved to the slow dynamics of the complex mode amplitudes near the bicritical point. In the present case this is not so, and our approach shows that the inclusion of the streaming flow when $C_g \ll 1$ lifts the degeneracy of the truncated amplitude equations and restores the original symmetry of the problem. Specifically, with the streaming flow included, equation (5.24a) is replaced by

$$\begin{aligned} dM_A/d\tau + 2M_A = {} & -\Omega \int_{-1}^{0}\int_0^{2\pi}\int_0^R [\mathrm{i}(\bar{A}_-B_+ - A_-\bar{B}_-)\boldsymbol{g}_{3-}\cdot\boldsymbol{u}e^{\mathrm{i}(n-m)\theta} \\ & \qquad + \text{c.c.}]r\,dr\,d\theta\,dz \\ & -\Omega \int_{-1}^{0}\int_0^{2\pi}\int_0^R [\mathrm{i}(\bar{A}_+B_- - A_-\bar{B}_-)\boldsymbol{g}_{4-}\cdot\boldsymbol{u}e^{\mathrm{i}(n+m)\theta} \\ & \qquad + \text{c.c.}]r\,dr\,d\theta\,dz, \end{aligned} \tag{5.26}$$

where $\boldsymbol{g}_{3+}$ and $\boldsymbol{g}_{4-}$ are as in (5.3); a similar expression obtains for the evolution of M_B. Thus the angular momenta M_A and M_B no longer vanish individually at large times (except of course in some particular cases, see below) and the spatial phases of the modes are no longer constant. Moreover:

A. M_A and M_B vanish at large times when the right-hand side of (5.26) (and of its counterpart for M_B) vanishes identically; this occurs for solutions that either are reflection-symmetric or satisfy $\bar{A}_+B_+ - A_-\bar{B}_- = \bar{A}_-B_- - A_-\bar{B}_+ = 0$.

B. Some obvious simplifications of the streaming flow equations yield a system of simplified equations that suffer from the same spurious symmetries (5.23) and thus are no better than (5.21)–(5.22). This happens, for instance, in the limit $Re \to 0$ in (5.10b), which is not realistic when $C_g \to 0$; see (5.13). In this case the streaming flow is slaved to the surface waves (much as the inviscid mean flow, see §2.1C) and its only effect is to change the values of the coefficients of the cubic terms in (5.21)–(5.23), without introducing new terms.

5.4. Dynamics of the CASF Equations near the Bicritical Point

If no simplifications are made, the instability thresholds from the flat state to surface waves consisting of pure modes are given by

$$1 + (\Gamma + \Lambda)^2 = \Upsilon^2 \quad \text{and} \quad \beta_1^2 + (\Gamma - \Lambda)^2 = \beta_2^2 \Upsilon^2. \tag{5.27}$$

For fixed $\Lambda \neq 0$, these yield two hyperbolas in the Υ-Γ plane, which intersect at the so-called bicritical point. The bifurcating families of pure modes are given (modulo rotations) by

$$A_+ = A_- = A, \qquad B_+ = B_- = 0, \quad \text{and} \quad A_- = A_- = 0, \qquad B_+ = B_- = B, \tag{5.28}$$

respectively. The corresponding amplitudes $|A|$ and $|B|$ are given by

$$1 + [\Gamma + \Lambda - (\alpha_0 + \alpha_1)|A|^2]^2 = \Upsilon^2 \quad \text{and} \quad \beta_1^2 + [\Gamma - \Lambda - (\alpha_5 + \alpha_6)|B|^2]^2 = \beta_2^2 \Upsilon^2, \tag{5.29}$$

respectively. Since both pure modes are standing they are decoupled from the streaming flow. The center manifolds at threshold, away from the bicritical point, are two-dimensional, but one degree of freedom plays no dynamic role since it is associated with the neutrally stable spatial phase of the wave. Near the bicritical point (Υ_c, Γ_c) the center manifold is four-dimensional and CASF equations take the form

$$\begin{aligned} da/d\tau &= [\gamma_1(\Upsilon - \Upsilon_c) + \gamma_2(\Gamma - \Gamma_c) + \gamma_3|a|^2 + \gamma_4|b|^2]a + \cdots, \\ db/d\tau &= [\gamma_5(\Upsilon - \Upsilon_c) + \gamma_6(\Gamma_c - \Gamma) + \gamma_7|b|^2 + \gamma_8|a|^2]b + \cdots, \end{aligned}$$

where $\gamma_1, \ldots, \gamma_8$ are real coefficients and the complex amplitudes a and b are given by

