Nearly inviscid Faraday waves in annular containers of moderately large aspect ratio

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Abstract

Nearly inviscid parametrically excited surface gravity–capillary waves in two-dimensional domains of finite depth and large aspect ratio are considered. Coupled equations describing the evolution of the amplitudes of resonant left- and right-traveling waves and their interaction with a mean flow in the bulk are derived, and the conditions for their validity established. Under suitable conditions the mean flow consists of an inviscid part together with a viscous mean flow driven by a tangential stress due to an oscillatory viscous boundary layer near the free surface and a tangential velocity due to a bottom boundary layer. These forcing mechanisms are important even in the limit of vanishing viscosity, and provide boundary conditions for the Navier–Stokes equation satisfied by the mean flow in the bulk. For moderately large aspect ratio domains the amplitude equations are nonlocal but decouple from the equations describing the interaction of the slow spatial phase and the viscous mean flow. Two cases are considered in detail, gravity–capillary waves and capillary waves in a microgravity environment.

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1. Introduction

The Faraday system, i.e., the study of surface gravity–capillary waves excited parametrically by the vertical oscillation of a container, has attracted a great deal of attention [1–4]. Despite this a number of issues remain outstanding. This is largely due to the fact that existing theory fails to provide a quantitative description of the experimental results in containers of large aspect ratio. One possible explanation, pursued by us in several papers [5,6] focuses on the fact that these theories include only the leading order effects of viscosity [7,8] despite the fact that for typical experimental parameter values this approach predicts an incorrect viscous dissipation time in the absence of parametric forcing. This is because the dissipation time for Faraday waves excited by typical oscillation frequencies is in fact dominated by dissipation in the bulk of the domain, and not in the boundary layers at solid walls as usually assumed. However, there is an additional important effect associated with the presence of viscosity that illustrates the singular nature of the required perturbation theory. This effect arises because the oscillatory viscous boundary layers at the free surface and the bottom of the container (as well as any lateral boundaries, if present) are capable of driving a large scale mean flow, hereafter a *viscous mean flow* or a *streaming flow*, due to a nonzero (time-averaged) Reynolds stress in these boundary layers. These flows have either been entirely ignored in the past or treated in an incomplete or inconsistent manner, but they are important because they can interact nontrivially with the surface waves responsible for them. This is so, for example, in systems of small to moderate aspect ratio provided at least two modes of oscillation are excited [9,10]. Large aspect ratio systems are yet more subtle because of the presence of an additional, *inviscid mean flow*. For inviscid free waves this mean flow is associated with spatial modulation of a single mode, as described by the celebrated Davey–Stewartson equations [11,12]. If viscosity is retained and the system forced, as in a shear flow, a similar set of equations but with complex coefficients can be derived [13]. In general the mean flow present will contain both types of contributions, even in nearly inviscid flows.

This paper is devoted to the derivation of the following equations governing the interaction between two parametrically excited counterpropagating wavetrains and the associated mean flow in a two-dimensional, annular Faraday system,

$$A_t - v_g A_x = \mathrm{i}\alpha A_{xx} - (\delta + \mathrm{i}d)A + \mathrm{i}(\alpha_3|A|^2 - \alpha_4|B|^2)A + \mathrm{i}\alpha_5\mu\bar{B} + \mathrm{i}\alpha_6\int_{-1}^{0} g(y)\langle\psi_y^{\mathrm{m}}\rangle^x \,\mathrm{d}y\, A \\ + \mathrm{i}\alpha_7\langle f^{\mathrm{m}}\rangle^x A + \mathrm{HOT}, \tag{1.1}$$

$$B_t + v_g B_x = \mathrm{i}\alpha B_{xx} - (\delta + \mathrm{i}d)B + \mathrm{i}(\alpha_3|B|^2 - \alpha_4|A|^2)B + \mathrm{i}\alpha_5\mu\bar{A} - \mathrm{i}\alpha_6\int_{-1}^{0} g(y)\langle\psi_y^{\mathrm{m}}\rangle^x \,\mathrm{d}y\, B \\ + \mathrm{i}\alpha_7\langle f^{\mathrm{m}}\rangle^x B + \mathrm{HOT}, \tag{1.2}$$

$$A(x+L,t) \equiv A(x,t), \qquad B(x+L,t) \equiv B(x,t), \tag{1.3}$$

together with the conditions under which these equations provide the correct description of Faraday waves in systems with reflection symmetry and one extended dimension. Here $L \gg 1$ is the aspect ratio of the system, measured in units of the layer depth. As part of the derivation explicit expressions for the coefficients are obtained. The complex amplitudes A and B are the amplitudes of the two counterpropagating waves driven parametrically by the forcing (with dimensionless amplitude μ), and the notation HOT indicates higher order terms. The first seven terms in these equations, accounting for inertia, propagation at the group velocity v_g, dispersion, damping, detuning, cubic nonlinearity and parametric forcing, are familiar from existing weakly nonlinear, nearly inviscid theories [14]. The last two terms account for coupling to the mean flow in the bulk (indicated by the superscript m) and are conservative. They are written in terms of (a local average $\langle\cdot\rangle^x$ of) the streamfunction ψ^{m} for the mean flow and the associated free surface elevation f^{m}. These quantities evolve according to the equations

$$\psi_{xx}^{\mathrm{m}} + \psi_{yy}^{\mathrm{m}} = \Omega^{\mathrm{m}}, \qquad \Omega_t^{\mathrm{m}} - [\psi_y^{\mathrm{m}} + (|A|^2 - |B|^2)g(y)]\Omega_x^{\mathrm{m}} + \psi_x^{\mathrm{m}}\Omega_y^{\mathrm{m}} = C_g(\Omega_{xx}^{\mathrm{m}} + \Omega_{yy}^{\mathrm{m}}) + \mathrm{HOT}, \tag{1.4}$$

$$\psi_x^{\mathrm{m}} - f_t^{\mathrm{m}} = \beta_1(|B|^2 - |A|^2)_x + \mathrm{HOT}, \qquad \psi_{yy}^{\mathrm{m}} = \beta_2(|A|^2 - |B|^2) + \mathrm{HOT} \quad \text{at } y = 0, \tag{1.5}$$

$$(1-S)f_x^{\mathrm{m}} - Sf_{xxx}^{\mathrm{m}} - \psi_{yt}^{\mathrm{m}} + C_g(\psi_{yyy}^{\mathrm{m}} + 3\psi_{xxy}^{\mathrm{m}}) = -\beta_3(|A|^2 + |B|^2)_x + \mathrm{HOT} \quad \text{at } y = 0, \tag{1.6}$$

$$\int_0^L \Omega_y^{\mathrm{m}}\,\mathrm{d}x = \psi^{\mathrm{m}} = 0, \qquad \psi_y^{\mathrm{m}} = -\beta_4[\mathrm{i}A\bar{B}\,\mathrm{e}^{2\mathrm{i}kx} + \mathrm{c.c.} + |B|^2 - |A|^2] + \mathrm{HOT} \quad \text{at } y = -1, \tag{1.7}$$

$$\psi^{\mathrm{m}}(x+L,y,t)\equiv\psi^{\mathrm{m}}(x,y,t),\qquad f^{\mathrm{m}}(x+L,t)\equiv f^{\mathrm{m}}(x,t),\tag{1.8}$$

$$\int_0^L f^{\mathrm{m}}(x,t)\,\mathrm{d}x=0,\tag{1.9}$$

valid outside of viscous boundary layers at the free surface and the bottom ($y=-1$). Here $C_g \ll 1$ is a dimensionless measure of viscosity. The resulting equations differ from the exact equations forming the starting point for the analysis in the presence of the forcing terms in the boundary conditions (1.5)–(1.7) and in two essential simplifications: the fast oscillation associated with the surface waves has been filtered out, and the boundary conditions are applied at the unperturbed location of the free surface, $y=0$. The mean flow itself is forced in two ways. The right-hand sides of the boundary conditions (1.5a) and (1.6) provide a *normal forcing mechanism*; this mechanism is the only one present in the strictly inviscid case and does not appear unless the aspect ratio is large. The right-hand sides of the boundary conditions (1.5b) and (1.7c) describe two *shear forcing mechanisms*, a tangential stress at the free surface and a tangential velocity at the bottom wall. Note that neither of these forcing terms vanishes in the limit of small viscosity (i.e., as $C_g \to 0$), cf. [15,16], in contrast to the strictly inviscid theory in which terms of this type do not arise.

The general coupled amplitude-mean-flow (hereafter GCAMF) equations summarized above are derived here by means of a consistent expansion that treats both the viscosity (i.e., the parameter C_g) and the inverse aspect ratio L^{-1} of the system as independent small parameters. However, in particular and physically relevant regimes in which these parameters are linked, the GCAMF equations simplify further. A particularly useful simplification arises when the system is large but not too large, in the sense that $L \ll C_g^{-1/2}$. In this regime, two cases are of special interest, corresponding, respectively, to nearly inviscid gravity–capillary waves and to pure capillary waves in a microgravity environment. Both systems are described by nonlocal amplitude equations of the type already studied in [17]; these equations determine the surface waves up to a spatial phase and decouple from the remaining equations governing the interaction between this phase and the (viscous) mean flow. If the system size is too large, different (hyperbolic) equations apply, but these are not discussed here (see [18,19] for a related problem). The remainder of this paper is organized as follows. In Section 2, we formulate the basic equations and explain the nature of the analysis that leads to the GCAMF equations. This analysis is performed explicitly in Section 3, with the simplifications alluded to above carried out in Section 4. The detailed properties of the resulting equations can only be ascertained numerically, and will be described in subsequent work. The paper concludes with a brief discussion in Section 5. Certain details of the analysis of the oscillatory boundary layers at the top and bottom of the layer that are required in the body of the paper can be found in Appendix A.

2. Formulation and other preliminaries

As a model of Faraday waves in annular containers, we consider a two-dimensional, laterally unbounded fluid layer above a horizontal plate that is vibrated vertically with an appropriately small amplitude. We use a Cartesian coordinate system with the x-axis along the unperturbed free surface and y vertically upwards, and nondimensionalize space and time with the unperturbed depth h and the gravity–capillary time $[g/h+T/(\rho h^3)]^{-1/2}$, where g is the gravitational acceleration, ρ the density and T the coefficient of surface tension. The nondimensional equations governing the system then are

$$\psi_{xx}+\psi_{yy}=\Omega,\qquad \Omega_t-\psi_y\Omega_x+\psi_x\Omega_y=C_g(\Omega_{xx}+\Omega_{yy}),\tag{2.1}$$

$$f_t-\psi_x-\psi_y f_x=(\psi_{yy}-\psi_{xx})(1-f_x^2)-4f_x\psi_{xy}=0\quad \text{at}\ y=f,\tag{2.2}$$

$$(1-S)f_x - S\left(\frac{f_x}{\sqrt{1+f_x^2}}\right)_{xx} - \psi_{yt} + \psi_{xt} f_x - (\psi_x + \psi_y f_x)\Omega + \tfrac{1}{2}(\psi_x^2+\psi_y^2)_x + \tfrac{1}{2}(\psi_x^2+\psi_y^2)_y f_x$$

$$-4\mu\omega^2 \cos(2\omega t) f_x = -C_g[3\psi_{xxy} + \psi_{yyy} - (\psi_{xxx}+\psi_{xyy}) f_x] + 2C_g\left[\frac{2\psi_{xy} f_x^2 + (\psi_{xx}-\psi_{yy}) f_x}{1+f_x^2}\right]_x$$

$$+2C_g \frac{(\psi_{xxy}-\psi_{yyy}) f_x^2 - \psi_{xyy}(1-f_x^2) f_x}{1+f_x^2} \quad \text{at} \quad y = f, \tag{2.3}$$

$$\int_0^L \Omega_y \, \mathrm{d}x = \psi = \psi_y = 0 \quad \text{at} \quad y = -1, \tag{2.4}$$

$$\int_0^L f \, \mathrm{d}x = 0, \tag{2.5}$$

subject to the requirement that ψ and f are both periodic in x with spatial period L (the nondimensional length of the annulus). Here ψ is the streamfunction, such that the velocity $(u, v) = (-\psi_y, \psi_x)$, Ω the vorticity, and f the free surface deflection required to satisfy volume conservation recalled in (2.5). The boundary condition (2.4a) is necessary in order that the pressure be periodic in x. The resulting problem depends on the aspect ratio L, the nondimensional vibration amplitude μ and frequency 2ω, the capillary–gravity number $C_g = \nu/[gh^3 + (Th/\rho)]^{1/2}$ and the gravity–capillary balance parameter $S = T/(T + \rho g h^2)$, where ν is the kinematic viscosity. Note that C_g and S are related to the usual capillary number $C = \nu[\rho/Th]^{1/2}$ and the Bond number $B = \rho g h^2/T$ by

$$C_g = \frac{C}{(1+B)^{1/2}}, \qquad S = \frac{1}{1+B}. \tag{2.6}$$

Thus

$$0 \le S \le 1, \tag{2.7}$$

and the extreme values $S = 0, 1$ correspond to the purely gravitational ($T = 0$) and the purely capillary ($g = 0$) limits, respectively.

