Weakly-nonlinear analysis of the Rayleigh-Taylor instability in a vertically vibrated, large aspect ratio container [1]

Victoria Lapuerta Francisco J. Mancebo José M. Vega

E.T.S.I. Aeronáuticos, Universidad Politécnica de Madrid. Plaza del Cardenal Cisneros, 3. 28040 Madrid. SPAIN

Abstract

We consider a horizontal liquid layer supported by air in a wide (as compared to depth) container, which is vertically vibrated with an appropriately large frequency, intending to counterbalance the Rayleigh-Taylor instability of the flat, rigid-body vibrating state. We apply a long-wave, weakly-nonlinear analysis that yields a generalized Cahn-Hilliard equation for the evolution of the fluid interface, with appropriate boundary conditions obtained by a boundary layer analysis. This equation shows that the stabilizing effect of vibration is like that of surface tension, and is used to analyze the linear stability of the flat state, and the local bifurcation at the instability threshold.

Key words: Rayleigh-Taylor instability, stabilization by forced vibration, free surface flows.
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1 Introduction and formulation

The Rayleigh-Taylor instability [1] appears when a heavy fluid is accelerated towards a lighter one and has a basic interest in Fluid Mechanics. The simplest configuration exhibiting this instability is that in which a horizontal heavy fluid layer (e.g., water or mineral oil) is supported by a lighter fluid (e.g., air); the destabilizing acceleration is provided by gravity. In this configuration, the instability can be counterbalanced by an imposed vertical vibration of the container, as already shown experimentally [2] and theoretically [3]-[4]. The main object of this paper is to provide a weakly nonlinear theory of this stabilizing effect in the limiting case when both the aspect ratio of the container and the vibrating frequency are appropriately large. In addition, we assume that the lower fluid (e.g. air) has negligible density and viscosity and thus can be neglected. See [5] for the general case.

Under the assumptions above we consider a wide cylindrical container with a horizontal size ℓ and depth h, which is placed in inverted position, with gravity acting downwards, and is vertically vibrated. We use the viscous time h^2/ν and the depth h for non-dimensionalization and a Cartesian coordinate system attached to the vibrating container, with the $z = 0$ plane on the unperturbed free surface, assumed to be horizontal. The nondimensional governing equations are

$$\nabla \cdot \boldsymbol{u} + \partial_z w = 0, \tag{1.1}$$

$$\partial_t(\boldsymbol{u}, w) + (\boldsymbol{u} \cdot \nabla + w\partial_z)(\boldsymbol{u}, w) = -(\nabla, \partial_z)p + (\Delta + \partial^2_{zz})(\boldsymbol{u}, w), \tag{1.2}$$

if $(x, y) \in \Omega$ and $f(x, y, t) < z < 1$, with boundary conditions

$$\boldsymbol{u} = \boldsymbol{0}, \quad w = 0 \quad \text{if } z = 1 \text{ and if } (x, y) \in \partial\Omega, \tag{1.3}$$

$$w = \partial_t f + \boldsymbol{u} \cdot \nabla f, \quad \partial_z \boldsymbol{u} + \nabla w = O\left(|\nabla \boldsymbol{u}||\nabla f| + (|\partial_z \boldsymbol{u}| + |\nabla w|)|\nabla f|^2\right) \quad \text{if } z = f, \tag{1.4}$$

$$\begin{aligned} -p + a\omega^2 f \cos(\omega t) + BC^{-2} f + C^{-2}\nabla \cdot [\nabla f/(1 + |\nabla f|^2)^{1/2}] \\ = -2\partial_z w + O\left(|\nabla \boldsymbol{u}| + (|\partial_z \boldsymbol{u}| + |\nabla w|)|\nabla f|\right) \quad \text{if } z = f, \end{aligned} \tag{1.5}$$

$$\nabla f \cdot \boldsymbol{n} = -D\partial_t f \quad \text{or} \quad f = 0 \quad \text{if } (x, y) \in \partial\Omega, \qquad \int_\Omega f(x, y, t)\, dxdy = 0, \tag{1.6}$$

