GLOBAL STABILITY OF A PREMIXED REACTION ZONE (TIME-DEPENDENT LIÑAN'S PROBLEM)*

CARLOS ALVAREZ PEREIRA† AND JOSÉ M. VEGA†

Abstract. Global stability properties of a premixed, three-dimensional reaction zone are considered. In the nonadiabatic case (i.e., when there is a heat exchange between the reaction zone and the burned gases) there is a unique, spatially one-dimensional steady state that is shown to be unstable (respectively, asymptotically stable) if the reaction zone is cooled (respectively, heated) by the burned mixture. In the adiabatic case, there is a unique (up to spatial translations) steady state that is shown to be stable. In addition, the large-time asymptotic behavior of the solution is analyzed to obtain sufficient conditions on the initial data for stabilization. Previous partial numerical results on linear stability of one-dimensional reaction zones are thereby confirmed and extended.

Key words. global stability, stabilization, reaction regions, premixed flames, nonadiabatic flames

AMS(MOS) subject classifications. 35B35, 35B40, 35B55, 80A25, 80A32

1. Introduction. We consider the time-dependent structure of a premixed n-dimensional ($n = 1, 2$, or 3) reaction zone which, after convenient nondimensionalization, is governed by

$$\partial u/\partial t = \Delta u - (u/2)\exp(mx_1 - u) \quad \text{for } (x, t) \in \mathbb{R}^n \times]0, T_0[, \tag{1.1}$$

$$u \to 0 \quad \text{if } m \leqq 0, \qquad u \text{ is bounded} \quad \text{if } m > 0 \quad \text{as } x_1 \to -\infty, \tag{1.2}$$

$$|u - x_1| \text{ is bounded} \quad \text{as } x_1 \to \infty, \tag{1.3}$$

$$u \text{ is bounded} \quad \text{as } x_2^2 + \cdots + x_n^2 \to \infty, \tag{1.4}$$

$$u(x, 0) = \varphi(x) > 0 \quad \text{for } x \in \mathbb{R}^n, \tag{1.5}$$

where conditions (1.2) and (1.3) are assumed to hold uniformly for $(x_2, \cdots, x_n) \in \mathbb{R}^{n-1}$ and for $t \in [0, T]$, for all $T \in [0, T_0[$ (for some $T_0 \leqq \infty$), and condition (1.4) is assumed to hold uniformly for $(x_1, t) \in I \times [0, T]$, for all bounded intervals $I \subset \mathbf{R}$ and all $T \in [0, T_0[$.

Here, Δ is the Laplacian operator, t and $x = (x_1, \cdots, x_n)$ ($n = 1, 2$, or 3) are the time and space variables, and $u \geqq 0$ is a reactant concentration. The parameter m is a measure of the heat flux (heat loss if $m > 0$ and heat gain if $m < 0$) from the reaction zone towards the burned mixture, which is located at $x_1 = -\infty$; m is assumed to satisfy $-\infty < m < 1$, for the chemical reaction to be frozen (i.e., for the reaction term $(u/2)\exp(mx_1 - u)$ to vanish) at the fresh mixture (i.e., at $x_1 = +\infty$). The initial state φ is assumed to satisfy the boundary conditions (1.2)–(1.4), which, of course, are expected to be superfluous; they are written to emphasize that the solution of the Cauchy problem (1.1)–(1.5) is physically meaningful only if it satisfies (1.2)–(1.4).

In this paper we will analyze the stability of steady states of (1.1)–(1.3) that depend only on the x_1 coordinate. Since the reaction term does not depend explicitly on the x_2 and x_3 coordinates, it makes sense (mathematically) to consider the (spatially) one- and two-dimensional cases in which $u = u(x_1, t)$ and $u = u(x_1, x_2, t)$, respectively. But since the underlying physical problem is spatially three-dimensional, to obtain conclusive stability results we must consider (1.1)–(1.5) in three space dimensions. It is not at all obvious (although it will be true under certain conditions for (1.1)–(1.5)) that initial inhomogeneities in the x_2 and x_3 coordinates disappear as $t \to \infty$. Some results in the literature [1] could perhaps be extended to include (1.1)–(1.5) if the spatial domain $\mathbf{R}^3$ were replaced by a cylinder $\Omega = \mathbb{R} \times \Omega_1$, with $\Omega_1 \subset \mathbf{R}^2$ bounded, and if boundary conditions of the Neumann type were imposed on $\mathbf{R} \times \partial\Omega_1$, provided that the size of Ω_1 is sufficiently small. But to assume that the characteristic lengths in the x_2 and x_3 directions are small (or even finite) is not justified from a physical point of view. Therefore, we will consider (1.1)–(1.5) mainly for $n = 3$, although the case where $n = 1$ will be considered also for technical reasons.

The one-dimensional, time-independent version of (1.1)–(1.3) was introduced by Liñán [2], in a pioneering work on counterflow diffusion flames in the large activation energy limit, and (1.1)–(1.5) is currently known in the literature as Liñán's problem. It has subsequently appeared in high-activation energy analysis of many other realistic problems that are significant in both combustion and chemical reactor theory. For example, it has appeared in the analysis of burning monopropellant drops [3]–[5], chambered diffusion flames [6], two-step sequential reactions [7], [8], and tubular nonadiabatic chemical reactors [9]; in all these instances, the parameter m is different from zero, but the adiabatic case ($m = 0$) appears in a large number of problems [10], such as the analysis of premixed flames [11]–[13] and porous catalysts [14], [15], to cite only two examples.

Problem (1.1)–(1.5) is also of interest if the nonlinearity $u \exp(mx_1 - u)$ is replaced by a more general one. For example, if we use Langmuir–Hinshelwood kinetic laws for the chemical reaction, instead of the Arrhenius law that has been used to derive (1.1), we obtain nonlinearities of the type [16], [17]

$$(1.6) \qquad u^p/(1+u)^q \quad \text{or} \quad [u^p/(a+u)^r] \exp(mx_1 - u),$$

where $p \geqq 0$, $q > p + 1$, $a > 0$, and $m < 1$ (the exponents p, q, and r are not necessarily integers). These generalizations will be considered in remarks after some of the main results.

A numerical analysis of the one-dimensional steady states of (1.1)–(1.3) has been done by Liñán [2]. His results were rigorously proven true by Hastings and Poore [18], [19], who showed that the solution is unique if either $-\infty < m < 0$ or $0 < m < \frac{1}{2}$, while there is no solution if $\frac{1}{2} \leqq m < \infty$ (if $m = 0$, there is a unique steady state up to translations in the space variable, as is easily seen by means of simple phase-plane arguments). To derive stability results, we will need slightly more precise information about the dependence of the steady state on m for $0 < m < \frac{1}{2}$, which will be obtained in the Appendix, where a simpler proof of the results by Hastings and Poore [18], [19] (partially based on their ideas) will also be given.

The first analysis of the stability of the steady states of (1.1)–(1.4) is due to Peters [20], who computed numerically the maximum eigenvalue of the (self-adjoint) linearized problem in the spatially one-dimensional case, and found that $m > 0$ is necessary and sufficient for a strictly positive eigenvalue to exist. More recently, Stewart and Buckmaster [21] performed an asymptotic analysis of the same linearized problem in the limit $m \to 0^+$, which is singular. Those results ignore the continuous spectrum of the linearized problem, which has been calculated, for related spatially one-dimensional problems on combustion, by Buckmaster, Nachman, and Taliaferro [22], by means of a general theory developed by Taliaferro [23]. Unfortunately, Taliaferro's results deal with a weak notion of linear stability (a steady state is said to be stable if the maximum of the spectrum is nonpositive and zero is not an eigenvalue) and, anyway, do not apply to the linearized problem associated with (1.1)–(1.3). Those results need completion also because they apply only to the one-dimensional case.

At this point, the boundary conditions (1.2), (1.3) deserve some attention. In this analysis of the steady state problem, Liñán [2] imposed the following conditions at $x_1 = \pm\infty$:

$$\partial u/\partial x_1 \to 0 \quad \text{as } x_1 \to -\infty, \qquad \partial u/\partial x_1 \to 1 \quad \text{as } x_1 \to \infty. \tag{1.7}$$

Stewart and Buckmaster [21] maintain conditions (1.7) for the time-dependent problem, while Peters [20] replaces them by

$$u \to c_1 \quad \text{as } x_1 \to -\infty, \qquad u - x_1 \to c_2 \quad \text{as } x_1 \to \infty \tag{1.8}$$

for some constants c_1 and c_2. In fact, conditions (1.2), (1.3), and (1.7) and (1.8), are equivalent when applied to the one-dimensional steady state problem (see the Appendix) and are seen to lead to equivalent linearized eigenvalue problems. But those three conditions are not equivalent when applied to the time-dependent problem. We will use conditions (1.2), (1.3), which are obtained from matching conditions in the singular perturbation analysis that leads to (1.1)–(1.5), as may be seen.

In this paper we will obtain precise global stability properties of the one-dimensional steady states of (1.1)–(1.5) for $n = 3$. First, existence, uniqueness, and some properties of the solution of (1.1)–(1.5) are considered in § 2. In § 3, sub- and supersolutions of (1.1)–(1.5), and some properties of the steady state, from the Appendix, are used to show that the (unique) steady state is stable and pointwise globally, asymptotically attracting if $-\infty < m < 0$, while it is unstable if $0 < m < \frac{1}{2}$. Comparison methods do not yield good enough results on the critical adiabatic case $m = 0$, which exhibits infinitely many steady states due to translation invariance. In § 4, a Lyapunov function argument and a nonlinear change of variables will be used to analyze the global stability of the steady states. In particular, we will obtain sufficient conditions on the initial data for the solution of (1.1)–(1.5) to approach the set of steady states as $t \to \infty$, and for it to approach a given steady state.