The basic assumption made below is that viscosity is small, namely

$$C_g \ll 1. \tag{2.8}$$

To understand the origin of the nearly inviscid and viscous mean flows in this limit, it suffices to look at the spectrum of the unforced problem, linearized around $\psi = f = 0$ [5]. The normal modes take the form

$$(\psi, f) = (\Psi, F)\, \mathrm{e}^{\lambda t + \mathrm{i}kx}. \tag{2.9}$$

In general, when $C_g \ll 1$ there are two types of such modes:

(A) The *nearly inviscid modes* (or surface modes) obey the dispersion relation

$$\lambda = \mathrm{i}\omega - (1+\mathrm{i})\alpha_1 C_g^{1/2} - \alpha_2 C_g + \mathrm{O}(C_g^{3/2}), \tag{2.10}$$

where

$$\omega = [(1 - S + Sk^2)k \tanh k]^{1/2}, \qquad \alpha_1 = \frac{k(\omega/2)^{1/2}}{\sinh(2k)}, \qquad \alpha_2 = \left[2 + \frac{5 + 3\tanh^2 k}{16\sinh^2 k}\right] k^2. \tag{2.11}$$

Eq. (2.10) provides a good approximation for both the frequency $\mathrm{Im}(\lambda)$ and the damping rate

$$\delta \equiv -\mathrm{Re}(\lambda) = \alpha_1 C_g^{1/2} + \alpha_2 C_g \tag{2.12}$$

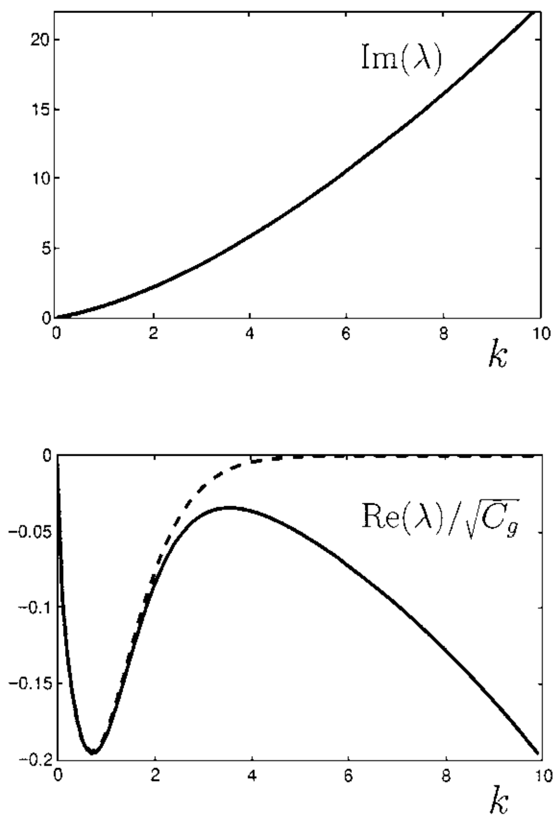

Fig. 1. The nearly inviscid dispersion relation, Im λ and Re λ vs. k for $C_g = 10^{-6}$, $S = 0.5$, from Eq. (2.10) using the $O(C_g^{1/2})$ results (dashed line) and the $O(C_g)$ results (solid line).

for small but fixed values of C_g, see Fig. 1. However, as noted in [5], if the third term in (2.10) is omitted the resulting approximation breaks down as soon as $k \gtrsim k_{\mathrm{m}} \sim |\ln C_g|$. Since these moderately large values of k are also of interest this term will be retained in what follows.

The eigenfunction associated with the dispersion relation (2.10) is given (up to a constant factor) by

$$(\Psi, F) = (\Psi_0, 1) + O(C_g^{1/2}) \quad \text{with} \quad \Psi_0 = \frac{\omega \sinh [k(y+1)]}{k \sinh k}. \tag{2.13}$$

These modes therefore exhibit a significant free-surface deflection and are irrotational in the bulk, outside two thin boundary layers (whose thickness is $O(C_g/\omega)^{1/2}$) attached to the bottom plate and the free surface. For small C_g their decay rate is $O(C_g^{1/2})$, i.e., these modes are all *near-marginal*. Note that the horizontal wavenumber k is only restricted by the periodicity condition and thus can take any value of the form $2\pi N/L$, where N is an integer; in the limit $L \to \infty$ the allowed wavenumbers become dense on the real line. However, the assumption (2.16) implies that the relevant nearly inviscid modes are either of long wavelength or are concentrated around the two counterpropagating modes. The long wave modes constitute the *nearly inviscid* mean flow; in the strictly inviscid case, this flow is the mean flow considered in inviscid theories [11,12]. However, because of its long wavelength this mean flow does not appear if the aspect ratio is of order unity [9,10].

(B) The *viscous modes* (or hydrodynamical modes) obey the dispersion relation

$$\lambda = -C_g[k^2 + q_n(k)^2] + O(C_g^2), \tag{2.14}$$

where for each $k > 0$, $q_n > 0$ is the nth root of $q \tanh k = k \tan q$, and hence decay on an $\mathrm{O}(C_g)$ timescale, i.e., more slowly than the surface modes when C_g is sufficiently small. Consequently, these modes are also near-marginal. Since the associated eigenfunction is

$$\Psi = \sin q_n \sinh(ky) - \sinh k \sin(q_n y) + \mathrm{O}(C_g), \qquad F = \mathrm{O}(C_g), \tag{2.15}$$

these modes do not result in any significant free-surface deformation at leading order. On the other hand, they are rotational throughout the domain and when forced at the edge of the oscillatory boundary layers attached to the bottom plate and the free surface by the mechanisms described by Schlichting [20] and Longuet-Higgins [21] (see Appendix A), they constitute the viscous mean flow. In view of its slow decay this flow must be included in any realistic nearly inviscid description. The assumption that follows implies that the relevant viscous modes are concentrated around a discrete set of values of k.

2.1. *Basic assumption*

The spatial Fourier transforms of the oscillatory part (in time) of ψ and f peak for all time around the wavenumbers $\pm\, k$, while those of the nonoscillatory part peak at wavenumbers $\pm\, 2mk$, with $m = 0, 1, \ldots$. (2.16)

Here and hereafter k denotes the wavenumber of the parametrically excited surface mode. If L is not too large, as specified in Eq. (2.19), this assumption is consistent, in the sense that the resulting equations do not generate arbitrarily small scales. This property is not guaranteed for larger L.

In addition to this assumption we also assume that

$$|\psi_x| + |\psi_y| \ll 1, \qquad |f| \ll 1, \qquad C_g \ll 1, \qquad L^{-1} \ll 1, \tag{2.17}$$

i.e., we focus on weakly nonlinear nearly inviscid waves in large aspect ratio systems. This restriction requires, in addition, that $\mu \ll 1$. Moreover, in view of the comment after Eq. (2.12), we also assume that

$$1 \lesssim k \lesssim |\ln C_g|, \tag{2.18}$$

which implies that $\delta = \mathrm{O}(C_g)^{1/2} \ll 1$ (see (2.12)) and $1 \lesssim \omega \lesssim [(1 - S)|\ln C_g| + S|\ln C_g|^3]^{1/2}$ (see (2.11)). As explained in Section 5, this assumption can be relaxed.

Within these assumptions several essentially different *distinguished limits* are possible, depending on the relative values of the small parameters C_g, μ and L^{-1}, and also on the order of magnitude of $1 - S$. In this paper, we assume that L is not too large, in the sense that

$$1 \ll L \ll \frac{v_g}{\delta + |d| + \mu}, \tag{2.19}$$

where v_g, δ and d are the (nondimensional) group velocity, damping rate and detuning of the surface waves, defined by (3.24), (2.12) and (3.28), respectively, and consider separately the two cases $S \ll 1$ and $S \sim 1$ (see Section 4).

3. The general coupled amplitude-mean flow equations

In the derivation that follows it is convenient to treat the small parameters C_g and L^{-1} as unrelated. Since viscosity is small, we must distinguish three regions in the physical domain, namely, the two oscillatory boundary layers

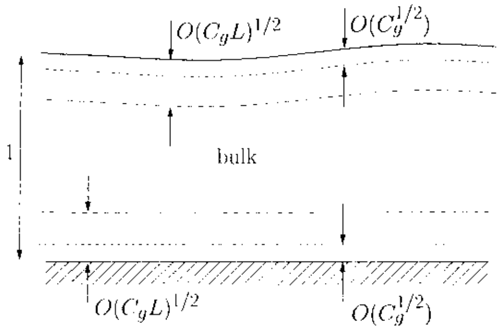

Fig. 2. Sketch of the primary and secondary boundary layers, indicating their widths in comparison to the layer depth.

(of thickness $O(C_g^{1/2})$) mentioned in Section 2 and the remaining part (or *bulk*) of the domain (see Fig. 2). The boundary layers must be considered in order to obtain the correct boundary conditions for the solution in the bulk. The description of these boundary layers can be found in Appendix A.

Now, according to assumption (2.16), the streamfunction in the bulk and the free surface deflection can be decomposed into three parts, namely: (i) the *two counterpropagating wavetrains* mentioned in Section 2, which are slowly modulated both in space and time around a basic frequency ω and wavenumbers $\pm k$; (ii) a *mean flow*, which depends weakly on time but can exhibit significant dependence on the space variables x and y; and (iii) the remaining part of the solution, which will be called *nonresonant*. Since we are not distinguishing between inviscid and viscous mean flows we must allow the mean flow to exhibit a significant dependence on the horizontal coordinate x. This is because the mean flow must include, among other things, any viscous modes with $O(k)$ wavenumber associated with the basic wavetrain. The assumption (2.16) is equivalent to the requirement that the mean flow variables exhibit well-defined averages in the fast variable x (see (3.19)).

Under these conditions the free-surface deflection, and the streamfunction and vorticity in the bulk, can be written in the form

$$f = \mathrm{e}^{\mathrm{i}\omega t}(A\,\mathrm{e}^{\mathrm{i}kx} + B\,\mathrm{e}^{-\mathrm{i}kx}) + \gamma_1 A\bar{B}\,\mathrm{e}^{2\mathrm{i}kx} + \gamma_2\,\mathrm{e}^{2\mathrm{i}\omega t}(A^2\,\mathrm{e}^{2\mathrm{i}kx} + B^2\,\mathrm{e}^{-2\mathrm{i}kx}) + f^+\mathrm{e}^{\mathrm{i}\omega t+\mathrm{i}kx} + f^-\,\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}kx} + \text{c.c.} + f^{\mathrm{m}} + \text{NRT}, \tag{3.1}$$

$$\psi = \Psi_0\,\mathrm{e}^{\mathrm{i}\omega t}(A\,\mathrm{e}^{\mathrm{i}kx} - B\,\mathrm{e}^{-\mathrm{i}kx}) + \gamma_3\Psi_{22}\,\mathrm{e}^{2\mathrm{i}\omega t}(A^2\,\mathrm{e}^{2\mathrm{i}kx} - B^2\,\mathrm{e}^{-2\mathrm{i}kx}) + \psi^+\mathrm{e}^{\mathrm{i}\omega t+\mathrm{i}kx} + \psi^-\,\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}kx} + \text{c.c.} + \psi^{\mathrm{m}} + \text{NRT}, \tag{3.2}$$

$$\Omega = \mathrm{i}\omega^{-1}\,\mathrm{e}^{\mathrm{i}\omega t}[(A\,\mathrm{e}^{\mathrm{i}kx} - B\,\mathrm{e}^{-\mathrm{i}kx})\Psi_0'\Omega^{\mathrm{m}}_x - \mathrm{i}k(A\,\mathrm{e}^{\mathrm{i}kx} + B\,\mathrm{e}^{-\mathrm{i}kx})\Psi_0\Omega^{\mathrm{m}}_y + \text{HOT}] + \text{c.c.} + \Omega^{\mathrm{m}} + \text{NRT}. \tag{3.3}$$