and with appropriate initial conditions, where $\boldsymbol{u}$ and w are the horizontal and vertical velocity, $p =$ pressure $+ [a\omega^2 \cos(\omega t) + B/C^2]z$ is a modified pressure, a and ω are the amplitude and frequency of the imposed vibration, ∇, $\nabla\cdot$ and Δ are the horizontal gradient, divergence and Laplacian operators and f is the vertical free surface deflection, assumed along the paper to be such that $|\nabla f| \ll |f|$. Ω is the transversal cross-section of the container, assumed to be large and homothetic to a fixed bounded domain, $\partial\Omega$ is its boundary and $\boldsymbol{n}$ is the outward unit normal to $\partial\Omega$. The aspect ratio (or dimensionless characteristic size of Ω) is $L = \ell/h$, and $B = \rho g h^2/\sigma$ and $C = \nu\sqrt{\rho/(\sigma h)}$ are the *Bond number* and the *capillary number* respectively, where g is the gravitational acceleration and σ is the surface tension coefficient. In the boundary condition (1.6a) we assume that either the contact line is moving or fixed. In the former case we assume that the static contact angle is 90° and employ the usual phenomenological law (see, e.g. [6] and references given therein) to account for contact line dynamics; the phenomenological constant D is positive and thus the motion of the contact line is dissipative.

The limit considered in this paper is

$$L \gg 1, \;\; \omega \gg 1, \;\; a \ll 1, \;\; BL^2 \equiv \hat{B} \sim 1, \quad a\omega C \equiv \hat{C} \sim 1, \quad a^2\omega^2 D \sim L^3, \tag{1.7}$$

where we are anticipating the appropriate values of the parameters to obtain a *distinguished limit*, namely a limit in which all terms appearing in the *evolution equation* obtained below are of the same order. This equation will allow us to analyze the Rayleigh-Taylor instability, which is a long-wave instability namely the wavelength of the most unstable mode is of the order of the aspect ratio. In addition, we must avoid the Faraday (parametric) instability [7] which would give short waves (with a wavelength small as compared to depth) along the free surface. If in addition to (1.7a-c), it is satisfied that $BC^{-2} \ll \omega^{3/2}$ and $C^{-2} \ll \omega^{1/2}$, as we will assume hereafter, the Faraday instability is avoided if $a^2\omega \leq 2.79\ldots$, see [8].

2 Asymptotic derivation of the nonlinear equation for the evolution of the free surface

Here we use the scalings (1.7d-f), the scaled horizontal coordinates $(\tilde{x}, \tilde{y})$ and the slow time variable T, defined as $\tilde{x} = X/L$, $\tilde{y} = y/L$, $T = a^2\omega^2 t/L^4$, and seek solutions of the form

$$(\boldsymbol{u}, w, \frac{p}{\omega L}) = \frac{a\omega}{L}(\boldsymbol{u}_0, \frac{w_0}{L}, p_0)e^{i\omega t} + \frac{a^2\omega}{L^3}HOA + \text{c.c.} + \frac{a^2\omega^2}{L^3}(\boldsymbol{u}_s, \frac{w_s}{L}, \frac{p_s}{\omega}) + \ldots,$$
$$f = aL^{-2}f_0 e^{i\omega t} + a^2L^{-4}HOA + \text{c.c.} + f_s + \ldots, \tag{2.1}$$

where $\boldsymbol{u}_0$, w_0, p_0, f_0, $\boldsymbol{u}_s$, w_s, p_s and f_s only depend on $\tilde{x}$, $\tilde{y}$, z and on the slow time variable T, c.c. denotes the complex conjugate and HOA stands for *higher order harmonics*, depending on the fast time variable as $e^{im\omega t}$, with $m \neq 0, \pm 1$. When these expansions are replaced into the original nonlinear problem (1.1)-(1.5), then we obtain

$$\tilde{\boldsymbol{\nabla}} \cdot \boldsymbol{u}_0 + \partial_z w_0 = 0, \quad i\boldsymbol{u}_0 = -\tilde{\boldsymbol{\nabla}} p_0, \quad \partial_z p_0 = 0, \tag{2.2}$$
$$\tilde{\boldsymbol{\nabla}} \cdot \boldsymbol{u}_s + \partial_z w_s = 0, \quad -\tilde{\boldsymbol{\nabla}} p_s + \partial^2_{zz}\boldsymbol{u}_s = (\bar{\boldsymbol{u}}_0 \cdot \tilde{\boldsymbol{\nabla}})\boldsymbol{u}_0 + \bar{w}_0\partial_z \boldsymbol{u}_0 + \text{c.c.}, \quad \partial_z p_s = 0, \tag{2.3}$$