2. Some preliminary results. In this section we analyze the well-posedness of problem (1.1)–(1.5), as well as some basic properties of its solutions.

The following notation will be used. Let $\Omega \subset \mathbb{R}^n$ be a convex, smooth domain and, for some $T > 0$, let $Q_T = \Omega \times]0, T[$. Let $W_p^q(\Omega)$ (respectively, $W_p^{2q,q}(Q_T)$) be the Sobolev space of those (classes of) functions, $u : \Omega \to \mathbb{R}$ (respectively, $u : Q_T \to \mathbb{R}$) such that $|D^i u|^p$ (respectively, $|D_t^i D_x^j u|^p$) is integrable in Ω (respectively, in Q_T) for all $i \leqq q$ (respectively, for all i and j such that $2i + j \leqq 2q$). The norms of $W_p^q(\Omega)$ and $W_p^{2q,q}(Q_T)$ will be denoted as

$$\|\cdot\|_{p,\Omega}^{(q)} \quad \text{and} \quad \|\cdot\|_{p,Q_T}^{(2q,q)},$$

respectively. $W_{p,\mathrm{loc}}^q(\mathbb{R}^n)$ (respectively, $W_{p,\mathrm{loc}}^{2q,q}(\mathbb{R}^n \times [0, T_0[)$) will be the linear space of those functions $u : \mathbb{R}^n \to \mathbb{R}$ (respectively, $u : \mathbb{R}^n \times [0, T_0[\to \mathbb{R}$) such that $u \in W_p^q(B)$ for all bounded balls $B \subset \mathbb{R}^n$ (respectively, $u \in W_p^{2q,q}(B \times]0, T[)$ for all bounded balls $B \subset \mathbb{R}^n$ and all $T \in]0, T_0[$). For any nonintegral positive number r, $C^r(\bar{\Omega})$ (respectively, $C^{r,r/2}(\bar{Q}_T)$) will be the Hölder space of those functions $u : \Omega \to \mathbb{R}$ (respectively, $u : Q_T \to \mathbb{R}$) having in Ω bounded, uniformly continuous derivatives up to order $[r]$ equal to the integral part of r (respectively, having in Q_T bounded, uniformly continuous derivatives $D_t^i D_x^j u$, for all i and j such that $2i + j < r$) and such that the $[r]$-derivative is uniformly Hölder continuous of order $r - [r]$ in Ω (respectively, the derivatives $D_t^i D_x^j u$ are uniformly Hölder continuous, of order $r - [r]$, in the x variable if $2i + j = [r]$, and of order $(r-2i-j)/2$ in the t variable if $r-2<2i+j<r$). The norms of $C^r(\bar{\Omega})$ and $C^{r,r/2}(\bar{Q}_T)$ (see, e.g., [24] for their precise definition) will be denoted as

$$|\cdot|_{\Omega}^{(r)} \quad \text{and} \quad |\cdot|_{Q_T}^{(r,r/2)},$$

respectively. Finally, $C^r(\mathbb{R}^n)$ (respectively, $C^{r,r/2}(\mathbb{R}^n \times [0, T_0[)$ will be the linear space of those functions $u:\mathbb{R}^n \to \mathbb{R}$ (respectively, $u:\mathbb{R}^n \times [0, T_0[\to \mathbb{R})$ such that $u \in C^r(\bar{B})$ for all bounded balls $B \subset \mathbb{R}^n$ (respectively, $u \in C^{r,r/2}(\bar{B} \times [0, T])$ for all bounded balls $B \subset \mathbb{R}^n$ and all $T \in]0, T_0[$).

We first show that (1.1)–(1.5) possesses a unique classical solution in $0 \leqq t < T_0$, if $-\infty < m < 1$, with $T_0 = \infty$ if $m \leqq 0$.

THEOREM 2.1. *If $-\infty < m < 1$, let $r > 0$ be a noninteger. If $\varphi \in C^{2+r}(\mathbb{R}^n)$ satisfies* (1.2)–(1.4), *then* (1.1)–(1.5) *possess a unique classical solution u in $\mathbb{R}^n \times [0, T_0[$, where $T_0 = \infty$ if $m \leqq 0$ and $T_0 = [2e(1-m)/m] \exp(-a_0)$ if $0 < m < 1$, with*

$$a_0 = \sup\{x_1 - \varphi(x): x \in \mathbb{R}^n\}.$$

Furthermore, $u \in C^{2+r,1+r/2}(\mathbb{R}^n \times [0, T_0[)$ and is such that

$$\sup\{0, x_1 - a(t)\} \leqq u(x,t) \leqq U(x,t) \quad \text{for all } (x,t) \in \mathbb{R}^n \times [0, T_0[, \tag{2.1}$$

where a is given by

$$2e(1-m)\, da/dt = \exp(ma), \qquad a(0) = a_0,$$

and $U > 0$ is the unique solution of

$$\partial U/\partial t = \Delta U \quad \text{in } \mathbb{R}^n \times [0, \infty[, \qquad U(\cdot, 0) = \varphi \quad \text{in } \mathbb{R}^n, \tag{2.2}$$

which satisfies (1.2)–(1.4).

Proof. The solution of (1.1)–(1.5) will be obtained as the limit of the sequence $\{u_k\}$ defined inductively by

$$\partial u_k/\partial t - \Delta u_k + (u_k/2)\exp(mx_1) = (u_{k-1}/2)[1 - \exp(-u_{k-1})]\exp(mx_1), \tag{2.3}$$

$$u_k(x, 0) = \varphi(x), \tag{2.4}$$

where $u_0 = U$ is given by (2.2) and each u_k satisfies (1.2)–(1.4). The coefficient of u_k in (2.3) is unbounded but positive. Therefore, the linear problem (2.3), (2.4) is dissipative, and each u_k is well defined with $u_k \in C^{2+r,1+r/2}(\mathbb{R}^n \times [0, \infty[)$. This is proven by using the estimates of Eidel'man [25, Thm. 3.1, p. 131] for the fundamental solution of (2.3), in standard proofs of the solvability of the Cauchy problem for linear parabolic equations (e.g., in the proof of Theorem 6.1 [24, p. 324]).

The sequence $\{u_k\}$ satisfies, for each $k \geqq 0$,

$$0 \leqq u_{k+1} \leqq u_k \quad \text{in } \mathbb{R}^n \times [0, \infty[, \tag{2.5}$$

as is seen inductively by means of the Phragmèn–Lindelöf (Ph-L) maximum principle [26], [27], when we take into account that the function $u \to u[1-\exp(-u)]$, appearing in the right-hand side of (2.3), is strictly increasing for $0 \leqq u < \infty$. Then the bounded, monotone sequence $\{u_k\}$ is pointwise convergent to a function u such that

$$0 \leqq u \leqq U \quad \text{in } \mathbb{R}^n \times [0, \infty[, \tag{2.6}$$

as it comes from (2.5).

Let us see that $u \in C^{2+r,1+r/2}(\mathbb{R}^n \times [0, \infty[)$, and that u is a classical solution of (1.1)–(1.5). For each bounded, open ball $B \subset \mathbb{R}^n$, let B' be another ball such that $\bar{B} \subset B'$. Local estimates of the solution of (2.3), (2.4) on $W_p^{2,1}$ and $C^{2+s,1+s/2}$ [24, pp. 355, 352] imply that, for each $T>0$, each integer $p \geqq 1$ and each noninteger $s \in [0, r]$, there exist constants, $c_1, \cdots, c_4$, depending only on B, B', T, p, and s, such that

$$(2.7) \qquad \|u_j - u_i\|^{(2,1)}_{p,B\times]0,T[} \leqq c_1 \|f_j - f_i\|^{(0,0)}_{p,B'\times]0,T[} + c_2 \|u_j - u_i\|^{(0,0)}_{p,B'\times]0,T[},$$

$$(2.8) \qquad |u_j - u_i|^{(2+s,1+s/2)}_{B\times]0,T[} \leqq c_3 |f_j - f_i|^{(s,s/2)}_{B'\times]0,T[} + c_4 |u_j - u_i|^{(s,s/2)}_{B'\times]0,T[}$$

for all integers $i, j \geqq 1$, where $f_k = (u_{k-1}/2)[1-\exp(-u_{k-1})]\exp(mx_1)$. Since $u_k \to u$ in $W^{0,0}_p(B'\times]0,T[) = L_p(B'\times]0,T[)$ (monotone convergence theorem [28]) then $f_k \to f = (u/2)[1-\exp(-u)]\exp(mx_1)$, and $\{u_k\}$ and $\{f_k\}$ are Cauchy sequences in the same space. Then $\{u_k\}$ is a Cauchy sequence in $W^{2,1}_p(B\times]0,T[)$ (by (2.7)) and thus it converges (to u) in the same space. Now, if we take $p > (n+2)/(2-r+[r])$, embedding theorems [24, p. 80] imply that $u_k \to u$ in $C^{\alpha,\alpha/2}(\bar{B}\times[0,T])$, where $\alpha = r-[r]$. Estimate (2.8) with $s=\alpha$ implies, by the same argument as above, that $u_k \to u$ in $C^{2+\alpha,1+\alpha/2}(\bar{B}\times[0,T])$ and, by repeating the argument if necessary (i.e., if $r>\alpha$), (2.8) implies that $u_k \to u$ in $C^{2+r,1+r/2}(\bar{B}\times[0,T])$. Then $u \in C^{2+r,1+r/2}(\bar{B}\times[0,T])$ for every bounded ball and every $T>0$ as stated, and u satisfies (1.1) and (1.5), as we see when taking limits in (2.3), (2.4).