Here the superscript m denotes terms associated with the *mean flow*, NRT denotes *nonresonant terms* and HOT denotes *higher order terms*. The function Ψ_0 is defined in (2.13). Moreover, the quantities A, B, $f^{\pm}$ and $\psi^{\pm}$ must all depend weakly on t and x, while f^{m}, ψ^{m} and Ω^{m} depend weakly on t but strongly on x (cf. Eq. (1.7)), i.e.,

$$|A_x| + |A_t| \ll |A| \ll 1, \qquad |B_x| + |B_t| \ll |B| \ll 1, \tag{3.4}$$

$$|f^{\pm}_x| + |f^{\pm}_t| \ll |f^{\pm}| \ll 1, \qquad |\psi^{\pm}_x| + |\psi^{\pm}_t| \ll |\psi^{\pm}| \ll 1, \tag{3.5}$$

$$|f^{\mathrm{m}}_t| \ll |f^{\mathrm{m}}| \ll 1, \qquad |\psi^{\mathrm{m}}_t| \ll |\psi^{\mathrm{m}}| \ll 1, \qquad |\Omega^{\mathrm{m}}_t| \ll |\Omega^{\mathrm{m}}| \ll 1, \tag{3.6}$$

while the periodic boundary equations on ψ and f imply that

$$A(x+L,t) \equiv \mathrm{e}^{-2\mathrm{i}kL}A(x,t), \qquad B(x+L,t) \equiv \mathrm{e}^{2\mathrm{i}kL}B(x,t). \tag{3.7}$$

The terms proportional to $e^{i\omega t \pm ikx}$ describe the two counterpropagating wavetrains. In order to distinguish between the leading order and higher order contributions to these waves we impose the additional condition

$$\int_{-1}^{0} \psi^{\pm}(x, y, t)\, dy = 0 \tag{3.8}$$

for all x and t. This condition serves as a definition of the complex amplitudes A and B, and readily implies that

$$|f^{+}| + |\psi^{+}| \ll |A|, \qquad |f^{-}| + |\psi^{-}| \ll |B|. \tag{3.9}$$

The coefficients γ_1, γ_2, γ_3 and the function Ψ_{22} in (3.1) and (3.2) are given by

$$\gamma_1 = \frac{(\sigma^2 + 1)\omega^2}{\sigma^2(1 - S + 4Sk^2)}, \tag{3.10}$$

$$\gamma_2 = \frac{(3 - \sigma^2)k(1 - S + Sk^2)}{2\sigma[(1 - S)\sigma^2 - Sk^2(3 - \sigma^2)]}, \tag{3.11}$$

$$\gamma_3 = \frac{3\omega[(1 - S)(1 - \sigma^2) + Sk^2(3 - \sigma^2)]}{2\sigma[(1 - S)\sigma^2 - Sk^2(3 - \sigma^2)]}, \tag{3.12}$$

$$\Psi_{22} = \frac{\sinh[2k(y + 1)]}{\sinh(2k)}, \tag{3.13}$$

where $\sigma = \tanh k$. Note that γ_2 and γ_3 diverge at $(1 - S)\sigma^2 = Sk^2(3 - \sigma^2)$, i.e., when the strictly inviscid eigenfrequency (2.11a) satisfies $\omega(2k) = 2\omega(k)$. In the present paper, we do not pursue this resonance further; see [22,23] for a strictly inviscid analysis, and [24–26] for nearly inviscid descriptions that ignore mean flow.

Substituting (3.1)–(3.7) into (2.1)–(2.5) with the boundary conditions derived in Appendix A, we obtain the evolution equations and boundary conditions for ψ^{m}, Ω^{m}, f^{m} listed in (1.4)–(1.9), together with the following equations and boundary conditions for the perturbations ψ^{+}, f^{+}:

$$\psi^{+}_{yy} - k^2\psi^{+} = -2ik(\Psi_0 A_x + \psi^{+}_x) - \Psi_0 A_{xx} - \left(\frac{k}{\omega}\right)\Psi_0\langle\Omega^{m}_y\rangle^x A + \text{HOT} \quad \text{in} \quad -1 < y < 0, \tag{3.14}$$

$$\begin{aligned} i\omega f^{+} - ik\psi^{+} = {} & -A_t + \Psi_0 A_x + \psi^{+}_x + [ik(\beta_5|A|^2 + \beta_6|B|^2) + ik\Psi_0'\langle f^{m}\rangle^x \\ & + ik\langle\psi^{m}_y\rangle^x + \beta_9 C_g]A + \text{HOT} \quad \text{at} \quad y = 0, \end{aligned} \tag{3.15}$$

$$\begin{aligned} ik(1 - S + Sk^2)f^{+} - i\omega\psi^{+}_y = {} & 3ikSA_{xx} + \Psi_0' A_t + ik(\beta_7|A|^2 + \beta_8|B|^2)A + C_g(3k^2\Psi_0' - \Psi_0''')A - ik\mu\omega^2\bar{B} \\ & + [i\omega\Psi_0''\langle f^{m}\rangle^x + ik\Psi_0\langle\Omega^{m}\rangle^x - ik\Psi_0'\langle\psi^{m}_y\rangle^x]A + \text{HOT} \quad \text{at} \quad y = 0, \end{aligned} \tag{3.16}$$

$$\psi^{+} = [(1 + i)\beta_{10}C_g^{1/2} + \beta_{11}C_g]A + \text{HOT} \quad \text{at} \quad y = -1, \tag{3.17}$$

$$\psi^{+}(x + L, y, t) \equiv \psi^{+}(x, y, t), \qquad f^{+}(x + L, t) \equiv f^{+}(x, t). \tag{3.18}$$

Here the mean value $\langle\cdot\rangle^x$ is defined by

$$\langle G(x, y, t)\rangle^x = (2\ell)^{-1}\int_{x-\ell}^{x+\ell} G(z, y, t)\, dz \quad \text{with} \quad 1 \ll \ell \ll L, \tag{3.19}$$

and is required to be independent of ℓ. In view of the assumption (2.16) this average is well-defined and can be thought of as a filter that filters out the smallest scales, $x \sim 1$. For this reason $\langle G(x, y, t)\rangle^x$ may still depend on the horizontal coordinate x, albeit weakly, so that

$$\langle f_x^{\rm m}\rangle^x \ll \langle f^{\rm m}\rangle^x, \qquad \langle \psi_x^{\rm m}\rangle^x \ll \langle \psi_y^{\rm m}\rangle^x, \qquad \langle \Omega_x^{\rm m}\rangle^x \ll \langle \Omega_y^{\rm m}\rangle^x.$$

These estimates have been used to drop higher order terms.

The coefficients $\beta_1, \ldots, \beta_8$ and the function g in (1.4)–(1.7), (3.15) and (3.16) are given by

$$\beta_1 = \frac{2\omega}{\sigma}, \qquad \beta_2 = \frac{8\omega k^2}{\sigma}, \qquad \beta_3 = \frac{(1-\sigma^2)\omega^2}{\sigma^2}, \qquad \beta_4 = \frac{3(1-\sigma^2)\omega k}{\sigma^2},$$
$$\beta_5 = \frac{\gamma_2\omega}{\sigma} + \frac{\gamma_3 k(1+\sigma^2)}{\sigma} + \frac{3\omega k}{2}, \quad \beta_6 = -\frac{\gamma_1\omega}{\sigma} - \omega k, \quad \beta_7 = \gamma_2\omega^2 + \frac{\gamma_3\omega k(\sigma^2-1)}{\sigma^2} - \frac{5\omega^2 k}{2\sigma} + \frac{3Sk^4}{2},$$
$$\beta_8 = \gamma_1\omega^2 + \frac{3\omega^2 k}{\sigma} + 3Sk^4, \qquad g(y) = \frac{2\omega k \cosh[2k(y+1)]}{\sinh^2 k}. \tag{3.20}$$

The coefficients β_9, β_{10} and β_{11} need not be calculated because they only contribute to the coefficient of A in the amplitude equation (3.23), and this coefficient follows readily from the (exact) dispersion relation (2.10). The corresponding equations and boundary conditions for ψ^- and f^- will not be needed below; they are obtained from (3.14)–(3.18) using the transformation

$$\psi^+ \to \psi^-, \qquad f^+ \to f^-, \qquad \psi^{\rm m} \to -\psi^{\rm m}, \qquad \Omega^{\rm m} \to -\Omega^{\rm m}, \qquad A \leftrightarrow B, \qquad x \to -x, \tag{3.21}$$

a consequence of the symmetry of Eqs. (2.1)–(2.5) under the reflection $x \to -x$.

In view of the condition (3.5) the terms on the right-hand side of Eqs. (3.14)–(3.18) are to be considered as inhomogeneous, while those on the left constitute a set of homogeneous equations solved by $(\psi^+, f^+) = (\Psi_0(y), 1)$. The solvability condition for this system yields the evolution equation for A (the *amplitude equation*) in the form

$$(\mathrm{i}k)^{-1}\Psi_0'(0)H_1 - (\mathrm{i}\omega)^{-1}\Psi_0(0)H_2 + \Psi_0'(-1)H_3 = \int_{-1}^{0} \Psi_0(y)H_0(y)\,\mathrm{d}y. \tag{3.22}$$

Here H_0, H_1, H_2 and H_3 denote the right hand sides of Eqs. (3.14)–(3.17), respectively. Using (3.8) this relation takes the explicit form

$$A_t - v_g A_x = \mathrm{i}\alpha A_{xx} - [(1+\mathrm{i})\alpha_1 C_g^{1/2} + \alpha_2 C_g]A + \mathrm{i}(\alpha_3|A|^2 - \alpha_4|B|^2)A + \mathrm{i}\alpha_5\mu\bar{B}$$
$$+\mathrm{i}\alpha_6\int_{-1}^{0} g(y)\langle\psi_y^{\rm m}\rangle^x\,\mathrm{d}y\,A + \mathrm{i}\alpha_7\langle f^{\rm m}\rangle^x A + \mathrm{HOT}. \tag{3.23}$$

To use this procedure to compute the coefficients of A_{xx} and $C_g A$, we would have to consider the expansions

$$\psi^+ = A_x\psi_1^+ + C_g^{1/2}A\psi_2^+ + \cdots, \qquad f^+ = A_x f_1^+ + C_g^{1/2}Af_2^+ + \cdots,$$

and explicitly calculate (ψ_1^+, f_1^+) and (ψ_2^+, f_2^+) from the equations that result from substituting these expansions into (3.8), (3.14)–(3.18) and setting the coefficients of A_x and $C_g^{1/2}A$ to zero; note that condition (3.8) is necessary to ensure uniqueness in these singular problems. In practice, it is simpler to deduce the coefficients α_1 and α_2, the group velocity v_g and the dispersion α directly from the dispersion relation (2.10) using (2.11) and

$$v_g = \omega'(k), \qquad \alpha = -\tfrac{1}{2}\omega''(k). \tag{3.24}$$

The remaining coefficients in (3.23), $\alpha_3, \ldots, \alpha_7$ must, however, be calculated from the solvability condition (3.22) and are found to be

$$\alpha_3 = \frac{\omega k^2[(1-S)(9-\sigma^2)(1-\sigma^2) + Sk^2(7-\sigma^2)(3-\sigma^2)]}{4\sigma^2[(1-S)\sigma^2 - Sk^2(3-\sigma^2)]} + \frac{[8(1-S)+5Sk^2]\omega k^2}{4(1-S+Sk^2)},$$

$$\alpha_4 = \frac{\omega k^2}{2}\left[\frac{(1-S+Sk^2)(1+\sigma^2)^2}{(1-S+4Sk^2)\sigma^2} + \frac{4(1-S)+7Sk^2}{1-S+Sk^2}\right], \qquad \alpha_5 = \omega k\sigma, \qquad \alpha_6 = \frac{k\sigma}{2\omega},$$

$$\alpha_7 = \frac{\omega k(1-\sigma^2)}{2\sigma}. \tag{3.25}$$

These expressions agree up to notation changes with their counterparts in strictly inviscid theories (see, e.g., [12] and references therein), and in particular confirm the results for the cubic coefficients obtained in Refs. [4,27]. Like γ_2 and γ_3, the coefficient α_3 diverges at the (excluded) resonant wavenumbers satisfying $\omega(2k) = 2\omega(k)$. The corresponding amplitude equation for the complex amplitude B is obtained from (3.23) using the reflection symmetry (3.21).