if $(x, y) \in \Omega$ and $f_s < z < 1$, and

$$\boldsymbol{u}_s = \boldsymbol{0}, \quad w_0 = w_s = 0 \quad \text{if } z = 1; \quad w_0 = if_0 + \boldsymbol{u}_0 \quad \tilde{\boldsymbol{\nabla}} f_s, \quad w_s = \partial_T f_s + \boldsymbol{u}_s \cdot \tilde{\boldsymbol{\nabla}} f_s, \tag{2.4}$$
$$\partial_z \boldsymbol{u}_s = 0, \quad p_0 = f_s/2, \quad p_s = (f_0 + \bar{f}_0)/2 + \hat{B}\hat{C}^{-2}f_s + \hat{C}^{-2}\tilde{\Delta} f_s \quad \text{if } z = f_s; \tag{2.5}$$
$$\partial_{\tilde{n}} f_s = -\hat{D}\partial_T f_s \quad \text{or} \quad f_s = 0, \quad \int_0^1 \boldsymbol{u}_s \cdot \tilde{\boldsymbol{n}}\, dz = 0 \quad \text{on } \partial\tilde{\Omega}. \tag{2.6}$$

Here the overbars stand for the complex conjugate and $\tilde{\Omega}$, $\tilde{\boldsymbol{\nabla}}$ and $\tilde{\Delta}$ are the cross-section, the $\boldsymbol{\nabla}$ and Δ operators written in terms of the re-scaled variables $\tilde{x}$ and $\tilde{y}$, $\tilde{\boldsymbol{n}}$ is the unit outward normal to $\partial\tilde{\Omega}$, $\tilde{n}$ is a coordinate along $\tilde{\boldsymbol{n}}$ and $\hat{D} = 2a^2\omega^2 D/[(2 + a^2\omega^2 C^2)L^3]$. These apply outside two thin viscous boundary layers, with $O(\omega^{-1/2})$ thicknesses, attached to the free surface and the upper plate and outside a lateral boundary layer, of $O(1)$ thickness, near the lateral walls, whose analysis provide the boundary conditions above, see [5] for details. On the other hand, we consider the following overall continuity equations, which are obtained upon integration of (2.2a) and (2.3a) in $f_s < z < 1$ and substitution of the boundary conditions (2.4c,d), $\tilde{\boldsymbol{\nabla}} \cdot (\int_{f_s}^1 \boldsymbol{u}_0\, dz\,) = if_0$, $\tilde{\boldsymbol{\nabla}} \cdot (\int_{f_s}^1 \boldsymbol{u}_s\, dz) = \partial_T f_s$. Using these, we may integrate (2.2b,c), (2.3b,c), (2.4d), (2.5a) and (2.5c) to obtain

$$\boldsymbol{u}_s = \frac{(z - f_s)^2 - (1 - f_s)^2}{8}\tilde{\boldsymbol{\nabla}}[4p_s + (|\tilde{\boldsymbol{\nabla}} f_s|^2)], \quad \partial_T f_s = -\tilde{\boldsymbol{\nabla}} \cdot [\frac{(1 - f_s)^3}{12}\tilde{\boldsymbol{\nabla}}(4p_s + |\tilde{\boldsymbol{\nabla}} f_s|^2)], \tag{2.7}$$
$$p_s = \hat{B}\hat{C}^{-2}f_s + [\hat{C}^{-2} + (1 - f_s)/2]\tilde{\Delta} f_s - |\tilde{\boldsymbol{\nabla}} f_s|^2/2 \quad \text{in } \tilde{\Omega}, \tag{2.8}$$

where we have taken into account that $(\tilde{\nabla} f_s \cdot \tilde{\nabla})\tilde{\nabla} f_s = \tilde{\nabla}(|\tilde{\nabla} f_s|^2)/2$. The evolution equation we are looking for is given by (2.7b)-(2.8). Also, invoking (2.6), the volume conservation condition (1.6b), (2.7a), re-scaling the time variable and dropping the subscript s we obtain

$$\partial_\tau f = -\tilde{\nabla} \cdot [(1-f)^3 \tilde{\nabla} U], \quad \text{with} \quad U = \lambda f + (1-\alpha f)\tilde{\Delta} f - \alpha |\tilde{\nabla} f|^2/2, \quad \text{in } \tilde{\Omega}, \tag{2.9}$$

$$\partial_{\tilde{n}} f = -\beta \partial_\tau f \quad \text{or} \quad f = 0, \quad \partial_{\tilde{n}} U = 0 \quad \text{on } \partial\tilde{\Omega}, \quad \int_{\tilde{\Omega}} f \, d\tilde{x} d\tilde{y} = 0, \tag{2.10}$$

where (see also (1.7d-f) and remind that $T = a^2\omega^2 t/L^4$)

$$\lambda = 2\hat{B}/(2+\hat{C}^2), \quad \alpha = \hat{C}^2/(2+\hat{C}^2), \quad \beta = \hat{D}(2+\hat{C}^2)/(6\hat{C}^2), \quad \tau = (2+\hat{C}^2)T/(6\hat{C}^2). \tag{2.11}$$