We now show that u satisfies (2.1) and, therefore, that it satisfies (1.2)–(1.4). It is enough to prove that $x_1 - a(t) \leqq u(x,t)$ for all $(x,t) \in \mathbb{R}^n \times [0, T_0[$ (see (2.6)); this is true since, for all $k \geqq 1$,

$$x_1 - a(t) \leqq u_k(x,t) \quad \text{for all } (x,t) \in \mathbb{R}^n \times [0, T_0[,$$

as is seen inductively when the Ph-L maximum principle is applied to $u_k(x,t) - x_1 + a(t)$, and it is taken into account that $w(x,t) = x_1 - a(t)$ satisfies

$$\partial w/\partial t \leqq \Delta w - \max\{0, (w/2)\exp(mx_1 - w)\}, \qquad w(x,0) \leqq \varphi(x)$$

for all $(x,t) \in \mathbb{R}^n \times [0, T_0[$, as is easily seen.

Finally, we see that u is the unique solution of (1.1)–(1.5) in $\mathbb{R}^n \times [0, T_0[$. To this end, first observe that any other solution of (1.1)–(1.5), u', is such that $u' \leqq u_k$ in $\mathbb{R}^n \times [0, T_0[$, for all $k \geqq 0$, as is seen inductively by means of the Ph-L maximum principle. Therefore,

$$(2.9) \qquad u' \leqq u \quad \text{in } \mathbb{R}^n \times [0, T_0[,$$

and $W = u - u'$ satisfies

$$(2.10) \qquad \begin{aligned} &\partial W/\partial t - \Delta W = (W/2)(\xi - 1)\exp(mx_1 - \xi) \quad \text{in } \mathbb{R}^n \times [0, T_0[, \\ &W(x,0) = 0 \quad \text{in } \mathbb{R}^n \end{aligned}$$

for some function $\xi : \mathbb{R}^n \times [0, T_0[\to \mathbb{R}$ such that $u' \leqq \xi \leqq u$ in $\mathbb{R}^n \times [0, T_0[$. Then the Ph-L maximum principle implies that $W \leqq 0$ in $\mathbb{R}^n \times [0, T_0[$ (observe that the coefficient of W in (2.10) is bounded above, since u and u' satisfy (1.2), (1.3)), and (see (2.9)) the conclusion follows.

Remarks 2.2. Some remarks about Theorem 2.1 are in order:

(A) The function u in the proof of Theorem 2.1 satisfies (1.1), (1.2), (1.4), (1.5) for all $(x,t) \in \mathbb{R}^n \times [0, \infty[$ if $-\infty < m < 1$, but if $0 < m < 1$ we have proved only that u satisfies (1.3) for $0 \leqq t < T_0$; for $t \geqq T_0$, u is a maximal (and not necessarily the unique) solution of (1.1), (1.2), (1.4), (1.5). It seems that this result cannot be improved significantly for arbitrary initial data. In fact, some numerical and asymptotic results (see [29]) suggest that for ($0 < m < 1$ and) appropriate initial data, the maximal solution of (1.1), (1.2), (1.4), (1.5), u, is such that $u(x,t) \to 0$ as $x_1 \to +\infty$, uniformly in $(x_2, \cdots, x_n) \in \mathbb{R}^{n-1}$, if $t > T_1$, for some finite T_1; for such initial data, (1.1)–(1.5) cannot have a solution for all $t \geqq 0$. On the other hand, the proof of Theorem 2.1 is easily extended to show that (1.1)–(1.5) possesses a unique solution in $\mathbb{R}^n \times [0, \infty[$ if the initial datum is such that $\varphi \geqq w(\cdot, 0)$ in $\mathbb{R}^n$, where $w \in C^{2+\alpha, 1+\alpha/2}(\mathbb{R}^n \times [0, \infty[)$ for some $\alpha > 0$ and

$$\partial w/\partial t \leqq \Delta w - (w/2) \exp(mx_1 - w) \quad \text{in } \mathbb{R}^n \times [0, \infty[.$$

(B) The conclusion of Theorem 2.1 (existence and uniqueness of the solution of (1.1)–(1.5) in $\mathbb{R}^n \times [0, T_0[$, for some $T_0 \leqq \infty$) remains true if the nonlinearity $u \exp(mx_1 - u)$ is replaced by a more general one, of the type

$$g(x_1)f(u),$$

where $g: \mathbb{R} \to \mathbb{R}$ and $f: [0, \infty[\to \mathbb{R}$ are positive C^1-functions and:

(i) $f(0) = 0$, $f'(u)$ is bounded in $0 \leqq u < \infty$.

(ii) $g(\xi)f(\xi + c) \to 0$ as $\xi \to \infty$, for any fixed $c \in \mathbb{R}$.

(iii) The boundary condition (1.2) at $x_1 = -\infty$ is replaced by "u bounded" if $g(x_1) \to 0$ (as $x_1 \to -\infty$), and "$u \to 0$" otherwise.

In particular, conditions (i) and (ii) are fulfilled by the first nonlinearity in (1.6) if $1 \leqq p < q$, and by the second if $a > 0$, $p \geqq 1$, and $-\infty < m < 1$.

(C) If, for $m \leqq 0$, the boundary condition (1.2) at $x_1 = -\infty$ is replaced by $u \leqq c$ as $x_1 \to -\infty$, uniformly in $x_2, \cdots, x_n$ and t, for some constant c such that $0 < c < 1$, then the conclusion of Theorem 2.1 remains true, as is easily seen. This fact will be used in § 3, where we will take a supersolution w_2 of (1.1), such that $\lim w_2(x) = \frac{1}{2}$ as $x_1 \to -\infty$, as initial datum.

Problem (1.1)–(1.5) defines a monotone flow, as shown in the following.

THEOREM 2.3. *Under the hypothesis of Theorem* 2.1, *if u_1 and u_2 are two solutions of* (1.1)–(1.5), *defined in $0 \leqq t < T_0$, such that $u_1(\cdot, 0) \leqq u_2(\cdot, 0)$ in $\mathbb{R}^n$, then $u_1(\cdot, t) \leqq u_2(\cdot, t)$ in $\mathbb{R}^n$, for all $t \in [0, T_0[$.*

Proof. The monotone sequences that (in the proof of Theorem 2.1) define u_1, u_2, $\{u_{1k}\}$, and $\{u_{2k}\}$ satisfy $u_{1k} \leqq u_{2k}$ in $\mathbb{R}^n \times [0, T_0[$ for all $k \geqq 0$, as is seen inductively by means of the Ph-L maximum principle. Thus we have the conclusion.

As is usual in the literature, a function $w \in C^{2,1}(\mathbb{R}^n \times [0, T_0[)$ is said to be a *supersolution* (respectively, a *subsolution*) of (1.1) in $0 \leqq t < T_0$ if $\partial w/\partial t \geqq \Delta w - (w/2) \exp(mx_1 - w)$ (respectively, $\partial w/\partial t \leqq \Delta w - (w/2) \exp(mx_1 - w)$) in $\mathbb{R}^n \times [0, T_0[$. A sub- or supersolution of (1.1) is said to be *steady* if it does not depend on time.

THEOREM 2.4. *Under the hypothesis of Theorem* 2.1, *if $w \geqq 0$ is a supersolution (respectively, a subsolution) of* (1.1) *in $0 \leqq t < T_0$ that satisfies* (1.2)–(1.4), *and if u is a solution of* (1.1)–(1.5), *defined in $0 \leqq t < T_0$, such that $u(\cdot, 0) \leqq w(\cdot, 0)$ (respectively, $u(\cdot, 0) \geqq w(\cdot, 0)$) in $\mathbb{R}^n$, $u(\cdot, t) \leqq w(\cdot, t)$ (respectively, $u(\cdot, t) \geqq w(\cdot, t)$) in $\mathbb{R}^n$, for all $t \in [0, T_0[$.*

Proof. If w is a supersolution (respectively, a subsolution) of (1.1), we define the sequence $\{u_k\}$, given by (2.3), (2.4), with $u_0 = w$. As in the proof of Theorem 2.1, it is seen that $0 \leqq u_{k+1} \leqq u_k \leqq w$ (respectively, $w \leqq u_k \leqq u_{k+1} \leqq U$) in $\mathbb{R}^n \times [0, T_0[$ for all $k \geqq 1$, and that $u_k \to u$ as $k \to \infty$; then the conclusion readily follows.

THEOREM 2.5. *Under the hypothesis of Theorem* 2.1, *if $u(\cdot, 0) = \varphi: \mathbb{R}^n \to \mathbb{R}$ is a steady supersolution (respectively, subsolution) of* (1.1), *and satisfies* (1.2)–(1.4), *then $\partial u/\partial t \leqq 0$ (respectively, $\partial u/\partial t \geqq 0$) in $\mathbb{R}^n \times [0, T_0[$.*

Proof. We consider only the case in which φ is a supersolution. Theorem 2.4 yields: $u(x, t) \leqq \varphi(x) = u(x, 0)$ for all $(x, t) \in \mathbb{R}^n \times [0, T_0[$. Then, for each constant $h \in]0, T_0[$, $w(x, t) = u(x, t + h)$ is a solution of (1.1)–(1.4) such that $w(\cdot, 0) = u(\cdot, h) \leqq u(\cdot, 0)$ in $\mathbb{R}^n$. Thus, Theorem 2.3 leads to $u(\cdot, t + h) = w(\cdot, t) \leqq u(\cdot, t)$ in $\mathbb{R}^n$, for all

$t \in [0, T_0 - h]$. Therefore, for each fixed $x \in \mathbb{R}^n$, the function $t \to u(x, t)$ is nonincreasing and $\partial u/\partial t \leqq 0$ as stated.