$$a = (A_- \bar{A}_-)^{1/2}, \qquad b = (B_+ \bar{B}_-)^{1/2}. \tag{5.30}$$

Thus the amplitudes and phases of a and b are precisely the amplitudes and spatial phases of the two pure standing-wave modes. If these equations are truncated at third order, they exhibit spurious symmetries that again lead to a spurious reduction of dimension unless $(m+n-1)$-th order terms are included, i.e., at the bicritical point the streaming flow also becomes slaved to the surface waves—this is because of its nonzero damping. However, if the Reynolds number Re of the streaming flow is large, the center manifold reduction only applies in an extremely small neighborhood of the bicritical point, and on larger neighborhoods defined by $\Upsilon - \Upsilon_c \sim Re^{-1}$, $\Gamma - \Gamma_c \sim Re^{-1}$ the timescale for the evolution of the streaming flow (§2.3.5) becomes comparable to the slow evolution of the center manifold variables; in this regime, some of the viscous modes associated with the streaming flow can no longer be considered slaved to the surface waves and enter explicitly into the description of the dynamics. This new source of complexity could also be responsible for the chaotic dynamics near the bicritical point reported in [45].

5.5. *Three-Mode Approximation of the Streaming Flow*

In this section we only consider the simplest approximation to the streaming flow that does not permit spurious symmetries. A look at the coupling terms in (5.8)–(5.9) shows that we need to consider at least the following hydrodynamic modes: the first axisymmetric, purely azimuthal one, and the first nonaxisymmetric modes with azimuthal wavenumbers $m-n$ and $m+n$. Thus we write the streaming flow and the associated pressure as

$$(\boldsymbol{u}^s, p^s) = V_1(v_1(r)\boldsymbol{e}_\theta, 0) + [W_1^-(\boldsymbol{v}_1(r,z), p_1(r,z))\mathrm{e}^{\mathrm{i}(m-n)\theta} + W_2^+(\boldsymbol{v}_2(r,z), p_2(r,z))\mathrm{e}^{\mathrm{i}(m+n)\theta} + \text{c.c.}],$$

where V_1 is real. Projecting the streaming flow equations onto these modes and rescaling the results leads to the following system of simplified equations:

$$A'_\pm(\tau) = -(1+\mathrm{i}\Gamma+\mathrm{i}\Lambda)A_\pm + \mathrm{i}(\alpha_0|A_\pm|^2 + \alpha_1|A_\mp|^2 + \alpha_2|B_\pm|^2 + \alpha_3|B_\mp|^2)A_\pm + \mathrm{i}\alpha_4\bar{A}_\mp B_\pm B_\mp + \mathrm{i}\Upsilon\bar{A}_\mp \mp \mathrm{i}(V_1A_\pm + W_1^\mp B_\pm + W_2^\pm B_\mp), \quad (5.31)$$

$$B'_\pm(\tau) = -(\beta_1+\mathrm{i}\Gamma-\mathrm{i}\Lambda)B_\pm + \mathrm{i}(\alpha_5|B_\pm|^2 + \alpha_6|B_\mp|^2 + \alpha_7|A_\pm|^2 + \alpha_8|A_\mp|^2)B_\pm + \mathrm{i}\alpha_9\bar{B}_\mp A_\pm A_\mp + \mathrm{i}\beta_2\Upsilon\bar{B}_\mp \mp \mathrm{i}(\beta_3V_1B_\pm + W_1^\mp A_\pm - W_2^\pm A_\mp), \quad (5.32)$$

$$V_1'(\tau) = -\varepsilon[V_1 + \beta_4(|A_-|^2 - |A_+|^2) + \beta_5(|B_-|^2 - |B_+|^2)], \quad (5.33)$$

$$W_1^{-\prime} = -\varepsilon(\delta_1W_1^- + \kappa_1A_-\bar{B}_+ - \bar{\kappa}_1\bar{A}_+B_-), \qquad W_1^+ = \bar{W}_1^-, \quad (5.34)$$

$$W_2^{+\prime} = -\varepsilon(\delta_2W_2^+ + \kappa_2A_+\bar{B}_- - \bar{\kappa}_2\bar{A}_-B_+), \qquad W_2^- = \bar{W}_2^+, \quad (5.35)$$