The resulting equations take the form (1.1) and (1.2) if we select a (large) integer N such that

$$-\pi < 2\pi N - kL \leq \pi, \tag{3.26}$$

and replace

$$A \to A\,\mathrm{e}^{\mathrm{i}(2\pi N/L-k)x}, \qquad B \to B\,\mathrm{e}^{-\mathrm{i}(2\pi N/L-k)x}, \tag{3.27}$$

and redefine v_g to be the group velocity at the wavenumber $2\pi N/L$. This change of variables shifts the wavenumber k to the nearest wavenumber commensurate with the imposed periodicity condition and leads to periodic boundary conditions on the (new) variables A and B as in (1.3); the resulting expressions for the damping rate δ and the effective detuning d present in Eqs. (1.1) and (1.2) are given by (2.12) and

$$d = \alpha_1 C_g^{1/2} - \left(\frac{2\pi N}{L} - k\right) v_g. \tag{3.28}$$

Both quantities are *small*. The change of variables has, however, no effect on the mean flow equations (1.4)–(1.9), except to replace the wavenumber k that appears explicitly in the boundary condition (1.7b) by $k = 2\pi N/L$, i.e., the solution in the bulk is now of the form

$$f = \mathrm{e}^{\mathrm{i}\omega t}[(A\,\mathrm{e}^{\mathrm{i}2\pi Nx/L} + B\,\mathrm{e}^{-\mathrm{i}2\pi Nx/L}) + \mathrm{HOT}] + \mathrm{c.c.} + f^{\mathrm{m}} + \mathrm{NRT}, \tag{3.29}$$

$$\psi = \mathrm{e}^{\mathrm{i}\omega t}[\Psi_0(A\,\mathrm{e}^{\mathrm{i}2\pi Nx/L} - B\,\mathrm{e}^{-\mathrm{i}2\pi Nx/L}) + \mathrm{HOT}] + \mathrm{c.c.} + \psi^{\mathrm{m}} + \mathrm{NRT}, \tag{3.30}$$

$$\Omega = \Omega^{\mathrm{m}} + \mathrm{HOT}. \tag{3.31}$$

Some remarks about the GCAMF equations derived above are now in order.

(a) The conservative nature of the terms describing the coupling to the mean flow implies that at leading order the mean flow does not take energy from the system, a result that is consistent with the small steepness of the associated surface displacement and its small velocity compared with the speed $|\nabla\psi|$ of the surface waves. This latter property follows from the fact that the mean flow is driven by quadratic terms in the complex amplitudes (see remarks (b) and (c)).

(b) The forcing terms on the right-hand sides of the boundary conditions (1.5a) and (1.6) are present when the aspect ratio is large, and are responsible for driving the strictly *inviscid* mean flow [11,12]. These terms vanish if the wavetrain is uniform, or if the surface waves are of standing wave type, namely if $|A| = |B|$, and $k \gg 1$ (so that $\beta_3 \ll 1$). Note that part of the strictly inviscid mean flow can be included explicitly in the expansion (3.2) because it is slaved to the waves. However, we choose not to do so here because there is always a part of this flow which solves a homogeneous problem (see Eq. (4.12)) and is not slaved. The shear nature of the remaining forcing terms, in Eqs. (1.5b) and (1.7c), leads us to retain the viscous term in (1.4) even when C_g is quite small. In fact, when C_g is very small, the mean flow itself generates additional boundary layers near the top and bottom of the container, and these must be thicker than the original boundary layers for the validity of the analysis. This puts an additional restriction on the validity of the GCAMF equations, namely

$$\left|\frac{\beta_2 k(|A|^2-|B|^2)}{C_g}\right|^{1/3}+\left|\frac{\beta_4 k(|A|^2+|B|^2)}{C_g}\right|^{1/2} \ll \left(\frac{C_g}{\omega}\right)^{1/2}. \tag{3.32}$$

In this expression the first term is an estimate of the inverse of the boundary layer thickness associated with the tangential stress boundary condition at the surface while the second term is the corresponding quantity due to the horizontal velocity boundary condition at $y=-1$.

(c) There is a third, less effective but inviscid, volumetric forcing mechanism associated with the second term in the vorticity equation (1.4), which looks like a horizontal force $(|A|^2-|B|^2)g(y)\Omega^{\mathrm{m}}$ and is sometimes called the *vortex force*. The term plays an important role in the generation of Langmuir circulation [28]. Although in the absence of mean flow this term vanishes, it can change the stability properties of such a flow and enhance or limit the effect of the remaining forcing terms. However, this is not the case in the limit considered here.

(d) The GCAMF equations (1.1)–(1.9) are invariant under reflection

$$\psi^{\mathrm{m}} \to -\psi^{\mathrm{m}}, \qquad \Omega^{\mathrm{m}} \to -\Omega^{\mathrm{m}}, \qquad A \leftrightarrow B, \qquad x \to -x. \tag{3.33}$$

The simplest reflection-symmetric solutions, i.e., solutions of the form $A(x,\cdot)=B(-x,\cdot)$, are the spatially uniform standing waves given by $A=B=R_0\,\mathrm{e}^{\mathrm{i}\theta}$, where θ is a constant and the amplitude R_0 is given by $\delta^2+[d+(\alpha_3-\alpha_4)R_0^2]^2=\alpha_5^2\mu^2$, with an associated reflection-symmetric streaming flow that is periodic in x with period π/k (see Eq. (1.7c)). Since this mean flow does not couple to the amplitudes A, B (i.e., the mean flow terms are absent from Eqs. (1.1) and (1.2)), the presence of this flow does not affect the standing waves. These much studied waves bifurcate from the flat state at $\mu=\mu_c=(\delta^2+d^2)^{1/2}/|\alpha_5|$, and do so supercritically if $d<0$ and subcritically if $d>0$, see Fig. 3. Note that μ can be of order μ_c without violating the conditions for the validity of the GCAMF equations, and that these equations describe correctly both cases $d<0$ and $d>0$. In the former case, the waves are stable near threshold, but may lose stability at finite amplitude through the action of the mean flow as the forcing amplitude increases. Like the secondary saddle-node bifurcation which stabilizes the spatially uniform standing waves when $d>0$ (see Fig. 3), this bifurcation is well within in the regime of validity of the GCAMF equations. Thus the mean flow is involved only in possible *secondary* instabilities of the primary standing wave branch.

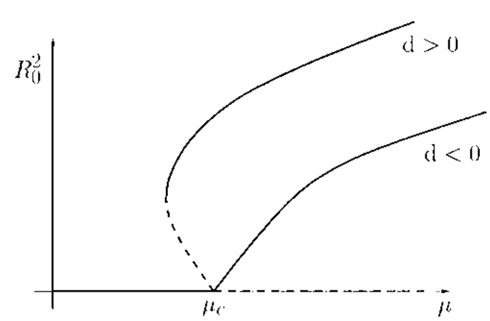

Fig. 3. The primary bifurcation from the flat state to the spatially uniform standing wave solutions. The GCAMF equations describe correctly all states with $|\mu-\mu_c|\sim\mu_c$, including the secondary saddle-node bifurcation present when $d>0$ and the solutions beyond it.

(e) The special case $d = 0$ (zero detuning) and $\mu = \mu_c$ defines a codimension-two point for the analysis since both L (or equivalently ω) and μ must be chosen appropriately. In this case the direction of branching is determined by higher order terms neglected in the analysis, such as the real parts of the coefficients of the cubic terms, and this is so for sufficiently small but nonzero values of d as well. In other words, the limit $d \to 0$ (although well-defined within the GCAMF equations) may not describe correctly the corresponding behavior of the underlying fluid equations appropriately close to threshold, i.e., for $|\mu - \mu_c| \ll \mu_c$. However, even in this case the GCAMF equations will correctly capture any secondary instabilities involving the mean flow, provided these occur at $\mu \sim \mu_c$. A similar remark applies to other codimension-two points as well.

(f) The GCAMF equations form a good starting point for any weakly nonlinear theory under the assumptions (2.8), (2.16), (2.18), (3.4) and (3.32), provided that second and third order internal resonances are avoided, and higher order terms consistently omitted. In fact, the condition (3.4) can be replaced by

$$|A_x| \ll k|A|, \qquad |A_t| \ll \omega|A|, \qquad |B_x| \ll k|B|, \qquad |B_t| \ll \omega|B|, \tag{3.34}$$

where A, B are themselves small. From the derivation of these equations it is clear that they apply whenever the parameters C_g, N^{-1} (or L^{-1}) and μ are small, but are otherwise unrelated to one another. Any relation between them, such as those assumed in Sections 4.1 and 4.2, will therefore lead to further simplifications.

(g) If an additional packet of nearly inviscid modes is present initially, with a basic frequency $\tilde{\omega} \neq \omega$ and sufficiently distinct wavenumber $\tilde{k}$ (see assumption (2.16)), the associated complex amplitudes interact with the original according to the equations

$$\begin{aligned}\tilde{A}_t - \tilde{v}_g\tilde{A}_x = {} & \mathrm{i}\tilde{\alpha}\tilde{A}_{xx} - (\tilde{\delta} + \mathrm{i}\tilde{d})\tilde{A} + \mathrm{i}(\tilde{\alpha}_3|\tilde{A}|^2 - \tilde{\alpha}_4|\tilde{B}|^2 + \tilde{\alpha}_8|A|^2 - \tilde{\alpha}_9|B|^2)\tilde{A} + \mathrm{i}\tilde{\alpha}_6\int_{-1}^{0}\tilde{g}(y)\langle\psi_y^{\mathrm{m}}\rangle^x\,\mathrm{d}y\,\tilde{A}\\ & +\mathrm{i}\tilde{\alpha}_7\langle f^{\mathrm{m}}\rangle^x\tilde{A} + \mathrm{HOT},\end{aligned} \tag{3.35}$$

$$\begin{aligned}\tilde{B}_t + \tilde{v}_g\tilde{B}_x = {} & \mathrm{i}\alpha\tilde{B}_{xx} - (\tilde{\delta} + \mathrm{i}\tilde{d})\tilde{B} + \mathrm{i}(\tilde{\alpha}_3|\tilde{B}|^2 - \tilde{\alpha}_4|\tilde{A}|^2 + \tilde{\alpha}_8|B|^2 - \tilde{\alpha}_9|A|^2)\tilde{B} - \mathrm{i}\tilde{\alpha}_6\int_{-1}^{0}\tilde{g}(y)\langle\psi_y^{\mathrm{m}}\rangle^x\,\mathrm{d}y\,\tilde{B}\\ & +\mathrm{i}\tilde{\alpha}_7\langle f^{\mathrm{m}}\rangle^x\tilde{B} + \mathrm{HOT},\end{aligned} \tag{3.36}$$

$$\tilde{A}(x + L, t) \equiv \tilde{A}(x, t), \qquad \tilde{B}(x + L, t) \equiv \tilde{B}(x, t), \tag{3.37}$$

i.e., such modes evolve under the influence of the ambient wavetrain, but are not directly excited by the parametric forcing. It is clear from the structure of these equations that this interaction cannot maintain the additional packet against viscous dissipation, and hence that both $\tilde{A}$ and $\tilde{B}$ decay exponentially on the timescale $t \sim \tilde{\delta}^{-1}$.

4. Coupled amplitude-mean flow equations for moderately large aspect ratios

The regime $1 \ll L \ll v_g/(\delta + |d| + \mu)$ provides perhaps the cleanest simplification of the GCAMF equations derived above. The resulting equations apply to systems of moderately large aspect ratios, and include in a particular regime the model equations studied at length by Martel et al. [17]. To derive such simplified equations we consider the distinguished limit

$$\frac{\delta L^2}{\alpha} = \Delta \sim 1, \qquad \frac{dL^2}{\alpha} = D \sim 1, \qquad \frac{\mu L^2}{\alpha} \equiv M \sim 1 \tag{4.1}$$

with $1 \lesssim k \lesssim |\ln C_g|$. Eq. (2.12) then implies that $C_g|\ln C_g|^2 \lesssim \delta \lesssim C_g^{1/2}$. In order to avoid unnecessarily involved expressions, we shall henceforth treat $|\ln C_g|$ as an O(1) quantity, thereby allowing some of the coefficients in the expansions (4.8), (4.9), (4.48) and (4.49) to be logarithmically small or large. The simplified equations are derived using a multiple scale method using x and t as *fast* variables and

$$\zeta = \frac{x}{L}, \qquad \tau = \frac{t}{L}, \qquad T = \frac{t}{L^2}, \tag{4.2}$$

as *slow* variables. In terms of these variables the local horizontal average $\langle\cdot\rangle^x$ defined in (3.19) becomes an average over the fast variable x. Note that assumption (4.1) imposes an implicit relation between L and C_g. In the following, we distinguish two sub-limits, depending on whether gravity is significant ($1 - S \sim 1$) or negligible ($1 - S \ll 1$). The resulting equations are valid in the whole range $1 \ll L \ll v_g/(\delta + |d| + \mu)$, and more specifically for $1 \ll L \ll C_g^{-1/2}$ if $k \sim 1$.