Remarks 2.6. (A) Theorems 2.3–2.5 stand when the nonlinearity of (1.1) is modified as in Remark 2.2B, and also, if $m \leqq 0$, when the boundary condition (1.2) at $x_1 = -\infty$ is modified as in Remark 2.2C.

(B) Theorems 2.3–2.5 give properties of the solution of (1.1)–(1.5) that are well known for scalar parabolic equations in bounded domains (see, e.g., [30]).

3. Global stability results in the nonadiabatic case ($m \neq 0$). In this section we analyze global stability properties of the (spatially one-dimensional) steady state of (1.1)–(1.4) under (spatially) three-dimensional perturbations, for $-\infty < m < \frac{1}{2}$, $m \neq 0$. Among the many different definitions of stability, we select the following [31]. Let X be the set of functions $u \in C^{2+r}(\mathbb{R}^3)$, for some $r > 0$, that satisfy (1.2)–(1.4), and let Σ be a family of subsets of X. A steady state u_s of (1.1)–(1.4), such that $u_s \in S$ for all $S \in \Sigma$, will be called *Σ-stable* if for any $S \in \Sigma$ there exists $S' \in \Sigma$ such that $u(\cdot, 0) \in S'$ implies that $u(\cdot, t) \in S$ for all $t > 0$; u_s will be said to be *Σ-unstable* if it is not Σ-stable. Below u_s will be a steady state that depends only on the x_1 coordinate, and the family Σ will be

$$\Sigma = \{S_{\alpha,\beta}\colon \alpha, \beta > 0\}, \tag{3.1}$$

where

$$S_{\alpha,\beta} = \{u \in X\colon u_s(x_1 - \alpha) < u(x) < u_s(x_1 + \beta), \text{ for all } x \in \mathbb{R}^3\}.$$

Observe that if $\alpha, \beta > 0$, then $u_s(x_1 - \alpha) < u_s(x_1 + \beta)$ for all $x_1 \in \mathbb{R}$ (Theorems A.4 and A.8 of the Appendix), and $S_{\alpha,\beta}$ is a nonempty open neighborhood of u_s in X with the order topology (i.e., the topology generated by the order intervals of the form $]u_1, u_2[= \{u \in X\colon u_1(x) < u(x) < u_2(x) \text{ for all } x \in \mathbb{R}^3\}$) defined for $u_1, u_2 \in X$; see [31].

In connection with asymptotic stability, u_s will be said to be *globally pointwise attracting* if $u(\cdot, 0) \in X$ implies that $u(\cdot, t) \to u_s$ pointwise as $t \to \infty$.

We first consider the case $m < 0$.

THEOREM 3.1. *If $m < 0$, then* (1.1)–(1.4) *possesses a unique, spatially one-dimensional steady state u_s that is Σ-stable* (Σ *defined by* (3.1)), *and globally pointwise attracting.*

Proof. We first show that (1.1)–(1.4) has a unique spatially one-dimensional steady state that is globally pointwise attracting. To this end, let us consider the functions $w_1, w_2\colon \mathbb{R} \to \mathbb{R}$, defined by

$$w_1(y) = \begin{cases} 0 & \text{if } y \leqq a, \\ A(y-a)^3/64 & \text{if } a < y \leqq a+4, \\ \bar{w}_1(y) & \text{if } a+4 < y, \end{cases}$$

where $A = 2(\sqrt{2} - 1)/3$ and $\bar{w}_1$ is the unique solution of

$$d^2\bar{w}_1/dy^2 = (\tfrac{3}{8})\bar{w}_1 \exp(A - \bar{w}_1), \quad \bar{w}_1(a+4) = A, \quad d\bar{w}_1(a+4)/dy = 3A/4,$$

$$w_2(y) = \begin{cases} \frac{1}{2} & \text{if } y \leqq -b, \\ \frac{1}{2} + (y+b)^3[1 - (y+b)/2] & \text{if } -b < y \leqq -b+1, \\ y + b & \text{if } -b+1 < y. \end{cases} \tag{3.2}$$

It is easily seen that w_1 satisfies (1.2), (1.3), w_2 satisfies the boundary conditions considered in Remark 2.2C of § 2, $w_1, w_2 \in C^{2+r}(\mathbb{R})$ for every $r \in]0, 1[$, w_1 is a steady subsolution of (1.1) if $a \geqq |m|^{-1} \ln \frac{4}{3}$, and w_2 is a steady supersolution of (1.1) if

$b \geqq 1+[1+\ln \frac{8}{3}]/|m|$. Also, for every function $\varphi:\mathbb{R}^3\to\mathbb{R}$ satisfying (1.2), (1.3) (uniformly for $x_2, x_3 \in \mathbb{R}$), and (1.4) (uniformly for x_1 on bounded intervals of $\mathbb{R}$), we have

$$w_1(x_1) \leqq \varphi(x) \leqq w_2(x_1) \quad \text{for all } x\in\mathbb{R}^3, \tag{3.3}$$

provided that a and b are sufficiently large, as is easily seen.

Now, for $i=1$ and 2, let $u_i:\mathbb{R}\times[0,\infty[\to\mathbb{R}$ be given by

$$\partial u_i/\partial t = \partial^2 u_i/\partial y^2 - (u_i/2)\exp(my-u_i) \quad \text{in } \mathbb{R}\times[0,\infty[, \tag{3.4}$$

$$u_1\to 0, \qquad 0<u_2\leqq \tfrac{1}{2} \quad \text{as } y\to-\infty \quad \text{for } 0\leqq t<\infty, \tag{3.5}$$

$$|u_i-y| \text{ bounded} \quad \text{as } y\to\infty, \qquad 0\leqq t<\infty, \tag{3.6}$$

$$u_i(y,0)=w_i(y) \quad \text{for } -\infty<y<\infty, \tag{3.7}$$

where conditions (3.5), (3.6) hold uniformly in $0\leqq t\leqq T$, for all $T\in\,]0,\infty[$. The functions u_1 and u_2 are uniquely defined by (3.4)–(3.7), and $u_1, u_2 \in C^{2+r,1+r/2}(\mathbb{R}\times[0,\infty[)$ (see Theorem 2.1 and Remark 2.2C). Furthermore, if the initial datum of (1.1)–(1.5) satisfies (3.3), then

$$u_1 \leqq u \leqq u_2 \quad \text{in } \mathbb{R}^3\times[0,\infty[,$$

as is seen when Theorem 2.3 is applied. Also, for each $y\in\mathbb{R}$, the functions $t\to u_1(y,t)$ and $t\to u_2(y,t)$ are monotonic (Theorem 2.5), and bounded since

$$w_1 \leqq u_1(\cdot,t) \leqq u_2(\cdot,t) \leqq w_2 \quad \text{in } \mathbb{R} \quad \text{for all } t\geqq 0, \tag{3.8}$$

as seen by means of Theorems 2.3 and 2.5. Then, for $i=1$ and 2, $u_i(\cdot,t)\to\tilde{u}_i$ pointwise as $t\to\infty$, for a certain function $\tilde{u}_i:\mathbb{R}\to\mathbb{R}$ such that (see (3.8))

$$w_1 \leqq \tilde{u}_1 \leqq \tilde{u}_2 \leqq w_2 \quad \text{in } \mathbb{R}. \tag{3.9}$$

Thus, according to Lemmas A.1 and A.3 of the Appendix, the conclusion will follow if we prove that $\tilde{u}_1$ and $\tilde{u}_2$ are steady states of (1.1), since these two functions satisfy the boundary conditions (A.4) for $\theta=1$ (see (3.9)).

To prove that, for $i=1$ and 2, $\tilde{u}_i$ is a steady state of (1.1) (i.e., that it satisfies (A.1)), let $\psi\in C_0^\infty(\mathbb{R})$ (the space of functions of $C^\infty(\mathbb{R})$ with compact support). We multiply (3.4) by ψ, integrate from $-\infty$ to ∞ in the y variable, and integrate by parts twice to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} &\psi(y)[\partial u_i(y,t)/\partial t]\,dy \\ &= \int_{-\infty}^{\infty}\psi''(y)u_i(y,t)\,dy - \int_{-\infty}^{\infty}\psi(y)f(u_i(y,t),y)\,dy,\end{aligned}$$

where $f(u,y)=(u/2)\exp(my-u)$. We further integrate from zero to T in the t variable and divide by T, to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} &\psi(y)\{[u_i(y,t)-u_i(y,0)]/T\}\,dy \\ &= \int_{-\infty}^{\infty}\psi''(y)\left(\int_0^T u_i(y,t)\,dt/T\right)dy \\ &\quad - \int_{-\infty}^{\infty}\psi(y)\left(\int_0^T f(u_i(y,t),y)\,dt/T\right)dy.\end{aligned} \tag{3.10}$$

But since, for each $y \in \mathbb{R}$, $u_i(y, t) \to \tilde{u}_i(y)$ as $t \to \infty$, we have

$$
(3.11) \qquad \begin{aligned} &[u_i(y, t) - u_i(y, 0)]/T \to 0, \qquad \int_0^T u_i(y, t)\, dt/T \to \tilde{u}_i(y), \\ &\int_0^T f(u_i(y, t), y)\, dt/T \to f(\tilde{u}_i(y), y) \quad \text{pointwise as } T \to \infty. \end{aligned}
$$

Furthermore, the left-hand sides in the limits (3.11) are uniformly bounded in every bounded interval of $\mathbb{R}$ (see (3.8)) and, in particular, in supp ψ. Then if we let $T \to \infty$ in (3.10), the dominated convergence theorem [28] yields

$$
\int_{-\infty}^{\infty} \psi''(y) \tilde{u}_i(y)\, dy = \int_{-\infty}^{\infty} \psi(y) f(\tilde{u}_i(y), y)\, dy,
$$

for all $\psi \in C_0^\infty(\mathbb{R})$. Therefore $\tilde{u}_i$ satisfies (A.1) as a distribution (observe that $\tilde{u}_i \in L_{2,\mathrm{loc}}(\mathbb{R})$, as we see by means of the dominated convergence theorem when taking into account (3.8)) and, since the function $y \to f(\tilde{u}_i(y), y)$ belongs to $L_{2,\mathrm{loc}}(\mathbb{R})$, $\tilde{u}_i \in W^2_{2,\mathrm{loc}}(\mathbb{R})$. Also, $\tilde{u}_i \in W^p_{2,\mathrm{loc}}(\mathbb{R})$ for all $p > 2$, as is seen by reiterating the argument. Then embedding theorems [28] imply that $\tilde{u}_i \in C^\infty(\mathbb{R})$ and satisfies (A.1) as stated.