where $\delta_1, \delta_2, \beta_1, \ldots, \beta_5$ are real, but κ_1, and κ_2 are generally complex. Note that if the forcing effect of the walls, described by the right-hand side of (5.14), is neglected, the streaming flow is only forced via the boundary condition (5.15), a fact consistent with setting to zero the imaginary parts of the coefficients κ_1 and κ_2 in (5.34)–(5.35). Note also that additional streaming flow modes forced by terms proportional to $A_\pm\bar{A}_\mp$, $B_\pm\bar{B}_\mp$, $|A_-|^2 + |A_+|^2$ and $|B_-|^2 + |B_+|^2$ and allowed by symmetry arguments are not included because they do not contribute to (5.31)–(5.32), an observation that follows from the form of the coupling terms in (5.8)–(5.9) and of the vector function $\boldsymbol{g}_1$; see (5.3). Likewise, the fact that the coefficients of $(W_1^\pm B_\mp, W_2^\pm B_\pm)$ in (5.31) coincide with those of $(W_1^\pm A_\pm, W_2^\pm A_\mp)$ in (5.32) follows from the form of the vector functions $\boldsymbol{g}_{3\pm}$ and $\boldsymbol{g}_{4\pm}$ (see (5.3)); note that the problem is not invariant under any transformation of the form $A_\pm \leftrightarrow B_\pm$.

If we let $A_+ = A_- = A$ and $B_+ = B_- = B$, then (after a transient) (5.31)–(5.35) become (cf. [87])

$$
\begin{aligned}
A'(\tau) &= -(1 + \mathrm{i}\Gamma + \mathrm{i}\Lambda)A + \mathrm{i}[(\alpha_0 + \alpha_1)|A|^2 + (\alpha_2 + \alpha_3)|B|^2]A + \mathrm{i}\alpha_4 \bar{A}B^2 \\
&\quad + \mathrm{i}\Upsilon \bar{A} - \mathrm{i}(W_1 + W_2)B, && (5.36) \\
B'(\tau) &= -(\beta_1 + \mathrm{i}\Gamma - \mathrm{i}\Lambda)B + \mathrm{i}[(\alpha_5 + \alpha_6)|B|^2 + (\alpha_7 + \alpha_8)|A|^2]B + \mathrm{i}\alpha_9 \bar{B}A^2 \\
&\quad + \mathrm{i}\beta_2\Upsilon \bar{B} - \mathrm{i}(W_1 - W_2)A, && (5.37) \\
W_1' &= -\varepsilon(\delta_1 W_1 + \kappa_1 A\bar{B} - \bar{\kappa}_1 \bar{A}B), && (5.38) \\
W_2' &= -\varepsilon(\delta_2 W_2 + \kappa_2 A\bar{B} - \bar{\kappa}_2 \bar{A}B), && (5.39)
\end{aligned}
$$

where $W_1 = W_1^{\mathrm{i}}$ and $W_2 = W_2^{\mathrm{i}}$ are purely imaginary. These equations contain as a particular case the simplified equations (3.24)–(3.25); in fact, they are the counterparts of these equations for mode interaction in a general rectangle, i.e., one that need not be close to a square. The resulting equations thus provide a convenient model of the Faraday system in rectangular containers (cf. [43]) that incorporates the effects of streaming flow.

6. Concluding Remarks

A general nearly inviscid, weakly nonlinear theory has been developed in Section 2 describing the interaction of N surface modes and the associated streaming flow in a vertically vibrated cylindrical container. The main result of the theory is a set of coupled amplitude-streaming flow (CASF) equations, summarized in (2.3)–(2.6). The amplitude equations (2.3) differ from the usual ones in the presence of terms that depend on weighted averages of the streaming flow velocity; thus only these terms have been explicitly derived (in §2.1). The streaming flow itself is governed by a continuity and a Navier-Stokes–like equation (2.4), both of which are similar to ones already used in existing studies of streaming flows but which are new in the present context; for this reason we have summarized their derivation as well (in §2.2.1). The boundary conditions (2.5)–(2.6) that drive these flows result from well-known forcing mechanisms, originally due to Schlichting and Longuet-Higgins, but again are new in the present context particularly since their derivation requires an analysis of three-dimensional oscillatory boundary layers. This analysis is well beyond the scope of the present paper and we have summarized the necessary results [59] in the Appendix. However, the "form" of the boundary conditions can be anticipated from general considerations. In Section 2.3 we have discussed the general properties of the CASF equations and their applicability to several outstanding experiments. We have emphasized that the excitation of streaming flow via finite amplitude instability provides an alternative saturation mechanism for the Faraday instability, and one that is particularly significant in the low viscosity limit in which the coefficients of the nonlinear terms are purely imaginary. Indeed, we have shown that *it is asymptotically inconsistent to retain cubic terms and neglect the streaming flow* as usually done in the literature, unless the state of the system has very specific symmetry properties. This observation remains true as the forcing frequency increases and, in particular, when the wavelength of the surface waves is small compared to the depth of the container (a frequent case in experiments). Since the Reynolds number associated with the streaming flow is never small, this flow is never slaved to the waves and hence is responsible for introducing qualitatively new ingredients into the dynamics of the system. We have explained these new ingredients in several cases (whose analysis was included for illustration) and noted that these could provide explanation for some striking behavior observed in Faraday experiments using low viscosity fluids.