4.1. Gravity or gravity–capillary waves

When $1 - S \sim 1$ the nearly inviscid and viscous mean flows can be clearly distinguished from one another as discussed in Section 2, and the viscous mean flow can be identified by taking appropriate averages of the whole mean flow over an intermediate timescale τ, i.e., the mean flow variables ψ^{m}, Ω^{m} and f^{m} take the form

$$\psi^{\mathrm{m}}(x, y, \zeta, \tau, T) = \psi^{v}(x, y, \zeta, T) + \psi^{i}(x, y, \zeta, \tau, T), \tag{4.3}$$

$$\Omega^{\mathrm{m}}(x, y, \zeta, \tau, T) = \Omega^{v}(x, y, \zeta, T) + \Omega^{i}(x, y, \zeta, \tau, T), \tag{4.4}$$

$$f^{\mathrm{m}}(x, \zeta, \tau, T) = f^{v}(x, \zeta, T) + f^{i}(x, \zeta, \tau, T) \tag{4.5}$$

with

$$\left|\int_0^\tau \psi^i_x \,\mathrm{d}\tau\right| + \left|\int_0^\tau \psi^i_\zeta \,\mathrm{d}\tau\right| + \left|\int_0^\tau \psi^i_y \,\mathrm{d}\tau\right| + \left|\int_0^\tau \Omega^i \,\mathrm{d}\tau\right| + \left|\int_0^\tau f^i \,\mathrm{d}\tau\right| \tag{4.6}$$

bounded as $\tau \to \infty$. Thus the nearly inviscid mean flow is purely oscillatory (i.e., it has a zero mean, see (4.6)) on the timescale τ. Since its frequency is of the order of L^{-1} (see (4.2)), which is large compared with C_g, the inertial term for this flow is large in comparison with the viscous terms (see Eq. (1.4)), except in two *secondary* boundary layers, of thickness of the order of $(C_g L)^{1/2}$ ($\ll 1$), attached to the bottom plate and the free surface. Note that, as required for the consistency of the analysis, these boundary layers are much thicker than the *primary* boundary layers associated with the surface waves (see Fig. 2), which provide the boundary conditions (1.5)–(1.7) for the mean flow. Moreover, the width of these secondary boundary layers remains small as $\tau \to \infty$ and (to leading order) the vorticity of this nearly inviscid mean flow remains confined to these boundary layers. This is because, according to condition (4.6), the nearly inviscid mean flow is purely oscillatory on the timescale τ. Consequently, condition (4.6) is essential for the validity of the analysis that follows, and the mathematical definition of the nearly inviscid mean flow through Eqs. (4.3)–(4.6) is the only consistent one; without this condition vorticity would diffuse outside the boundary layers and affect the structure of the whole 'nearly inviscid' solution even at leading order. In fact, vorticity does diffuse (and is convected) from the boundary layers, but this vorticity transport is included in the viscous mean flow. The vorticity associated with the nearly inviscid mean flow is readily seen to be of, at most, the order of

$$||A|^2 - |B|^2|, \qquad (|A|^2 + |B|^2)(C_g L)^{-1/2} \tag{4.7}$$

in the upper and lower secondary boundary layers, respectively; the jump in the associated streamfunction ψ^i across each boundary layer is $\mathrm{O}(C_g L)$ times smaller. This jump only affects higher order terms; as a consequence the secondary boundary layers can be completely ignored and no additional contributions to the boundary conditions on the nearly inviscid flow need be included in (1.5) and (1.7). Outside these boundary layers, the complex amplitudes and the flow variables associated with the nearly inviscid mean flow are expanded as

$$(A, B) = L^{-1}(X_0, Y_0) + L^{-2}(X_1, Y_1) + \cdots, \qquad (\psi^i, f^i) = L^{-2}(\phi_0^i, F_0^i) + L^{-3}(\phi_1^i, F_1^i) + \cdots, \tag{4.8}$$

$$\Omega^i = L^{-3} W_0^i + \cdots, \qquad (\psi^v, \Omega^v) = L^{-2}(\phi_0^v, W_0^v) + \cdots, \qquad f^v = L^{-3} F_0^v + \cdots. \tag{4.9}$$

Substitution of (4.1)–(4.6), (4.8) and (4.9) into (1.1)–(1.9) leads to the following:

(i) From (1.4)–(1.7) at leading order

$$\phi_{0xx}^i + \phi_{0yy}^i = 0 \quad \text{in} \quad -1 < y < 0, \qquad \phi_0^i = 0 \quad \text{at} \quad y = -1, \qquad \phi_{0x}^i = 0 \quad \text{at} \quad y = 0, \tag{4.10}$$

together with $F_{0x}^i = 0$. Thus

$$\phi_0^i = (y + 1)\Phi_0^i(\zeta, \tau, T), \qquad F_0^i = F_0^i(\zeta, \tau, T). \tag{4.11}$$

At second order, the boundary conditions (1.5a) and (1.6) yield

$$\phi_{1x}^i(x, 0, \zeta, \tau, T) = F_{0\tau}^i - \Phi_{0\zeta}^i + \beta_1(|Y_0|^2 - |X_0|^2)_\zeta,$$
$$(1 - S)F_{1x}^i - SF_{1xxx}^i = \Phi_{0\tau}^i - (1 - S)F_{0\zeta}^i - \beta_3(|X_0|^2 + |Y_0|^2)_\zeta$$

at $y = 0$. Since the right-hand sides of these two equations are independent of the fast variable x and both ϕ_1^i and F_1^i must be bounded in x, it follows that:

$$\Phi_{0\zeta}^i - F_{0\tau}^i = \beta_1(|Y_0|^2 - |X_0|^2)_\zeta, \qquad \Phi_{0\tau}^i - v_p^2 F_{0\zeta}^i = \beta_3(|X_0|^2 + |Y_0|^2)_\zeta, \tag{4.12}$$

where

$$v_p = (1 - S)^{1/2} \tag{4.13}$$

is the phase velocity of long wavelength surface gravity waves. Eq. (4.12) must be integrated with the following additional conditions, which result from (1.8), (1.9) and (4.6),

$$\Phi_0^i(\zeta + 1, \tau, T) \equiv \Phi_0^i(\zeta, \tau, T), \qquad F_0^i(\zeta + 1, \tau, T) \equiv F_0^i(\zeta, \tau, T), \tag{4.14}$$

$$\int_0^1 F_0^i \, d\zeta = 0, \qquad \left|\int_0^\tau \Phi_{0\zeta}^i \, d\tau\right| + \left|\int_0^\tau F_0^i \, d\tau\right| = \text{bounded as } \tau \to \infty. \tag{4.15}$$

(ii) The leading order contributions to Eqs. (1.1) and (1.2) yield $X_{0\tau} - v_g X_{0\zeta} = Y_{0\tau} + v_g Y_{0\zeta} = 0$. Thus

$$X_0 = X_0(\xi, T), \qquad Y_0 = Y_0(\eta, T), \tag{4.16}$$

where ξ and η are the characteristic variables

$$\xi = \zeta + v_g \tau, \qquad \eta = \zeta - v_g \tau. \tag{4.17}$$

Moreover, according to (1.3)

$$X_0(\xi + 1, T) \equiv X_0(\xi, T), \qquad Y_0(\eta + 1, T) \equiv Y_0(\eta, T). \tag{4.18}$$

Substitution of these expressions into (4.12) followed by integration of the resulting equations yields

$$\Phi_0^i = \frac{\beta_1 v_p^2 + \beta_3 v_g}{v_g^2 - v_p^2}[|X_0|^2 - |Y_0|^2 - \langle |X_0|^2 - |Y_0|^2 \rangle^\zeta] + v_p[F^+(\zeta + v_p\tau, T) - F^-(\zeta - v_p\tau, T)], \tag{4.19}$$

$$F_0^i = \frac{\beta_1 v_g + \beta_3}{v_g^2 - v_p^2}[|X_0|^2 + |Y_0|^2 - \langle |X_0|^2 + |Y_0|^2 \rangle^\zeta] + F^+(\zeta + v_p\tau, T) + F^-(\zeta - v_p\tau, T)], \tag{4.20}$$

where $\langle\cdot\rangle^\zeta$ denotes the mean value over the slow spatial variable ζ, i.e.,

$$\langle G \rangle^\zeta = \int_0^1 G \, d\zeta, \tag{4.21}$$

and the functions $F^\pm$ are such that

$$F^\pm(\zeta + 1 \pm v_p\tau, T) \equiv F^\pm(\zeta \pm v_p\tau, T), \quad \langle F^\pm \rangle^\zeta = 0. \tag{4.22}$$

The particular solution of (4.19) and (4.20) yields the usual inviscid mean flow included in nearly inviscid theories (see [12] and references therein); the averaged terms are a consequence of the conditions (4.15), i.e., of volume conservation (cf. [12]) and the requirement that the nearly inviscid mean flow has a zero mean on the timescale τ; the latter condition is never imposed in strictly inviscid theories but is essential in the limit we are considering, as explained above. To avoid the breakdown of the solution (4.19) and (4.20) at $v_p = v_g$, we assume in addition that

$$|v_p - v_g| \sim 1. \tag{4.23}$$

The functions $F^\pm$ remain undetermined at this stage. In fact, they are not needed below because the evolution of both the viscous mean flow and the complex amplitudes is decoupled from these functions. However, at next order one finds that $F^\pm$ remain constant on the timescale T, but decay exponentially due to viscous effects (resulting from viscous dissipation in the secondary boundary layer attached to the bottom plate) on the timescale $t \sim (L/C_g)^{1/2}$.

(iii) The evolution equations for X_0 and Y_0 on the timescale T are readily obtained from Eqs. (1.1)–(1.3), invoking (4.1)–(4.6), (4.19), (4.20) and (4.22) and eliminating secular terms (i.e., requiring $|X_1|$ and $|Y_1|$ to be bounded on the timescale τ):

$$X_{0T} = i\alpha X_{0\xi\xi} - (\Delta + iD)X_0 + i[(\alpha_3 + \alpha_8)|X_0|^2 - \alpha_8\langle |X_0|^2\rangle^\xi - \alpha_4\langle |Y_0|^2\rangle^\eta]X_0 + i\alpha_5 M\langle \bar{Y}_0\rangle^\eta$$
$$+ i\alpha_6 \int_{-1}^0 g(y)\langle\langle \phi_{0y}^v\rangle^x\rangle^\zeta \, dy \, X_0, \tag{4.24}$$

$$Y_{0T} = i\alpha Y_{0\eta\eta} - (\Delta + iD)Y_0 + i[(\alpha_3 + \alpha_8)|Y_0|^2 - \alpha_8\langle |Y_0|^2\rangle^\eta - \alpha_4\langle |X_0|^2\rangle^\xi]Y_0 + i\alpha_5 M\langle \bar{X}_0\rangle^\xi$$
$$- i\alpha_6 \int_{-1}^0 g(y)\langle\langle \phi_{0y}^v\rangle^x\rangle^\zeta \, dy \, Y_0, \tag{4.25}$$

$$X_0(\xi + 1, T) \equiv X_0(\xi, T), \qquad Y_0(\eta + 1, T) \equiv Y_0(\eta, T). \tag{4.26}$$

Here ξ and η are the comoving variables defined in (4.17), and $\langle\cdot\rangle^x$, $\langle\cdot\rangle^\zeta$, $\langle\cdot\rangle^\xi$ and $\langle\cdot\rangle^\eta$ denote mean values over the variables x, ζ, ξ and η, respectively. Note that ζ averages over functions of X_0 are equivalent to ξ averages, while those over functions of Y_0 are equivalent to η averages. The real coefficient α_8 is given by

$$\alpha_8 = \frac{\alpha_6(2\omega/\sigma)(\beta_1 v_p^2 + \beta_3 v_g) + \alpha_7(\beta_1 v_g + \beta_3)}{v_g^2 - v_p^2}. \tag{4.27}$$

Eqs. (4.24) and (4.25) are independent of $F^\pm$ because of the second condition in (4.22).