Finally, $\tilde{u}_1 = \tilde{u}_2 = u_s$ is Σ-stable, as comes out when Theorems 2.3 and 2.4 are applied, and it is taken into account that, if $\alpha, \beta \geqq 0$, then the functions $x \to u_s(x_1 - \alpha)$ and $x \to u_s(x_1 + \beta)$ are steady sub- and supersolutions of (1.1), respectively, as is easily seen.

Corollary 3.2. *If $m < 0$ and $n = 3$, then* (1.1)–(1.4) *has a unique steady state u_s which depends only on the x_1 variable.*

Proof. The steady state of Theorem 3.1 is necessarily the unique steady state of (1.1)–(1.4) since it is globally attracting.

Remarks 3.3. (A) In Theorem 3.1 we have shown that, for every initial datum φ satisfying (1.2)–(1.4), the solution of (1.1)–(1.5) is such that $u(\cdot, t) \to u_s$ pointwise as $t \to \infty$. It may be seen that the convergence is uniform on compact subsets of $\mathbb{R}^3$, but it is not uniform in $\mathbb{R}^3$ for arbitrary initial data. For example, if φ depends only on the x_1 variable, $\varphi(x_1) - x_1$ has a limit as $x_1 \to \infty$, and $\lim (\varphi(x_1) - x_1) \neq \lim (u_s(x_1) - x_1)$ as $x_1 \to \infty$, then the solution of (1.1)–(1.5) satisfies, for each $t > 0$, $\lim (u(x_1, t) - x_1) = \lim (\varphi(x_1) - x_1) \neq \lim (u_s(x_1) - x_1)$ as $x_1 \to \infty$, as may be seen.

(B) Corollary 3.2 shows that, in addition to the (spatially one-dimensional) steady state of (1.1)–(1.4) found by Liñán [2], there are no other steady states, possibly depending on the x_2 and x_3 coordinates. For a more precise information about the (unique) steady state of (1.1)–(1.4), see Theorem A.4 in the Appendix.

We now consider the case $m > 0$, in which (1.1)–(1.4) possesses a unique spatially one-dimensional steady state (see Theorem A.8 in the Appendix), that is expected to be unstable, according to the numerical results by Peters [20].

Theorem 3.4. *If $0 < m < \frac{1}{2}$ and $n = 3$, let u_s be the (unique) spatially one-dimensional steady state of* (1.1)–(1.4). *Then*:

(A) *If the initial state* (1.5) *satisfies $\varphi(x) \geqq u_s(x_1 + \alpha)$ for some $\alpha > 0$ and all $x \in \mathbb{R}^3$, then the solution of* (1.1)–(1.5) *is uniquely defined for all $t \geqq 0$ and such that, for each $x \in \mathbb{R}^3$, $\lim u(x, t) = \infty$ as $t \to \infty$.*

(B) *If a solution of* (1.1)–(1.5) *is defined for all $t \geqq 0$ and the initial state satisfies $\varphi(x) \leqq u_s(x_1 - \alpha)$ for all $x \in \mathbb{R}^3$ and some $\alpha > 0$, then $\lim u(\cdot, t) = 0$ pointwise as $t \to \infty$.*

(C) *The steady state u_s is Σ-unstable (Σ defined by* (3.1)).

Remark 3.5. Under the hypothesis of Theorem 3.4B, the solution of (1.1)–(1.5) is uniquely defined in $0 \leqq t < \infty$ whenever the initial state satisfies $u(x, 0) \geqq \varphi(x)$ for all $x \in \mathbb{R}^n$ (φ as given in Theorem 3.4B), as is seen when taking the solution considered in Theorem 3.4B as w in Remark 2.2A. If the solution of Theorem 3.4B is not assumed to exist for all $t \geqq 0$ but the other hypothesis is maintained, then the maximal solution of (1.1), (1.2), (1.4), (1.5) (that exists for $0 \leqq t < \infty$; see Remark 2.2A) satisfies the conclusion, as is seen after slight modifications in the proof.

Proof of Theorem 3.4. (A) To prove that u is uniquely defined for all $t \geqq 0$, observe that $w = u_s$ satisfies the required properties of Remark 2.2A. It is sufficient to prove the remaining parts of the statement when $\varphi(x) = u_s(x_1 + \alpha)$ (Theorem 2.3); then $u(x, t) = u(x_1, t)$ does not depend on the x_2 and x_3 coordinates, and satisfies $\partial u(x_1, t)/\partial t \geqq 0$ for all $(x_1, t) \in \mathbf{R} \times [0, \infty[$ (apply Theorem 2.5 and take into account that the function $x_1 \to u_s(x_1 + \alpha)$ is a subsolution of (1.1) since $\alpha > 0$). To prove that $\lim u(x, t) \to \infty$ pointwise as $t \to \infty$ suppose, on the contrary, that for some finite c, $x_1^0 \in \mathbf{R}$, $u(x_1^0, t) \leqq c$ for all $t > 0$. Then

$$u(x_1, t) \leqq c \quad \text{for all } (x_1, t) \in]-\infty, x_1^0] \times [0, \infty[,$$

$$u(x_1, t) \leqq c + x_1 - x_1^0 \quad \text{for all } (x_1, t) \in [x_1^0, \infty[\times [0, \infty[,$$

as is seen by applying the Ph-L maximum principle on the intervals $]-\infty, x_1^0[$ and $]x_1^0, \infty[$. Then, for each $x_1 \in \mathbb{R}$, the increasing function $t \to u(x_1, t)$ is bounded above and, by the argument of the proof of Theorem 3.1, $u(x_1, t) \to \bar{u}_s(x_1)$ pointwise as $t \to \infty$, where $\bar{u}_s$ is a solution of (A.1), (A.2) such that $u_s(x_1) < \bar{u}_s(x_1)$ for all $x_1 \in \mathbb{R}$. But this is not possible, according to Theorem A.4.

(B) As in the proof of part A, it is sufficient to prove the result when $\varphi(x) = u_s(x_1 - \alpha)$. Then the solution does not depend on the x_2 and x_3 coordinates, but is defined for all $t \geqq 0$ (Remark 3.5), and, by the argument of the proof of part A ($t \to u(x_1, t) \geqq 0$ is now decreasing), $u(x_1, t) \to \bar{u}_s(x_1)$ pointwise as $t \to \infty$, where $\bar{u}_s$ satisfies (A.1) and $0 \leqq \bar{u}_s(x_1) < u_s(x_1)$ for all $x_1 \in \mathbb{R}$. Then $\bar{u}_s(x_1) = 0$ for all $x_1 \in \mathbf{R}$ (Lemma A.10 in the Appendix) and the conclusion follows.

(C) Apply parts A and B above.

4. Global stability results in the adiabatic case ($m = 0$). Let us now consider the critical case $m = 0$. Again, we are interested in the stability properties of the spatially one-dimensional steady states of (1.1)–(1.4) (there are infinitely many due to translation invariance; see Theorem A.2 in the Appendix) under spatially three-dimensional perturbations. The last part of the proof of Theorem 3.1 is readily extended to yield Theorem 4.1 below.

THEOREM 4.1. *If $m = 0$, then every spatially one-dimensional steady state of* (1.1)–(1.4) *is Σ-stable* (Σ as given in (3.1)).

The remaining part of Theorem 3.1 cannot hold in this case, since there is no *unique* steady state now. We can easily be convinced that comparison methods alone cannot lead us further in the analysis of asymptotic stability properties if $m = 0$. Linear stability of the steady states of (1.1)–(1.4) is easily analyzed for $n = 1$. Although linear stability results do not solve the problem, they are enlightening, and help us to avoid the pursuit of ideas that cannot work in this case. For $n = 1$, the linear eigenvalue problem associated with a given steady state u_s of (1.1)–(1.3) is

$$u'' - f'(u_s)u = \omega u \quad \text{in } -\infty < x < \infty, \tag{4.1}$$

where $f(u) = (u/2) \exp(-u)$. The steady state u_s is easily seen to be such that u_s''/u_s', u_s'''/u_s', and $f(u_s)$ are bounded in $-\infty < x < \infty$. We consider (4.1) in $L2(\mathbb{R})$ (where (4.1) is self-adjoint) and in $C(\bar{\mathbb{R}})$ (the space of real, bounded, uniformly continuous functions on $\mathbb{R}$ with the sup norm). The function $u_s' \in C(\bar{\mathbb{R}})$ satisfies