We have used the CASF equations in several different contexts. In the first two we have explored the consequences of small changes in symmetry on the dynamics of Faraday waves. This idea is not new. In the Hamiltonian context it is well known that changes in symmetry can couple modes that would otherwise be uncoupled, thereby causing instability [52]. This is the idea behind the so-called elliptical instability. Likewise, Crawford [89] noted that the Faraday system with Neumann boundary conditions possesses several hidden symmetries and suggested an interesting experiment on Faraday waves in nonsquare containers that nonetheless possess D_4 symmetry [90]. The required change in the domain destroys these (unphysical) symmetries and permits new types of behavior. We have seen here that the inclusion of viscous effects has similar consequences. The boundary conditions are no longer Neumann, and if the D_4 symmetry is itself broken, coupling to streaming flow is enhanced. Specifically, our investigation in Section 3 of the mode-mode interaction in almost-square containers showed that streaming flow is always associated with the surface waves dynamics unless the state of the system possesses a reflection symmetry *for all time*; however, even these reflection-symmetric states may lose stability at finite amplitude to modes that break their symmetry and hence drive a streaming flow. In Section 3.3 we constructed a simple model to illustrate this phenomenon; we expect this model to be qualitatively valid when the streaming flow Reynolds number is not too large. A similar study of mode interactions in almost circular containers (in §4.2) showed that breaking of the invariance of the system under rotation is essential in order that the surface wave amplitudes couple to the streaming flow. For simplicity we retained a reflection symmetry when perturbing the shape of the container. We found, once again, that only states lacking reflection symmetry were accompanied by streaming flow, but that such flows could be excited in secondary instabilities of reflection-symmetric states. In the generic case in which the perturbed cross section has no reflection symmetry at all, all states of the system involve the streaming flow. The role of the streaming flow can be seen more clearly in the two simplified models constructed for low effective Reynolds number or in the high frequency limit. The simplest, one-mode approximation to the streaming flow, considered in Section 4.5, allowed us to examine analytically the different secondary instabilities of a reflection-symmetric state, and to classify the resulting dynamics. In particular we found that a symmetry-breaking Hopf bifurcation could only occur as a result of the coupling to the streaming flow, and we identified several codimension-two bifurcations involving this bifurcation. These could of course be responsible for much complex dynamics that would not occur in the absence of streaming flow. We hope that these predictions will stimulate experimental studies of this set-up. As a final example, we considered the interaction of two modes with distinct azimuthal wavenumbers, this time in a circular domain. In this case streaming flows are always excited unless the state of the system is both quasi-standing *and* reflection-symmetric. Thus such mode interactions are much more likely to generate streaming flows. We have found that inclusion of such flows avoids the spurious symmetries that are an artifact of a truncation of the amplitude equations at cubic order, and provides a much more realistic description of the system that does not have to rely on high order terms arising from spatial resonance [62].

Despite their complexity, the CASF equations provide a substantial simplification of the original equations (1.1)–(1.4): The oscillations on the fast timescale $t \sim 1$ have been filtered out, the effect of the viscous boundary layers has been replaced by effective boundary conditions on the flow in the bulk, and the motion of the free surface has been eliminated. Since direct numerical simulations of the full CASF equations are well beyond the scope of the present paper, we have resorted to investigating the properties of several model systems motivated by existing experiments, and have used these to suggest possible explanations for the discrepancy between the experiments and theories that omit streaming flows. In particular we emphasize that, in the nearly inviscid Faraday system, streaming flows enter into the theoretical description already at third order in the amplitude, and hence that their omission is inconsistent with the retention of other cubic terms. Indeed in many cases the streaming flow provides *the* saturation mechanism for the Faraday instability, particularly in multimode situations. We hope, therefore, that the present paper will stimulate both experimental and theoretical studies of the role of streaming flows in the nearly inviscid Faraday system.