Eqs. (4.24) and (4.25) depend on the horizontal velocity of the viscous mean flow, $-\phi_{0y}^v$. The equations and boundary conditions governing the evolution of this flow are derived by substituting Eqs. (4.3)–(4.6), (4.8)–(4.11), (4.19) and (4.20) into (1.4)–(1.7), obtaining

$$\phi_{0xx}^v + \phi_{0yy}^v = W_0^v, \tag{4.28}$$

$$W^v_{0T} - [\phi^v_{0y} + \langle |X_0|^2 - |Y_0|^2\rangle^\zeta g(y)]W^v_{0x} + \phi^v_{0x}W^v_{0y} = Re^{-1}(W^v_{0xx} + W^v_{0yy}) \quad \text{in} \quad -1 < y < 0, \tag{4.29}$$

$$\phi^v_{0x} = 0, \qquad \phi^v_{0yy} = \beta_2\langle |X_0|^2 - |Y_0|^2\rangle^\tau \quad \text{at} \quad y = 0, \tag{4.30}$$

$$\langle\langle W^v_{0y}\rangle^x\rangle^\zeta = \phi^v_0 = 0, \qquad \phi^v_{0y} = -\beta_4[\mathrm{i}\langle X_0\bar{Y}_0\rangle^\tau\,\mathrm{e}^{\mathrm{i}4\pi Nx/L} + \text{c.c.} + \langle |Y_0|^2 - |X_0|^2\rangle^\tau] \quad \text{at} \quad y = -1, \tag{4.31}$$

$$\phi^v_0(x + L, \zeta + 1, y, T) \equiv \phi^v_0(x, \zeta, y, T), \tag{4.32}$$

where $X_0 \equiv X_0(\zeta + v_g\tau, T)$, $Y_0 \equiv Y_0(\zeta - v_g\tau, T)$ are given by (4.24) and (4.25) and $\langle\cdot\rangle^\tau$ denotes averages over the timescale τ. The *effective Reynolds number* associated with this viscous mean flow is

$$Re = \frac{1}{C_g L^2}. \tag{4.33}$$

Some remarks about these equations and boundary conditions are now in order.

1. The viscous mean flow is associated with only a small free-surface deflection, $f^v \sim L^{-3}$ (see (4.9)), which plays no role in the evolution of this flow, as expected of a flow involving the excitation of viscous modes (see Section 2).
2. According to the scaling (4.1) and the definitions (2.11), (2.12) and (4.33), the effective Reynolds number *Re* is large, and ranges from logarithmically large values if $k \sim |\ln C_g|$ to $O(C_g^{-1/2})$ if $k \sim 1$. However, even in the latter limit we must retain the viscous terms in (4.29) in order to account for the second boundary conditions in (4.30) and (4.31). Of course, if $Re \gg 1$ vorticity diffusion is likely to be confined to thin layers, but the structure and location of all these layers cannot be anticipated in an obvious way (see below) and in this case we must rely on numerical computations for realistically large values of *Re*.
3. The viscous mean flow is driven by the short gravity–capillary waves through the averaged terms in the boundary conditions (4.30) and (4.31). The quantity $\langle |X_0|^2 - |Y_0|^2\rangle^\tau = \langle |X_0|^2\rangle^\xi - \langle |Y_0|^2\rangle^\eta$ (=0, see below) depends only on T, but $\langle X_0\bar{Y}_0\rangle^\tau$ (which will play a major role below) depends on both ζ and T (unless either X_0 or Y_0 is spatially uniform). Thus, because of the boundary condition (4.31), ϕ^v_0 and W^v_0 depend on both the fast and slow horizontal spatial variables x and ζ. Unfortunately, the dependence of ϕ^v_0 and W^v_0 on x cannot be obtained in closed form (except, of course, in the uninteresting limit $Re \to 0$), and we must rely, once again, on numerical computations for realistically large values of L.
4. Observe that the boundary conditions (4.30b) and (4.31c) contain inhomogeneous forcing terms that are averages over the intermediate timescale τ. Like the oscillatory terms $F^\pm$ in Eqs. (4.19) and (4.20) the omitted terms oscillate on this timescale and hence generate secondary boundary layers. The contributions from these boundary layers are all subdominant and have no effect at the order considered.
5. The dominant forcing of the viscous mean flow comes from the lower boundary. This forcing vanishes exponentially when $k \gg 1$ leaving only a narrow range of wavenumbers within which such a mean flow is forced while $\delta = O(C_g)$, see Fig. 1. Thus in most cases in which viscous mean flow is present one may assume that $\delta = O(C_g^{1/2})$. Note, however, that in fully three-dimensional situations (such as that in [29]) in which lateral walls are included a viscous mean flow will be present even when $k \gg 1$ because the forcing of the mean flow in the oscillatory boundary layers attached to the lateral walls remains.

The form of the parametric forcing terms in (4.24) and (4.25) allows a further simplification of the system (4.24), (4.25), (4.28)–(4.32). With the change of variables

$$X_0 = \tilde{X}_0\,\mathrm{e}^{-2\pi\mathrm{i}N\theta/L}, \qquad Y_0 = \tilde{Y}_0\,\mathrm{e}^{2\pi\mathrm{i}N\theta/L}, \tag{4.34}$$

where $\theta \equiv \theta(T)$ obeys

$$\theta'(T) = -(2\pi N)^{-1} L\alpha_6 \int_{-1}^{0} g(y)\langle\langle\phi^v_{0y}\rangle^x\rangle^\zeta \, \mathrm{d}y, \tag{4.35}$$

the mean flow decouples, and Eqs. (4.24)–(4.26) become

$$\tilde{X}_{0T} = \mathrm{i}\alpha\tilde{X}_{0\xi\xi} - (\Delta + \mathrm{i}D)\tilde{X}_0 + \mathrm{i}[(\alpha_3 + \alpha_8)|\tilde{X}_0|^2 - \alpha_8\langle|\tilde{X}_0|^2\rangle^\xi - \alpha_4\langle|\tilde{Y}_0|^2\rangle^\eta]\tilde{X}_0 + \mathrm{i}\alpha_5 M\langle\bar{\tilde{Y}}_0\rangle^\eta, \tag{4.36}$$

$$\tilde{Y}_{0T} = \mathrm{i}\alpha\tilde{Y}_{0\eta\eta} - (\Delta + \mathrm{i}D)\tilde{Y}_0 + \mathrm{i}[(\alpha_3 + \alpha_8)|\tilde{Y}_0|^2 - \alpha_8\langle|\tilde{Y}_0|^2\rangle^\eta - \alpha_4\langle|\tilde{X}_0|^2\rangle^\xi]\tilde{Y}_0 + \mathrm{i}\alpha_5 M\langle\bar{\tilde{X}}_0\rangle^\xi, \tag{4.37}$$

$$\tilde{X}_0(\xi + 1, T) \equiv \tilde{X}_0(\xi, T), \qquad \tilde{Y}_0(\eta + 1, T) \equiv \tilde{Y}_0(\eta, T). \tag{4.38}$$

Except for differences in notation these equations are identical to the equations already extensively investigated by Martel et al. [17]. In constructing their nonlocal amplitude equations Martel et al. deliberately ignored the possible presence of viscous mean flow in order to write down a tractable system of equations. Consequently, they considered their equations to be a phenomenological description of the Faraday system rather than a quantitatively precise one. The present paper shows that the equations originally written down in Ref. [17] do in fact provide a *quantitative* description of this system, and establishes the conditions under which they do so. In addition, the systematic derivation of these equations indicates that the omitted viscous mean flow does in fact play a role in that it affects the spatial phase of the pattern, and the manner in which it does so. In view of the exact relation

$$\frac{\mathrm{d}\langle|\tilde{X}_0|^2 - |\tilde{Y}_0|^2\rangle^\zeta}{\mathrm{d}T} = -2\Delta\langle|\tilde{X}_0|^2 - |\tilde{Y}_0|^2\rangle^\zeta, \quad \Delta > 0,$$

we may assume that

$$\langle|\tilde{X}_0|^2 - |\tilde{Y}_0|^2\rangle^\zeta \equiv \langle|\tilde{X}_0|^2\rangle^\xi - \langle|\tilde{Y}_0|^2\rangle^\eta = 0, \tag{4.39}$$

and rewrite the viscous mean flow equations (4.28)–(4.32) in the form

$$\phi^v_{0xx} + \phi^v_{0yy} = W^v_0, \tag{4.40}$$

$$W^v_{0T} - \phi^v_{0y}W^v_{0x} + \phi^v_{0x}W^v_{0y} = Re^{-1}(W^v_{0xx} + W^v_{0yy}) \quad \text{in} \quad -1 < y < 0, \tag{4.41}$$

$$\phi^v_{0x} = \phi^v_{0yy} = 0 \quad \text{at} \quad y = 0, \tag{4.42}$$

$$\langle\langle W^v_{0y}\rangle^x\rangle^\zeta = \phi^v_0 = 0, \qquad \phi^v_{0y} = 2\beta_4 R_0(\zeta, T)\sin\left[\frac{4\pi N(x - \theta - \theta_0)}{L}\right] \quad \text{at} \quad y = -1, \tag{4.43}$$

$$\phi^v_0(x + L, \zeta + 1, y, T) \equiv \phi^v_0(x, \zeta, y, T), \tag{4.44}$$

where the functions $R_0 = R_0(\zeta, T)$ and $\theta_0 = \theta_0(\zeta, T)$ are defined by

$$R_0\,\mathrm{e}^{-4\pi \mathrm{i}N\theta_0/L} = \langle\tilde{X}_0\bar{\tilde{Y}}_0\rangle^\tau \equiv \sum_{-\infty}^{+\infty} x_n(T)\bar{y}_{-n}(T)\,\mathrm{e}^{4\pi \mathrm{i}n\zeta}, \tag{4.45}$$

and $\{x_n\}$, $\{y_n\}$ are the Fourier coefficients in the expansions

$$\tilde{X}_0(\xi, T) = \sum_{-\infty}^{+\infty} x_n(T)\,\mathrm{e}^{2\pi \mathrm{i}n\xi}, \qquad \tilde{Y}_0(\eta, T) = \sum_{-\infty}^{+\infty} y_n(T)\,\mathrm{e}^{2\pi \mathrm{i}n\eta}. \tag{4.46}$$

Eqs. (4.35), (4.40)–(4.44) describe the resulting coupled evolution of the spatial phase θ of the pattern and of the viscous mean flow, and constitute a separate dynamical system forced by the amplitude dynamics studied by Martel et al. [17] via the functions R_0 and θ_0 appearing in the boundary condition (4.43). Note that the viscous mean flow is forced by the bottom boundary layer only, and that this forcing vanishes exponentially when $k \gg 1$ (see remarks (3) and (5) above).

It is worth remarking that the condition (4.39) prevents the existence of spatially uniform progressive waves (i.e., solutions of the type $|A|$ = constant, $|B|$ = constant, $|A| \neq |B|$) as solutions of the nonlocal equations (4.36) and (4.37). However, a number of other solution types is possible. These split naturally into solutions lying in the invariant subspace $|A| = |B|$ and those with $|A| \neq |B|$. With periodic boundary conditions the former can be either symmetric with respect to a spatial reflection $x \to -x$ or nonsymmetric. As discussed in more detail by Martel et al. [17] solutions of the former type may be uniform and steady, nonuniform and steady, time-periodic and chaotic. The same is also true for the nonsymmetric solutions, but in this case the spatial asymmetry is responsible for the presence of a net drift of the solution. This is a consequence of the periodic boundary conditions (4.38), and introduces an additional, typically small frequency into the solution. Drifts of this type are called type I in order to distinguish them from type II drifts that are due to an asymmetry between the *amplitudes* of left- and right-traveling wavetrains. Type II drifts are present even if both $|A|$ and $|B|$ are reflection-symmetric about some point x (not necessarily the same), provided only that $|A| \neq |B|$, modulo translation. Moreover, when both of these two types of asymmetry are present multiply periodic drifts will result, as discussed and illustrated by Martel et al. As a result the variety of possible solutions to even the simplest set of equations, the decoupled amplitude equations, is quite substantial, and each such solution is accompanied in addition by a viscous mean flow. This mean flow responds to type I drifts in the amplitudes $\tilde{X}_0$, $\tilde{Y}_0$ through the amplitude R_0, and to type II drifts through the dependence of the forcing on the phase θ_0. The explicit computation of the relevant coefficients performed in this paper can be used to identify physically relevant regimes in the classification of Ref. [17].

4.2. Capillary waves in the microgravity limit

As $1 - S \to 0$, the phase velocity of the long (gravity) waves v_p vanishes (see Eq. (4.13)) and a part of the nearly inviscid mean flow defined above resonates with the viscous mean flow. In fact, because of the decomposition (4.3)–(4.7) this resonant interaction is captured completely in the viscous mean flow, which will now involve a significant free surface deformation.