$$u_s''' - f'(u_s)u_s' = 0 \quad \text{in } -\infty < x < \infty, \tag{4.2}$$

and thus is an eigenfunction of (4.1) associated with $\omega = 0$. Then the general solution of the homogeneous equation (4.1) is easily calculated for $\omega = 0$, and it is seen that $\omega = 0$ is a simple eigenvalue in $C(\bar{\mathbb{R}})$ and that it is not an eigenvalue in $L2(\mathbb{R})$. Also, if Re $\omega > 0$, then any bounded eigenfunction of (4.1) belongs to $L2(\mathbb{R})$, as is seen from its asymptotic behavior as $x \to \pm\infty$ (see, e.g., [32]); then $\omega \in \mathbb{R}$ also in $C(\bar{\mathbb{R}})$, and any eigenfunction u of (4.1) is such that

$$\omega \int_{-\infty}^{\infty} u^2\,dx = -\int_{-\infty}^{\infty} [u' - uu_s''/u_s']^2\,dx, \tag{4.3}$$

as seen after multiplication of (4.1) by u, integration from $-\infty$ to ∞, substitution of (4.2), and integration by parts. To obtain (4.3) observe that, since $u \in L2(\mathbb{R})$, $u'' \in L2(\mathbb{R})$ (see (4.1)), and $u' \in L2(\mathbb{R})$ as shown by interpolation inequalities (see, e.g., [28, p. 70]). Equation (4.3) implies that every eigenvalue of (4.1) in $C(\bar{\mathbb{R}})$ or in $L2(\mathbb{R})$ is such that Re $\omega \leqq 0$. If the continuous spectrum of (4.1), σ, were such that max Re $\sigma < 0$, then standard results on linear stability [33, p. 108, Exercise 6] would show that if $u(\cdot, 0)$ is in a certain neighborhood (in $C(\bar{\mathbb{R}})$) of u_s, then the solution of (1.1)–(1.5) for $n = 1$ approaches exponentially a translate of u_s as $t \to \infty$. Unfortunately, the continuous spectrum of (4.1), in $L2(\mathbb{R})$ and in $C(\bar{\mathbb{R}})$, is $\sigma =]-\infty, 0] \subset \mathbb{R}$ [33, p. 140], and the result above does not apply. Observe that the spectrum of (4.1) is equal to that of the heat equation (which also has infinitely many steady states in $C(\bar{\mathbb{R}})$), which, as is well known, exhibits erratic behavior as $t \to \infty$ for appropriate initial conditions in every neighborhood (in $C(\bar{\mathbb{R}})$) of each steady state (see, e.g., [26, p. 349]). Finally, let us point out that problem (1.1)–(1.3) for $n = 1$ has some features in common with one-dimensional reaction-diffusion problems exhibiting travelling fronts, which have received considerable attention in the literature (see [33] and [34] and references given therein).

We first consider problem (1.1)–(1.5) in one space dimension. The first part of the following theorem contains an invariant principle that holds for a general class of semilinear parabolic equations *in a bounded domain*, as is well known [33, § 4.3]. There are some more recent extensions of this principle (see, e.g., [35], [36]) that, unfortunately, do not apply to (1.1)–(1.5). Observe also that the result of Theorem 4.2B implies stabilization of certain solutions of (1.1)–(1.5) in a very weak sense, and resembles well-known results for travelling fronts, such as those appearing in the celebrated Kolmogorov–Petrovsky–Piscounov model equation [33, p. 134].

THEOREM 4.2. *If $n = 1$, let the hypothesis of Theorem* 2.1 *be satisfied, and let u_s be a steady state of* (1.1)–(1.3). *If the initial state* (1.5) *is such that $u_s(x - \alpha) \leqq \varphi(x) \leqq u_s(x + \beta)$, $\varphi'(x) > 0$ in $-\infty < x < \infty$, for some finite constants α and β, and $\varphi' - u_s' \in W_2^6(\mathbb{R})$, then the unique solution of* (1.1)–(1.3), (1.5) *is such that*

$$u_s(x - \alpha) \leqq u(x, t) \leqq u_s(x + \beta), \qquad u_x(x, t) \geqq 0 \quad \textit{for all } (x, t) \in \mathbb{R} \times [0, \infty[, \tag{4.4}$$

$$u_x - u_s' \in W_2^{6,3}(\mathbb{R} \times [0, T[) \quad \textit{for all } T \in [0, \infty[, \tag{4.5}$$

and satisfies the following properties:

(A) *There exists a C^2 bounded function $\xi : [0, \infty[\to \mathbb{R}$ such that $u(x, t) - u_s(x + \xi(t)) \to 0$, uniformly on bounded intervals of $\mathbb{R}$, as $t \to \infty$.*

(B) *$\xi'(t) \to 0$ as $t \to \infty$.*

Proof. The first inequalities (4.4) are readily obtained by applying Theorem 4.1. Then (4.5) is obtained by standard estimates on $W_2^{2m,m}$ spaces (see, e.g., [24, Chap. IV, § 9]) applied to the (linear) parabolic Cauchy problem for $u_x - u_s'$ which is obtained by differentiating (1.1) with respect to x. Then $\lim (u_x - u_s') = 0$ as $x \to \pm\infty$, uniformly in $0 \leqq t < T$ for all $T \in [0, \infty[$, and the second inequality (4.4) is readily obtained when the Ph-L maximum principle is applied to the equation obtained by differentiating (1.1) with respect to x, and we take into account that $\varphi'(x) \geqq 0$ in $-\infty < x < \infty$. We now prove properties A and B.

(A) We define the energy integral

$$H(t) = \int_{-\infty}^{\infty} [(u_x - u_s')^2 + (1 + u_s - u_s u + u_s^2) \exp(-u_s) - (1+u) \exp(-u)]\, dx,$$

which, when using (4.5), is easily seen to satisfy

$$H'(t) = -2 \int_{-\infty}^{\infty} u_t^2 \, dx.$$

Then the function $t \to H(t)$ is monotonically decreasing; since it is bounded below (see (4.4)), it has a limit as $t \to \infty$, and

(4.6) $$\int_{-\infty}^{\infty} (u_x - u_s')^2 \, dx \quad \text{and} \quad \int_0^t dt \int_{-\infty}^{\infty} u_t^2 \, dx \quad \text{are bounded in } 0 \leqq t < \infty.$$

On the other hand, when differentiating (1.1) with respect to t, multiplying by u_t, integrating in the x variable from $-\infty$ to ∞, and taking into account (4.4), (4.5) we obtain

(4.7) $$\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} u_t^2 \, dx \leqq -\int_{-\infty}^{\infty} u_{tx}^2 \, dx + k \int_{-\infty}^{\infty} u_t^2 \, dx \quad \text{in } 0 \leqq t < \infty$$

for a certain positive, finite constant k. When we take into account (4.6), this inequality yields

(4.8) $$\frac{d}{dt} \int_{-\infty}^{\infty} u_t^2 \, dx \text{ is bounded above} \quad \text{in } 0 \leqq t < \infty.$$

Then, (4.6) and (4.8) imply that

(4.9) $$\int_{-\infty}^{\infty} u_t^2 \, dx \to 0 \quad \text{as } t \to \infty.$$

Now, when using Hölder's inequality, (4.6) and (4.9) yield

$$0 \leqq \left| \int_{-\infty}^{x} u_x u_t \, dx \right| \leqq \left[\int_{-\infty}^{\infty} (u_x - u_s')^2 \, dx \int_{-\infty}^{\infty} u_t^2 \, dx \right]^{1/2} + \left[\int_{-\infty}^{x} u_s'^2 \, dx \int_{-\infty}^{\infty} u_t^2 \, dx \right]^{1/2} \to 0 \quad \text{as } t \to \infty,$$

uniformly on each bounded interval of $\mathbf{R}$. Then, when multiplying (1.1) by u_x and integrating in the x variable from $-\infty$ to x, we easily obtain

(4.10) $$u_x^2 - 1 + (1+u) \exp(-u) \to 0 \quad \text{as } t \to \infty,$$

uniformly on each bounded interval of $\mathbf{R}$.

Finally, for each $t > 0$, let us define $\xi(t)$ as the unique solution of the equation

(4.11) $$u(0, t) = u_s(\xi(t)).$$

Since $u_s'(x)>0$ in $-\infty<x<\infty$, $t\to\xi(t)$ is a well-defined C^2-function in $0\leqq t<\infty$ (Inverse Function Theorem) and (see (4.4))

$$-\infty<-\alpha\leqq\xi(t)\leqq\beta<\infty \quad \text{in } 0\leqq t<\infty. \tag{4.12}$$

Then, since u satisfies (4.10), (4.11), $u_x(x,t)\geqq 0$ for all $(x,t)\in\mathbb{R}\times[0,\infty[$ (see (4.4)) and, for each fixed $t\geqq 0$, the function $x\to u_s(x+\xi(t))$ satisfies (A.6), standard results on continuous dependence on parameters of the solution of the Cauchy problem for ordinary differential equations [32] imply that $u(x,t)-u_s(x+\xi(t))\to 0$, uniformly on bounded intervals of $\mathbb{R}$, as $t\to\infty$.