Appendix A. The Boundary Conditions for the Mean Flow in the Bulk

These boundary conditions result from matching conditions between the solution in the bulk and in the oscillatory boundary layers attached to the solid boundary and free surface. The well-known formulae in the literature (first obtained by Schlichting [7] and Longuet-Higgins [8]) apply only to strictly two-dimensional problems, while the streaming flows considered in this paper are genuinely three-dimensional. The necessary results are derived in [59] and summarized here.

The appropriate boundary conditions at (the edge of the Stokes boundary layer attached to) a static solid wall Γ_S are given in terms of the mean flow velocity, $\boldsymbol{u}^m$ (=$\boldsymbol{u}^i + \boldsymbol{u}^s$= the inviscid plus the viscous mean flow velocities, with the notation in this paper), and are

$$\boldsymbol{u}^m \cdot \boldsymbol{n} = o(\varepsilon^2), \qquad \tilde{\boldsymbol{u}}^m = -\varepsilon^2 (2\Omega)^{-1} [(2+3\mathrm{i})(\tilde{\boldsymbol{\nabla}} \cdot \overline{\boldsymbol{V}})\boldsymbol{V} + (\overline{\boldsymbol{V}} \cdot \tilde{\boldsymbol{\nabla}})\boldsymbol{V} + \text{c.c.}] + \boldsymbol{o}(\varepsilon^2).$$

As in Section 2.2.2, $\tilde{\boldsymbol{\nabla}}\cdot$ and $\tilde{\boldsymbol{\nabla}}$ are *the intrinsic divergence and gradient operators along the solid boundary* Γ_S, $\boldsymbol{n}$ is the outward unit normal to Γ_S, and $\tilde{\boldsymbol{u}}^m$ is the tangential projection of $\boldsymbol{u}^m$ along Γ_S. The quantity ε is defined in terms of the *velocity of the outer inviscid flow* at Γ_S, assumed to be of the form

$$\boldsymbol{v} = \varepsilon(\boldsymbol{V} \exp(\mathrm{i}\Omega t) + \text{c.c.}) + \boldsymbol{o}(\varepsilon), \tag{A.1}$$

where $|\boldsymbol{V}| = O(1)$ as $\varepsilon \to 0$. Note that $\boldsymbol{V}$ is tangent to Γ_S and that $\tilde{\boldsymbol{u}}^s$ is independent of viscosity and of the curvature of Γ_S to leading order.

Similarly, the appropriate boundary conditions for the streaming flow to be imposed at a (horizontal) unperturbed free surface, $z = 0$, are

$$\boldsymbol{u}^m \cdot \boldsymbol{e}_z = \varepsilon^2[\tilde{\boldsymbol{\nabla}} \cdot (\bar{F}\tilde{\boldsymbol{V}}) + \text{c.c.}] + o(\varepsilon^2), \tag{A.2}$$

$$\partial\tilde{\boldsymbol{u}}^m/\partial z = \varepsilon^2[\tilde{\boldsymbol{\nabla}}(\tilde{\boldsymbol{\nabla}} \cdot (\bar{F}\tilde{\boldsymbol{V}})) + 2(\tilde{\boldsymbol{\nabla}}\bar{F} \cdot \tilde{\boldsymbol{\nabla}})\tilde{\boldsymbol{V}} + 2(\tilde{\boldsymbol{\nabla}} \cdot \tilde{\boldsymbol{V}})\tilde{\boldsymbol{\nabla}}\bar{F} + \text{c.c.}] + \boldsymbol{o}(\varepsilon^2). \tag{A.3}$$

Here, as in Section 2.2.2, $\tilde{\boldsymbol{\nabla}}\cdot$ and $\tilde{\boldsymbol{\nabla}}$ are *the horizontal divergence and gradient operators*, and $\tilde{\boldsymbol{u}}^m$ and $\tilde{\boldsymbol{V}}$ are the horizontal projections of $\boldsymbol{u}^m$ and $\boldsymbol{V}$, respectively, with $\boldsymbol{V}$ given by (A.1). The (oscillatory) deflection f of the free surface is taken to be

$$f = \varepsilon(F\mathrm{e}^{\mathrm{i}\Omega t} + \text{c.c.}) + o(\varepsilon).$$

Note that the right-hand sides of (A.2) and (A.3) are again independent of viscosity. In fact, for planar unperturbed free surfaces such as those considered in this paper, the 3-D oscillatory boundary layer problem had already been solved by Liu, in a relatively unknown paper [91].

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