We suppose that

$$(1 - S)L^2 = \Lambda \sim 1, \tag{4.47}$$

and consider the expansions

$$(A, B) = L^{-1}(X_0, Y_0) + L^{-2}(X_1, Y_1) + \cdots, \qquad (\psi^i, f^i) = L^{-2}(\phi_0^i, F_0^i) + L^{-3}(\phi_1^i, F_1^i) + \cdots, \tag{4.48}$$

$$\Omega^i = L^{-3}W_0^i + \cdots, \qquad (\psi^v, \Omega^v) = L^{-2}(\phi_0^v, W_0^v) + \cdots, \qquad f^v = L^{-1}F_0^v + \cdots. \tag{4.49}$$

The resulting analysis proceeds as in Section 4.1. The main differences are that Eqs. (4.12), (4.19) and (4.20) must be replaced, respectively, by

$$\Phi_{0\zeta}^i - F_{0\tau}^i = \beta_1(|Y_0|^2 - |X_0|^2)_\zeta, \qquad \Phi_{0\tau}^i = \beta_3(|X_0|^2 + |Y_0|^2)_\zeta, \tag{4.50}$$

$$\Phi_0^i = \left(\frac{\beta_3}{v_\mathrm{g}}\right)[|X_0|^2 - |Y_0|^2 - \langle |X_0|^2 - |Y_0|^2\rangle^\zeta], \tag{4.51}$$

$$F_0^i = \left(\frac{\beta_1 v_\mathrm{g} + \beta_3}{v_\mathrm{g}^2}\right)[|X_0|^2 + |Y_0|^2 - \langle |X_0|^2 + |Y_0|^2\rangle^\zeta]. \tag{4.52}$$

Because of the conditions (4.15) the solution of the homogeneous part of (4.50) vanishes identically. Thus X_0 and Y_0 are still given by (4.24)–(4.26), but S must now be replaced by 1 everywhere in the expressions for the coefficients $\alpha_3, \ldots, \alpha_6$ and α_8, and the parameters Δ, D and M. Eqs. (4.28) and (4.29) and the conditions (4.31) and (4.32) still apply, but the boundary conditions (4.30) must be replaced by

$$\phi^v_{0x} = F^v_{0T}, \qquad \phi^v_{0yy} = \beta_2 \langle |X_0|^2 - |Y_0|^2 \rangle^\tau \quad \text{at} \quad y = 0. \tag{4.53}$$

Moreover, from (1.6)

$$\phi^v_{0yT} = \Lambda F^v_{0\zeta} - F^v_{0\zeta\zeta\zeta} \quad \text{at} \quad y = 0. \tag{4.54}$$

If we redefine the complex amplitudes as in (4.34) the amplitude equations (4.36)–(4.38) still uncouple from the viscous mean flow, and for large times (on the timescale T) the result (4.39) implies that the phase shift θ and the viscous mean flow evolve according to

$$\theta'(T) = -(2\pi N)^{-1} L\alpha_6 \int_{-1}^{0} g(y) \langle\langle \phi^v_{0y} \rangle^x \rangle^\zeta \, \mathrm{d}y, \tag{4.55}$$

$$\phi^v_{0xx} + \phi^v_{0yy} = W^v_0, \tag{4.56}$$

$$W^v_{0T} - \phi^v_{0y} W^v_{0x} + \phi^v_{0x} W^v_{0y} = Re^{-1}(W^v_{0xx} + W^v_{0yy}) \quad \text{in} \quad -1 < y < 0, \tag{4.57}$$

$$\phi^v_{0x} = F^v_{0T}, \qquad \phi^v_{0yy} = 0, \qquad \phi^v_{0yT} = \Lambda F^v_{0\zeta} - F^v_{\zeta\zeta\zeta} \quad \text{at} \quad y = 0, \tag{4.58}$$

$$\langle\langle W^v_{0y} \rangle^x \rangle^\zeta = \phi^v_0 = 0, \qquad \phi^v_{0y} = 2\beta_4 R_0(\zeta, T) \sin\left[\frac{4\pi N(x - \theta - \theta_0)}{L}\right] \quad \text{at} \quad y = -1, \tag{4.59}$$

$$\phi^v_0(x + L, \zeta + 1, y, T) \equiv \phi^v_0(x, \zeta, y, T) \tag{4.60}$$

with the functions $R_0 = R_0(\zeta, T)$ and $\theta_0 = \theta_0(\zeta, T)$ still given by (4.45) and (4.46) in terms of the solutions of the decoupled system (4.36)–(4.38). Thus the structure of the problem in the microgravity limit and in the presence of gravity is fundamentally the same, and the study of the decoupled system (4.36)–(4.38) by Martel et al. [17] applies to both.

5. Concluding remarks

In this paper, we have given a systematic derivation of the basic equations governing the interaction between parametrically excited surface gravity–capillary waves in nearly inviscid fluids and a mean flow. We have argued that in such fluids, depending on the aspect ratio of the container, the hydrodynamic (or bulk) modes decay more slowly than the surface waves and that such modes cannot therefore be omitted from a consistent weakly nonlinear description of these systems. Since the excitation of these modes manifests itself as a (viscous) mean flow a description in terms of equations of the type summarized in (1.1)–(1.9) is inevitable; we determined here explicitly the conditions under which this is the case. In general, traveling waves are associated with the presence of an inviscid mean flow as well [12], and, consequently, the (total) mean flow in these equations includes contributions from both sources. Under the conditions of Section 4, the viscous mean flow is driven by a tangential velocity boundary condition imposed on the largely inviscid flow in the bulk. This boundary condition describes the net effect on the bulk of the presence of an oscillatory viscous boundary layer attached to the bottom of the container, as first discussed by Schlichting [20]. In general, we found that the lower boundary is more effective at driving the viscous mean flow than a similar boundary layer at the free surface which provides a stress boundary condition on the mean flow in the bulk [21]. In contrast, the purely inviscid flow that may be present is a consequence of the mechanism by which the waves are excited [30].

A careful examination of the analysis that led us to Eqs. (1.1)–(1.9) shows that these in fact apply under the conditions

$$k(|\psi_x| + |\psi_y|) \ll \omega, \qquad |f| + |f_x| \ll 1, \qquad L^{-1} \ll k, \tag{5.1}$$

or equivalently

$$k(|A| + |B|) + |f_x^{\mathrm{m}}| \ll 1, \qquad k|\psi_x^{\mathrm{m}}| \ll \omega, \tag{5.2}$$

obtained from (2.17), and the condition

$$L \ll \frac{v_{\mathrm{g}}}{\delta + |d| + |\alpha_5|\mu} \tag{5.3}$$

that relaxes somewhat the requirement (2.19). Here v_{g} is the (nondimensional) group velocity of the surface waves, defined in (3.24), α_5 is given in (3.25) and we assumed that the smallest spatial scale is k^{-1}. The condition (5.1) can be stated succinctly as requiring that the nonlinearity be weak and the aspect ratio of the system be large compared to the nondimensional wavelength of the surface waves; the condition (5.3) requires that the terms accounting for inertia and propagation at the group velocity in the amplitude equations (1.1) and (1.2) be much larger than the remaining terms. In addition, the requirements

$$(1 - S)k^2 + Sk^4 \gg C_g^2, \qquad k^{3/2}(1 - S + Sk^2)^{-1/2} \ll C_g^{-1}, \tag{5.4}$$

or equivalently

$$C_g \ll \omega, \qquad C_g^{1/2}\omega^{3/2} \ll 1 - S + \frac{S\omega}{C_g}, \tag{5.5}$$

are imposed implicitly both on the carrier wavenumber k as well as on all wavenumbers associated with the (viscous) mean flow. These conditions guarantee that the thickness of the associated boundary layers will be small compared to the depth (if $k \ll 1$) or compared to the wavelength (if $k \gg 1$), see Fig. 2. Since the lowest wavenumber of the mean flow is $k = 2\pi/L$, condition (5.4) implies, in particular, that

$$(1 - S)L^{-2} + (2\pi)^2 SL^{-4} \gg C_g^2. \tag{5.6}$$

Additional assumptions, such as the requirement that all second and third resonances are avoided (Wilton ripples) and that in Section 4 $v_{\mathrm{p}} \neq v_{\mathrm{g}}$, appear in the course of the analysis.

The resulting GCAMF equations were derived with one further but essential assumption, namely that the spatial Fourier transforms of both the basic wavetrains and of the associated mean flow remain peaked around a set of discrete wavenumbers (two in the case of the carrier wavenumbers, and infinite in the case of the mean flow) for all time. This assumption concerns the small scale structure of the solution and it excludes the generation of small scales that may arise if the aspect ratio is too large. In fact, it is not necessary to assume that these wavenumbers are commensurate; it is only necessary that some scale separation is present so that the averages introduced are well defined. In a numerical solution starting with given initial conditions this assumption may either fail, indicating that the aspect ratio is too large or be found to hold for timescales of interest. The equations derived here describe the latter situation. We have also seen that under certain specific conditions it is possible to distinguish unambiguously the two types of mean flow (viscous and inviscid), and described in Section 4 a particularly useful instance in which this can be done. The equations derived there by means of an additional multiple scale analysis led to a surprisingly simple description of the resulting system, consisting of a pair of decoupled, albeit nonlocal equations of the type already studied at length in [17], together with a set of equations governing the interaction of the spatial phase of the wave amplitudes and the viscous mean flow. Since the Reynolds number of this flow can be (indeed must be) substantial these equations must be treated numerically as already done in other circumstances [31,32]. Such solutions will be reported elsewhere. To the extent the presence of lateral walls may be ignored the results may provide a quantitative description of the plethora of experimental results on the Faraday system with nearly inviscid fluids [3,14,29,33].

It is useful to consider an explicit experimental realization of the theory described here. We focus on an annular container with a 110 mm diameter filled with extremely clean (see [34] and references therein) water to an 8 mm depth (as in [29]), but with a forcing frequency of 10.6 Hz. Using $T = 72$ dyn/cm, we calculate the gravity–capillary time (see Section 2) to be 0.027 s and hence that $\omega = 0.87$. The remaining dimensionless parameters of the theory then take the values $L = 43.2$, $S = 0.1$, $C_g = 4.2 \times 10^{-4}$, $k = 1$, $\delta = 3.8 \times 10^{-3}$, $d = -0.0102$ and $v_g = 0.767$.

Under these conditions the requirements (2.17)–(2.19) for the validity of the theory are fulfilled, and two-dimensional waves of small steepness are described by the GCAMF equations derived above. An experiment of the above type, designed to minimize three-dimensional effects, could therefore test the predictions of these equations.

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Appendix A. Boundary conditions on the bulk flow

A.1. Stokes boundary layer near the bottom plate

For convenience, we take the small parameter ε as the order of magnitude of the complex amplitudes, namely, we assume that

$$|A| \sim |B| \sim \varepsilon, \qquad |\psi^{\mathrm{m}}| \sim \varepsilon^2 \tag{A.1}$$

in (3.2). We introduce the stretched coordinate

$$\tilde{y} = \frac{y+1}{C_g^{1/2}}, \tag{A.2}$$

and seek a solution in the form

$$(C_g^{-1/2}\psi, C_g^{1/2}\Omega) = \varepsilon(\tilde{\psi}_1, \tilde{\Omega}_1)\,\mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.} + \varepsilon^2[(\tilde{\psi}_2, \tilde{\Omega}_2) + \mathrm{OT}] + \cdots, \tag{A.3}$$

where for $j = 1$ and 2, $\tilde{\psi}_j$ and $\tilde{\Omega}_j$ depend weakly on time and OT stands for *oscillatory terms* on the time scale $t \sim 1$. Substitution of (A.2) and (A.3) into (2.1) and (2.4) yields

$$\tilde{\psi}_{1\tilde{y}\tilde{y}} = \tilde{\Omega}_1, \qquad \Omega_{1\tilde{y}\tilde{y}} = \mathrm{i}\omega\Omega_1 \quad \text{in } 0 < \tilde{y} < \infty, \qquad \tilde{\psi}_1 = \tilde{\psi}_{1\tilde{y}} = 0 \quad \text{at } \tilde{y} = 0, \tag{A.4}$$

$$\tilde{\psi}_{2\tilde{y}\tilde{y}} = \tilde{\Omega}_2, \qquad \tilde{\Omega}_{2\tilde{y}\tilde{y}} = -\bar{\tilde{\psi}}_{1\tilde{y}}\tilde{\Omega}_{1x} + \tilde{\psi}_{1x}\bar{\tilde{\Omega}}_{1\tilde{y}} + \text{c.c.} \quad \text{in } 0 < \tilde{y} < \infty,$$

$$\tilde{\psi}_2 = \tilde{\psi}_{2\tilde{y}} = 0 \quad \text{at } \tilde{y} = 0, \tag{A.5}$$

where the overbar stands for the complex conjugate. In addition, we have

$$\tilde{\Omega}_1 = \tilde{\Omega}_2 = 0 \quad \text{as} \quad \tilde{y} \to \infty \tag{A.6}$$

as required by matching with the solution in the bulk, which is completed below. Integration of (A.4) yields

$$\tilde{\psi}_1 = K_1(x) \frac{\sqrt{\mathrm{i}\omega}\tilde{y} + \exp(-\sqrt{\mathrm{i}\omega}\tilde{y}) - 1}{\sqrt{\mathrm{i}\omega}}, \tag{A.7}$$

where (A.6) has been taken into account; $K_1(x)$ is determined by matching conditions between (3.2) and (A.3) at order ε (see also (A.1) and (A.2)) and is given by

$$\varepsilon K_1 = \Psi_0'(-1)(A\,\mathrm{e}^{ikx} - B\,\mathrm{e}^{-ikx}). \tag{A.8}$$