(B) We first observe that

$$\int_{-\infty}^{\infty} u_{xx}^2\,dx \text{ is bounded} \quad \text{in } 0\leqq t<\infty,$$

as obtained from (1.1) when taking into account (4.4) and (4.9). Then for each $(x,t)\in\mathbb{R}\times[0,\infty[$ we have

$$\begin{aligned}[u_x(x,t)-u_s'(x)]^2 &= 2\int_{-\infty}^{x}(u_x-u_s')(u_{xx}-u_s'')\,dx\\ &\leqq 2\left[\int_{-\infty}^{\infty}(u_x-u_s')^2\,dx\int_{-\infty}^{\infty}(u_{xx}-u_s'')^2\,dx\right]^{1/2},\end{aligned}$$

and (see (4.6))

$$u_x(x,t) \text{ is uniformly bounded} \quad \text{in } \mathbb{R}\times[0,\infty[. \tag{4.13}$$

On the other hand, when integrating (4.7) from zero to ∞ and taking into account (4.6) and (4.9), we obtain

$$\int_0^t dt\int_{-\infty}^{\infty} u_{xt}^2\,dx \text{ is bounded} \quad \text{in } 0\leqq t<\infty. \tag{4.14}$$

In addition, when differentiating (1.1) twice with respect to x and to t, multiplying by u_{xt}, integrating in the x variable from $-\infty$ to ∞, and taking into account (4.4), (4.5), and (4.13), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}u_{xt}^2\,dx\leqq k_1\int_{-\infty}^{\infty}u_{xt}^2\,dx+k_2\left[\int_{-\infty}^{\infty}u_{xt}^2\,dx\int_{-\infty}^{\infty}u_t^2\,dx\right]^{1/2}$$

for certain finite constants k_1 and k_2. If we integrate this inequality from zero to t, and take into account (4.6) and (4.14), we get that

$$\int_{-\infty}^{\infty} u_{xt}^2\,dx \text{ is bounded} \quad \text{in } 0\leqq t<\infty.$$

Then (4.9) yields

$$0\leqq u_t(0,t)^2=2\int_{-\infty}^{0}u_tu_{xt}\,dx\leqq 2\left[\int_{-\infty}^{\infty}u_t^2\,dx\int_{-\infty}^{\infty}u_{tx}^2\,dx\right]^{1/2}\to 0 \quad \text{as } t\to\infty,$$

and, since $u_t(0,t)=u_s'(\xi(t))\xi'(t)$ and $\xi(t)$ satisfies (4.12), the conclusion follows.

Observe that Theorem 4.2 does not imply that u approaches a steady state of (1.1)-(1.3) as $t\to\infty$. Nevertheless, if $u(x,0)-u_s(x)\to 0$ as $x\to+\infty$, for a certain steady state u_s, then u approaches u_s as $t\to\infty$, as is proven below. To this end, let us assume that the hypotheses of Theorem 4.2 are satisfied (less than that is needed in the following analysis, but more generality will not be necessary and is avoided for the sake of brevity). Let us introduce the function $\rho:\mathbb{R}\times[0,\infty[\to\mathbb{R}$ by

$$u_s(\rho(x,t))=u(x,t).$$

Since $u_s'(x)>0$ in $-\infty<x<\infty$, ρ is a well-defined function of $C^{r,r/2}(\mathbb{R}\times[0,\infty[)$ for some $r>3$ (Inverse Function Theorem) and satisfies

$$\partial\rho/\partial t=\partial^2\rho/\partial x^2+g(\rho)[(\partial\rho/\partial x)^2-1] \quad \text{in } \mathbb{R}\times[0,\infty[, \tag{4.15}$$

as is easily seen, where

$$g(\rho)=u_s''(\rho)/u_s'(\rho)$$

is positive and uniformly bounded and

$$g'(\rho)=[u_s'(\rho)u_s'''(\rho)-u_s''(\rho)^2]/u_s'(\rho)^2$$

is uniformly bounded. To prove that, take into account that u_s satisfies (A.1) and (A.6).

In addition, the function ρ satisfies

$$x-\alpha\leqq\rho(x,t)\leqq x+\beta, \qquad \rho_x(x,t)\geqq 0 \quad \text{in } \mathbb{R}\times[0,\infty[, \tag{4.16}$$

from (4.4).

The required result will be easily obtained from the following two lemmas.

LEMMA 4.3. *Under the assumptions above, if* $0\leqq\rho_x(x,0)\leqq 1$ (*respectively,* $1\leqq\rho_x(x,0)<\infty$) *in* $-\infty<x<\infty$, *then* $0\leqq\rho_x(x,t)\leqq 1$ (*respectively,* $1\leqq\rho_x(x,t)<\infty$) *for all* $(x,t)\in\mathbb{R}\times[0,\infty[$.

Proof. Let us first show that there is a finite constant k such that

$$0\leqq\rho_x(x,t)\leqq k \quad \text{for all } (x,t)\in\mathbb{R}\times[0,\infty[. \tag{4.17}$$

ρ_x is nonnegative (see (4.16)) and, since $\rho_x(x,0)$ is bounded in $-\infty<x<\infty$, $u_s'(x)\exp(-x/\sqrt{2})$ has a limit as $x\to-\infty$ (as is seen from the asymptotic behavior of (A.1), (A.2)), and ρ satisfies (4.16), we have

$$0\leqq u_x(x,0)=u_s'(\rho(x,0))\rho_x(x,0)\leqq k_1\exp(x/\sqrt{2}) \quad \text{in } -\infty<x<\infty \tag{4.18}$$

for a certain finite constant k_1. Then we can see that

$$u_x(x,t)\leqq k_2\exp(x/\sqrt{2}) \quad \text{for all } (x,t)\in\,]-\infty,x_0[\,\times[0,\infty[, \tag{4.19}$$

where x_0 is any point of $\mathbb{R}$ such that $u_s(x_0+\beta)<1$ (then $u(x,t)\leqq 1$ in $]-\infty,x_0]\times[0,\infty[$; see (4.16)) and $k_2=\max\{k_1,\sup\{u_x(x_0,t): t\geqq 0\}\}$ (k_2 is finite; see (4.13)). To prove that (4.19) holds, apply the Ph-L maximum principle in $-\infty<x\leqq x_0$ to the equation obtained when (1.1) is differentiated with respect to x, and take into account (4.13) and (4.18). Then $\rho_x(x,t)=u_x(x,t)/u_s'(\rho(x,t))$ satisfies (4.17) since (i) $u_s'(x)\exp(-x/\sqrt{2})$ and $u_s'(x)$ are bounded below by a strictly positive constant in $]-\infty,x_0]$ and in $[x_0,\infty[$, respectively; (ii) the function $x\to u_s'(x)$ is strictly increasing in $-\infty<x<\infty$; (iii) ρ satisfies (4.16); and (iv) u_x satisfies (4.13) and (4.19).

Then the conclusion of the lemma readily follows when the Ph-L maximum principle is applied to the equation obtained when (4.15) is differentiated with respect to x, and it is taken into account that g and g' are uniformly bounded, and that (4.17) holds.

LEMMA 4.4. *If, in addition to the assumptions of Theorem* 4.2, *the initial condition* (1.5) *is such that* $\varphi(x)=u_s(x)$ *in* $k<x<\infty$, *for some finite constant* k, *then* $u(x,t)\to u_s(x)$ *pointwise as* $t\to\infty$.

Proof. Let the function $\rho^0:\mathbb{R}\to\mathbb{R}$ be defined by $u_s(\rho^0(x))=\varphi(x)$; as above, ρ^0 is a well-defined function such that $x-\alpha\leqq\rho^0(x)\leqq x+\beta$ in $-\infty<x<\infty$.

It is easily seen also that there exist two functions, $\rho_1^0, \rho_2^0\in C^\infty(\mathbb{R})$, such that

$$x-\alpha\leqq\rho_1^0(x)\leqq\rho^0(x)\leqq\rho_2^0(x)\leqq x+\beta, \qquad -\infty<x<\infty, \tag{4.20}$$

$$\rho_1^0(x)=x-\alpha, \qquad \rho_2^0(x)=x+\beta \qquad \text{in } -\infty<x<k_1, \tag{4.21}$$

$$\rho_1^0(x)=\rho_2^0(x)=\rho^0(x)=x \qquad \text{in } k_2<x<\infty, \tag{4.22}$$

$$1\leqq d\rho_1^0/dx<\infty, \qquad 0\leqq d\rho_2^0/dx\leqq 1 \quad \text{in } -\infty<x<\infty \tag{4.23}$$

for some finite constants k_1 and k_2. Then the assumptions of Theorem 4.2 are satisfied for the functions u_1 and u_2 given by (1.1)–(1.3) and

$$u_i(x,0)=u_s(\rho_i^0(x)) \quad \text{in } -\infty<x<\infty \quad \text{for } i=1,2.$$

Furthermore, since $u_1(\cdot,0)\leqq u(\cdot,0)\leqq u_2(\cdot,0)$ and $u_1(\cdot,0)\leqq u_s\leqq u_2(\cdot,0)$ in $\mathbb{R}$ (see (4.20)–(4.23)), Theorem 2.3 yields

$$u_1(\cdot,t)\leqq u(\cdot,t)\leqq u_2(\cdot,t), \qquad u_1(\cdot,t)\leqq u_s\leqq u_2(\cdot,t) \quad \text{in } \mathbb{R} \tag{4.24}$$

for all $t\geqq 0$. Then the conclusion follows if we prove that

$$u_i(x,t)\to u_s(x) \quad \text{pointwise as } t\to\infty. \tag{4.25}$$

To this end, let us define, for $i=1$ and 2, the function $\rho_i:\mathbb{R}\times[0,\infty[\to\mathbb{R}$ by $u_i(x,t)=u_s(\rho_i(x,t))$; that function is again well defined and $\rho_i(x,0)=\rho_i^0(x)$ in $-\infty<x<\infty$. Then (apply Lemma 4.3 and take into account (4.23)),

$$1\leqq\partial\rho_1/\partial x<\infty, \qquad 0\leqq\partial\rho_2/\partial x\leqq 1 \quad \text{in } \mathbb{R}\times[0,\infty[, \tag{4.26}$$

and, since ρ_i satisfies (4.15) and $\partial u_i(x,t)/\partial t=u_s'(\rho_i(x,t))\partial\rho_i/\partial t$, we have, for $i=1$ and 2 and for all $t\geqq 0$,