3 Analysis of the evolution equation

Eq.(2.9) is somewhat similar to the Cahn-Hilliard equation. Since $0 < \alpha < 1$, the problem (2.9) is uniformly parabolic and thus has a unique solution satisfying given initial conditions [9] whenever $|f| =$ bounded and $f < 1$; this latter condition means that no dry spot appears at the upper plate. Note that the first boundary condition is somewhat non-standard, but it is dissipative because $\beta \geq 0$ and thus standard results for Dirichlet and Neumann boundary conditions are somewhat straightforwardly extended when this condition applies. Also, (2.9)-(2.10) admits a Lyapunov function which, using general results from [10], allows to show that for large time the solutions either develop dry spots or converge to the set of steady sates, see [5] for details.

The linear stability of the simplest steady state of (2.9)-(2.10), $f = 0$, is analyzed as usually, by first linearizing around $f = 0$ and then replacing $f(\tilde{x}, \tilde{y}, \tau)$ by $F(\tilde{x}, \tilde{y})e^{\mu\tau}$ in the resulting problem, to obtain the linear eigenvalue problem

$$-\tilde{\Delta} U = \mu F, \quad \tilde{\Delta} F + \lambda F = U \quad \text{in } \tilde{\Omega}, \tag{3.1}$$

$$\partial_{\tilde{n}} F = -\mu\beta F \quad \text{or} \quad F = 0, \quad \partial_{\tilde{n}} U = 0 \quad \text{on } \partial\tilde{\Omega}, \quad \int_{\tilde{\Omega}} F \, d\tilde{x} d\tilde{y} = 0. \tag{3.2}$$

For convenience we consider the linear problem posed by (3.1a) and (3.2b), which uniquely provides U in terms of F, in the form $U = \mu G(F) + \text{constant}$, where G is the Green operator associated with the problem $-\tilde{\Delta} U = F$ in $\tilde{\Omega}$, $\partial_{\tilde{n}} U = 0$ on $\partial\tilde{\Omega}$, $\int_{\tilde{\Omega}} U \, d\tilde{x} d\tilde{y} = 0$. Eqs. (3.1)-(3.2) can be rewritten as the following generalized eigenvalue problem

$$\tilde{\Delta} F + \lambda F = \mu G(F) + \text{const. in } \tilde{\Omega}, \quad \partial_{\tilde{n}} F = -\mu\beta F \text{ or } \quad F = 0 \text{ on } \partial\tilde{\Omega}, \quad \int_{\tilde{\Omega}} F \, d\tilde{x} d\tilde{y} = 0. \tag{3.3}$$

Thus μ can be also calculated as a generalized eigenvalue of this problem. Since G is compact, selfadjoint and positive (i.e. $\int_{\tilde{\Omega}} F G(F) \, d\tilde{x} d\tilde{y} \geq 0$) in the space Y_1 defined below, the spectrum of (3.3) is readily seen to be real, discrete and bounded above [11]. And using standard variational arguments [11], the largest eigenvalue of this problem is found to be given by

$$-\mu_0 = \min_{F\in Y_1} \frac{\int_{\tilde\Omega}[|\tilde{\nabla} F|^2 - \lambda F^2]\, d\tilde{x}d\tilde{y}}{\int_{\tilde\Omega} FG(F)\, d\tilde{x}d\tilde{y} + \beta \int_{\partial\tilde\Omega} F^2\, ds} \quad \text{with } Y_1 = \{F \in H^1(\tilde\Omega) : \int_{\tilde\Omega} F\, d\tilde{x}d\tilde{y} = 0\} \tag{3.4}$$