Substitution into the second equation (A.5) now yields

$$\tilde{\Omega}_{2\tilde{y}\tilde{y}} = \sqrt{\mathrm{i}\omega}\bar{K}_1 K_{1x}[-\mathrm{e}^{-\sqrt{\mathrm{i}\omega}\tilde{y}} + (-1 + \sqrt{\mathrm{i}\omega}\tilde{y})\,\mathrm{e}^{-\sqrt{-\mathrm{i}\omega}\tilde{y}} + 2\,\mathrm{e}^{-\sqrt{2\omega}\tilde{y}}] + \text{c.c.}, \tag{A.9}$$

and we need only integrate this equation twice, using (A.6), and integrate (A.5a) once, using (A.5c), to obtain

$$\tilde{\psi}_{2\tilde{y}} = \bar{K}_1 K_{1x}\left[\frac{3-3\mathrm{i}}{2\omega} + \frac{\mathrm{e}^{-\sqrt{\mathrm{i}\omega}\tilde{y}}}{\mathrm{i}\omega} + \left(\frac{3\mathrm{i}-1}{\omega} + \frac{\tilde{y}}{\sqrt{-\mathrm{i}\omega}}\right)\mathrm{e}^{-\sqrt{-\mathrm{i}\omega}\tilde{y}} - \frac{1+\mathrm{i}}{2\omega}\mathrm{e}^{-\sqrt{2\omega}\tilde{y}}\right] + \text{c.c.} \tag{A.10}$$

With this expression for $\tilde{\psi}_{2\tilde{y}}$, we can apply matching conditions between (3.2) and (A.3) at order ε^2 (see also (A.1), (A.2) and (A.8)) to obtain

$$\begin{aligned} &\psi_x^{\mathrm{m}}(x,-1,t) = \mathrm{o}(\varepsilon^2), \\ &\psi_y^{\mathrm{m}}(x,-1,t) = \varepsilon^2\left[\frac{3(1-\mathrm{i})\bar{K}_1 K_{1x}}{2\omega} + \text{c.c.}\right] = -3\omega k(1-\sigma^2)\sigma^{-2}(\mathrm{i}A\bar{B}\,\mathrm{e}^{2\mathrm{i}kx} + \text{c.c.} + |B|^2 - |A|^2), \end{aligned} \tag{A.11}$$

where we have used the relation $\Psi_0'(-1)^2 = \omega^2(1-\sigma^2)\sigma^{-2}$ obtained from (2.13). The boundary condition (1.7b) now follows with β_4 as given in (3.20).

A.2. *Oscillatory boundary layer near the free surface*

For convenience, we assume again that (A.1) holds, introduce a stretched coordinate attached to the interface

$$\tilde{y} = \frac{y - f(x,t)}{C_g^{1/2}}, \tag{A.12}$$

and seek a solution in the form

$$\begin{aligned} &\psi = \varepsilon(\tilde{\psi}_1 + C_g^{1/2}\tilde{\psi}_2 + C_g\tilde{\psi}_3)\,\mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.} + \varepsilon^2(\tilde{\psi}_4 + C_g^{1/2}\tilde{\psi}_5 + C_g\tilde{\psi}_6 + \text{OT}) + \cdots, \\ &\Omega = \varepsilon(\tilde{\Omega}_1 + \cdots)\,\mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.} + \varepsilon^2(\tilde{\Omega}_2 + \cdots + \text{OT}) + \cdots, \end{aligned} \tag{A.13}$$

i.e., we anticipate that $\Omega \sim \varepsilon$. We also rewrite (3.1) in the form

$$f = \varepsilon(f_1 + C_g^{1/2} f_2)\,\mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.} + \cdots. \tag{A.14}$$

Substitution of (A.12)–(A.14) into (2.1) and (2.2) now yields the following system of equations and boundary conditions:

(A) *For the oscillatory part of* (A.13)

$$\tilde{\psi}_{1\tilde{y}\tilde{y}} = \tilde{\psi}_{2\tilde{y}\tilde{y}} = \tilde{\psi}_{3\tilde{y}\tilde{y}} + \tilde{\psi}_{1xx} - \tilde{\Omega}_1 = \tilde{\Omega}_{1\tilde{y}\tilde{y}} - \mathrm{i}\omega\tilde{\Omega}_1 = 0 \quad \text{in} \quad -\infty < \tilde{y} < 0, \tag{A.15}$$

$$\tilde{\psi}_{1x} - \mathrm{i}\omega f_1 = \tilde{\psi}_{2x} - \mathrm{i}\omega f_2 = \tilde{\psi}_{3\tilde{y}\tilde{y}} - \tilde{\psi}_{1xx} = 0 \quad \text{at} \quad \tilde{y} = 0, \tag{A.16}$$

$$\tilde{\psi}_{1\tilde{y}} = \tilde{\Omega}_1 = 0 \quad \text{as} \quad \tilde{y} \to -\infty. \tag{A.17}$$

Thus

$$\tilde{\psi}_{1\tilde{y}} = 0, \qquad \tilde{\psi}_{1x} = \mathrm{i}\omega f_1, \qquad \tilde{\psi}_{2\tilde{y}} = K_2, \qquad \tilde{\psi}_{2x} = \mathrm{i}\omega f_2, \qquad \tilde{\psi}_{3\tilde{y}} = K_3, \qquad \tilde{\Omega}_1 = 2\mathrm{i}\omega f_{1x}\,\mathrm{e}^{\sqrt{\mathrm{i}\omega}\tilde{y}} \tag{A.18}$$

with K_2 and K_3 independent of $\tilde{y}$.

(B) *For the slowly varying part of* (A.13)

$$\begin{aligned}&\tilde{\psi}_{4\tilde{y}\tilde{y}} = \tilde{\psi}_{5\tilde{y}\tilde{y}} = \tilde{\psi}_{6\tilde{y}\tilde{y}} + \tilde{\psi}_{4xx} - \tilde{\Omega}_2 - (2\bar{f}_{1x}K_{2x} + \bar{f}_{1xx}K_2 + \text{c.c.}) = 0 \quad \text{in} \quad -\infty < \tilde{y} < 0,\\ &\tilde{\Omega}_{2\tilde{y}\tilde{y}} + (\bar{K}_2\tilde{\Omega}_{1x} - \tilde{y}\bar{K}_{2x}\tilde{\Omega}_{1\tilde{y}} + \text{c.c.}) = 0 \quad \text{in} \quad -\infty < \tilde{y} < 0,\end{aligned} \tag{A.19}$$

$$\tilde{\psi}_{4x} = \tilde{\psi}_{6\tilde{y}\tilde{y}} - \tilde{\psi}_{4xx} + (-2\bar{f}_{1x}K_{2x} + \bar{f}_{1xx}K_2 + \text{c.c.}) = 0 \quad \text{at} \quad \tilde{y} = 0, \tag{A.20}$$

$$\tilde{\psi}_{4\tilde{y}} = \tilde{\Omega}_{2\tilde{y}} = 0 \quad \text{as} \quad \tilde{y} \to -\infty, \tag{A.21}$$

where we have taken into account (A.18). Thus

$$\begin{aligned}&\tilde{\psi}_4 = 0, \qquad \tilde{\psi}_{5\tilde{y}} = K_4,\\ &\tilde{\psi}_{6\tilde{y}\tilde{y}} = 2[\bar{K}_{2x}f_{1x}(\sqrt{\mathrm{i}\omega}\tilde{y} - 2) - f_{1xx}\bar{K}_2]\,\mathrm{e}^{\sqrt{\mathrm{i}\omega}\tilde{y}} + f_{1xx}\bar{K}_2 + 6f_{1x}\bar{K}_{2x} + \text{c.c.},\end{aligned} \tag{A.22}$$

where K_4 is again independent of $\tilde{y}$. For matching with the solution in the bulk we need the $\tilde{y} \to -\infty$ limit of horizontal velocity and stress:

$$\psi_y(x, f, t) = \varepsilon\tilde{\psi}_{2\tilde{y}}\,\mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.} + \varepsilon^2(\tilde{\psi}_{5\tilde{y}} + \text{OT}) + \cdots = \varepsilon K_2\,\mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.} + \varepsilon^2(K_4 + \text{OT}) + \cdots, \tag{A.23}$$

$$\begin{aligned}\psi_{yy}(x, f, t) &= \varepsilon\tilde{\psi}_{3\tilde{y}}\,\mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.} + \varepsilon^2(\tilde{\psi}_{6\tilde{y}} + \text{OT}) + \cdots\\ &= \varepsilon K_3\,\mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.} + \varepsilon^2(6\bar{f}_{1x}K_{2x} + \bar{f}_{1xx}K_2 + \text{c.c.} + \text{OT}) + \cdots.\end{aligned} \tag{A.24}$$

On the other hand, for the solution in the bulk $\psi_y(x, f(x,t), t) = \psi_y(x, 0, t) + \cdots$ and $\psi_{yy}(x, f(x,t), t) = \psi_{yy}(x, 0, t) + \psi_{yyy}(x, 0, t)f(x, t) + \cdots$ or, according to (3.1) and (3.2)

$$\psi_y(x, f, t) = \Psi_0'(0)\,\mathrm{e}^{\mathrm{i}\omega t}(A\,\mathrm{e}^{\mathrm{i}kx} - B\,\mathrm{e}^{-\mathrm{i}kx}) + \text{c.c.} + \cdots, \tag{A.25}$$

$$\begin{aligned}\psi_{yy}(x, f, t) = {}&\Psi_0''(0)\,\mathrm{e}^{\mathrm{i}\omega t}(A\,\mathrm{e}^{\mathrm{i}kx} - B\,\mathrm{e}^{-\mathrm{i}kx}) + \text{c.c.}\\ &+ [\Psi_0'''(0)(A\,\mathrm{e}^{\mathrm{i}kx} - B\,\mathrm{e}^{-\mathrm{i}kx})(\bar{A}\,\mathrm{e}^{-\mathrm{i}kx} + \bar{B}\,\mathrm{e}^{\mathrm{i}kx}) + \text{c.c.} + \psi_{yy}^{\mathrm{m}}(x, 0, t) + \text{OT}] + \cdots.\end{aligned} \tag{A.26}$$

Identification of (3.1) with (A.14) and (A.23) with (A.25) yields

$$\varepsilon f_1 = A\,\mathrm{e}^{\mathrm{i}kx} + B\,\mathrm{e}^{-\mathrm{i}kx}, \qquad \varepsilon K_2 = \Psi_0'(0)(A\,\mathrm{e}^{\mathrm{i}kx} - B\,\mathrm{e}^{-\mathrm{i}kx}). \tag{A.27}$$

Finally, matching expressions (A.24) with (A.26) gives

$$\psi^{\rm m}_{yy}(x,0,t) = [5k^2\Psi_0'(0) - \Psi_0'''(0)](A\,{\rm e}^{ikx} - B\,{\rm e}^{-ikx})(\bar{A}\,{\rm e}^{ikx} + \bar{B}\,{\rm e}^{-ikx}) + {\rm c.c.} = \frac{8\omega k^2(|A|^2 - |B|^2)}{\sigma},$$
$$\psi^{\rm m}_{x}(x,0,t) = {\rm o}(\varepsilon^2), \tag{A.28}$$

and hence the coefficient β_2 in Eq. (3.20).

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