if the first boundary condition in (3.3b) holds, where s is an arch length parameter along $\partial\tilde\Omega$. If the the second boundary condition in (3.3b) holds, the largest eigenvalue of (3.3) is obtained from (3.4) replacing β by 0 and the space Y_1 by $Y_2 = \{F \in H^1(\tilde\Omega) : \int_{\tilde\Omega} F\, d\tilde{x}d\tilde{y} = 0,\ F = 0 \text{ on } \partial\tilde\Omega\}$. Note that because $\beta \geq 0$ and $\int_{\tilde\Omega} FG(F)\, d\tilde{x}d\tilde{y} > k_0 \int_{\tilde\Omega} |F|^2\, d\tilde{x}d\tilde{y}$ the functionals that are minimized in (3.4) are bounded and continuous (in fact, analytic). Since the lowest eigenvalue of the problem obtained taking $\mu = 0$ in (3.3) is given by

$$\lambda_0 = \min_{F\in Y_j} \frac{\int_{\tilde\Omega} |\tilde{\nabla} F|^2\, d\tilde{x}d\tilde{y}}{\int_{\tilde\Omega} F^2\, d\tilde{x}d\tilde{y}} \quad \text{for } j\text{=1 or 2,} \tag{3.5}$$

depending on which boundary condition of (3.3b) applies, we obtain the following

Property 3.1. *If $\lambda < \lambda_0$ then all eigenvalues of (3.1)-(3.2) are strictly negative, and if $\lambda > \lambda_0$ then (3.1)-(3.2) possesses a strictly positive eigenvalue.*

Proof. The first assertion follows from (3.4) and (3.5). And the second assertion follows from the first one because, according to the characterization (3.4), μ_0 (i) depends continuously on λ and (ii) strictly increases as λ increases.

This property and the definitions (1.7d-f) imply that *the flat state is linearly, asymptotically stable if and only if the Bond number is such that*

$$B < B_c \equiv \lambda_0[1 + C^2a^2\omega^2/2]/L^2, \tag{3.6}$$

where λ_0 is as defined in (3.5). This expression yields the *Rayleigh-Taylor instability limit* of the flat state under vertical vibration. When taken into account the non-dimensionalization in §1, we readily obtain that the first term (namely, the 1) in the bracket in the right hand side of (3.6) is due to the effect of surface tension, and the second term is due to vibration. Thus, eq.(3.6) shows that *the stabilizing effect of vibration is like that of surface tension.*

Let us analize the non-flat steady states of (2.9)-(2.10) without dry spots. These steady states are given by

$$(1 - \alpha f)\tilde{\Delta} f + \lambda f - \alpha|\tilde{\nabla} f|^2/2 = \text{constant}, \quad f < 1 \quad \text{in } \tilde\Omega, \tag{3.7}$$

$$\partial_{\tilde n} f = 0 \quad \text{or} \quad f = 0 \quad \text{on } \partial\tilde\Omega, \quad \int_{\tilde\Omega} f\, d\tilde{x}d\tilde{y} = 0. \tag{3.8}$$

The following *global* result gives sufficient conditions for non-existence of non-flat steady states without dry spots.

Property 3.2. Let $\lambda_0 > 0$ be as defined in (3.5). If $\alpha < 2/3$ and $\lambda < \lambda_0(1 - 3\alpha/2)$ then (3.7)-(3.8) only possesses the flat solution $f = 0$.

Proof. The solutions of (3.7)-(3.8) satisfy $\int_{\tilde{\Omega}}[(1 - 3\alpha f/2)|\tilde{\nabla} f|^2 - \lambda f^2]\, d\tilde{x} d\tilde{y} = 0$, as readily obtained upon multiplication of (3.7a) by f, integration in $\tilde{\Omega}$, integration by parts and substitution of (3.8). And we only need to use the variational definition (3.5) of λ_0 to obtain the stated result.

As seen in Property 3.1 above, the basic steady state $f = 0$ is stable provided that $\lambda < \lambda_0$. Local bifurcation at $\lambda = \lambda_0$ is readily analyzed by the Lyapunov-Schmidt method. It is seen that the bifurcation from the trivial branch of (3.7),(3.8) at $\lambda = \lambda_0$ is transcritical if $\int_{\tilde{\Omega}} F_0^3 \neq 0$, and is of pitchfok type if $\int_{\tilde{\Omega}} F_0^3 = 0$, where F_0 is the function that minimizes the functional (3.5). In particular, if the domain $\tilde{\Omega}$ is a rectangle or a circle then the bifurcation at $\lambda = \lambda_0$ is a subcritical pitchfork one. See [5] for details.

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