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty}[u_i(x,t)-u_s(x)]\,dx &= \int_{-\infty}^{\infty}u_s'(\rho_i(x,t))\frac{\partial\rho_i(x,t)}{\partial x}\,dx\\
&=\int_{-\infty}^{\infty}u_s'(\rho_i(x,t))\frac{\partial^2\rho_i(x,t)}{\partial x^2}\,dx\\
&\qquad+\int_{-\infty}^{\infty}u_s''(\rho_i(x,t))\left[\left(\frac{\partial\rho_i(x,t)}{\partial x}\right)^2-1\right]dx\\
&=1-\int_{-\infty}^{\infty}u_s''(\rho_i(x,t))\,dx\\
&=\int_{-\infty}^{\infty}u_s''(\rho_i(x,t))\left[\frac{\partial\rho_i(x,t)}{\partial x}-1\right]dx,\end{aligned}$$

where the manipulations on the improper integral required to obtain the first equality are easily seen to be justified. The third equality is obtained by integration by parts in the first integral of the left-hand side, when taking into account that $\rho_{ix}(x,t)$ is bounded and $u_s'(\rho_i(x,t))\to 0$ as $x\to-\infty$, and that $\rho_{ix}(x,t)\to 1$ and $u_s'(\rho_i(x,t))\to 1$ as $x\to\infty$, for all $t\geqq 0$; the last equality is obtained when taking into account that

$$\int_{-\infty}^{\infty}u_s''(\rho_i(x,t))\frac{\partial\rho_i(x,t)}{\partial t}\,dx=\int_{-\infty}^{\infty}\frac{d}{dx}[u_s'(\rho_i(x,t))]\,dx=1.$$

Then the functions $t \to \int_{-\infty}^{\infty} (u_s - u_1)\,dx$ and $t \to \int_{-\infty}^{\infty} (u_2 - u_s)\,dx$ are monotonically decreasing (see (4.26)) and nonnegative (see (4.24)) in $0 \leqq t < \infty$.

Therefore, for $i = 1$ and 2,

$$\int_{-\infty}^{\infty} |u_i - u_s|\,dx \text{ is bounded} \quad \text{in } 0 \leqq t < \infty;$$

since, in addition, u_1 and u_2 satisfy property A of Theorem 4.2, (4.25) readily follows, and the proof is complete.

Finally, we prove the main result of this section.

THEOREM 4.5. *If $m = 0$ and $n = 3$, let u_s be a spatially one-dimensional steady state of* (1.1)–(1.3), *and let the assumptions of Theorem* 2.1 *hold. If the initial state* (1.5) *is such that*

$$u_s(x_1 - \alpha) \leqq \varphi(x) \leqq u_s(x_1 + \beta) \quad \textit{for all } x \in \mathbb{R}^3$$

for some finite constants α and β, and

$$\lim\,(\varphi(x) - u_s(x_1)) = 0 \quad \textit{as } x_1 \to \infty,$$

uniformly for $(x_2, x_3) \in \mathbb{R}^2$, then the solution of (1.1)–(1.5) *is such that* $\lim u(\cdot, t) \to u_s$ *pointwise as $t \to \infty$.*

Proof. From the assumptions above, it is clear that, for each $\varepsilon > 0$, there exist two functions, $\varphi_1^\varepsilon, \varphi_2^\varepsilon \in C^\infty(\mathbb{R})$, that satisfy (1.2), (1.3), and

$$\varphi_1^\varepsilon(x_1) \leqq \varphi(x) \leqq \varphi_2^\varepsilon(x_1) \quad \text{for all } x \in \mathbb{R}^3,$$

$$\text{(4.27)} \qquad \varphi_1^\varepsilon(x_1) = u_s(x_1 - \alpha), \qquad \varphi_2^\varepsilon(x_1) = u_s(x_1 + \beta) \quad \text{in } -\infty < x_1 < k_1,$$

$$\varphi_1^\varepsilon(x_1) = u_s(x_1 - \varepsilon), \qquad \varphi_2^\varepsilon(x_1) = u_s(x_1 + \varepsilon) \quad \text{in } k_2 < x_1 < \infty,$$

for some finite constants k_1 and k_2. For $i = 1$ and 2, let us define the functions $u_i^\varepsilon : \mathbb{R} \times [0, \infty[\to \mathbb{R}$ by (1.1)–(1.3) (with $n = 1$) and

$$u_i^\varepsilon(x_1, 0) = \varphi_i^\varepsilon(x_1) \quad \text{in } -\infty < x_1 < \infty.$$

Then (apply Theorem 2.3 and take into account (4.27))

$$\text{(4.28)} \qquad u_1^\varepsilon(x_1, t) \leqq u(x, t) \leqq u_2^\varepsilon(x_1, t) \quad \text{for all } (x, t) \in \mathbb{R}^3 \times [0, \infty[,$$

and (apply Lemma 4.4)

$$\text{(4.29)} \quad u_1^\varepsilon(x_1, t) \to u_s(x_1 - \varepsilon), \qquad u_2^\varepsilon(x_1, t) \to u_s(x_1 + \varepsilon) \quad \text{pointwise} \quad \text{as } t \to \infty.$$

Since (4.28) and (4.29) are true for all $\varepsilon > 0$, the conclusion follows.

Remarks 4.6. Some remarks about the result above are in order.

(A) Theorem 4.5 is true also in one and two space dimensions (after obvious modifications).

(B) The results of this section and, in particular, of Theorem 4.5, stand when the nonlinearity of (1.1) is replaced by a positive C^3-function $f : [0, \infty[\to \mathbb{R}$ such that

(i) $f(0) = 0$, $f'(u)$ is bounded in $0 \leqq u < \infty$;

(ii) $\int_0^\infty f(u)\,du$ exists and is equal to 1,

as may be seen. In particular, conditions (i) and (ii) are fulfilled by the first nonlinearity in (1.6) if either $p = 1$ or 2, or $p \geqq 3$ and $q > p + 1$, and by the second if $m = 0$, $a > 0$, and $p = 1$ or 2 or $p \geqq 3$, after multiplication by an appropriate positive constant.

5. Conclusions. In § 2 we showed that problem (1.1)–(1.5) has a unique classical solution in $0 \leqq t < T_0$, with $T_0 = \infty$ if $m \leqq 0$. If $0 < m < 1$, then $T_0 = \infty$ for appropriate initial conditions, but the solution is not expected to exist in $0 \leqq t < \infty$ for arbitrary initial data, as pointed out in Remark 2.2A.

Global stability properties for $m \neq 0$ were considered in § 3, where some previous partial numerical results on linear stability were confirmed and extended. In particular, the unique spatially one-dimensional steady state of (1.1)-(1.4) was shown to be unstable if $0 < m < \frac{1}{2}$ and globally, asymptotically stable in a certain sense if $m < 0$; in the latter case, it was shown also that (1.1)-(1.4) does not have other steady states, depending on the x_2 and/or the x_3 coordinates.

In § 3 we obtained sufficient conditions on the initial data for the solution of (1.1)-(1.5) to approach a given one-dimensional steady state.

Finally, let us point out that some questions about existence of more steady states and about the dynamics of (1.1)-(1.5) for $m \geqq 0$ remain unsolved. It seems that their solution requires more powerful mathematical tools (and perhaps some numerics on the two- and three-dimensional problems to get predictions) than those used in this paper. We think that any effort towards a complete understanding of (1.1)-(1.5) is worthwhile since, as was pointed out in the Introduction, Liñán's problem is ubiquitous in combustion theory.

Appendix. Spatially one-dimensional steady states of (1.1)–(1.4). We consider the one-dimensional steady states of (1.1)-(1.4) that satisfy the (slightly more general if $m \leqq 0$) boundary value problem

$$u'' = (u/2) \exp(mx - u) \quad \text{in } -\infty < x < \infty, \tag{A.1}$$

$$u \text{ bounded at } x = -\infty, \qquad |u - x| \text{ bounded at } x = +\infty, \tag{A.2}$$

where $u > 0$ in $\mathbf{R}$.

If $m \neq 0$, for each constant θ such that $0 < \theta < \infty$, (A.1) is invariant under the transformation

$$x \to \theta x - (2/m) \ln \theta, \qquad m \to m/\theta, \tag{A.3}$$

while the boundary conditions (A.2) become

$$u \text{ bounded at } x = -\infty, \qquad |u - \theta x| \text{ bounded at } x = \infty. \tag{A.4}$$

Therefore, problem (A.1)-(A.4), which will be considered below for convenience, is not essentially more general than (A.1), (A.2).

LEMMA A.1. *Every positive solution of* (A.1), (A.4) *satisfies*

$$\begin{aligned} &u \to u_0, \quad u' \to 0 \quad \text{as } x \to -\infty, \quad 0 < u' < \theta \quad \text{in } -\infty < x < \infty, \\ &u - \theta x \to c, \qquad u' \to \theta \quad \text{as } x \to +\infty \end{aligned} \tag{A.5}$$

for some finite constants $u_0 \geqq 0$ and c, with $u_0 = 0$ if $m \leqq 0$.

Proof. Since $u'' > 0$ in $-\infty < x < \infty$, the function $x \to u'(x)$ is strictly increasing, and the limits of u' at $x = -\infty$ and $x = +\infty$ exist; these limits are zero and θ, respectively, for (A.4) to be satisfied. Then $u' > 0$ in $-\infty < x < \infty$, and the limit of u at $x = -\infty$ exists, and it vanishes if $m \leqq 0$, for (A.4) to be satisfied. Finally, since $0 < u' < \theta$ in $-\infty < x < \infty$, the function $x \to u(x) - \theta x$ is strictly decreasing, and bounded at $x = +\infty$, and thus it must have a finite limit.

We first consider the case $m = 0$.

THEOREM A.2. *If $m = 0$ and $\theta = 1$, then* (A.1), (A.4) *possess a solution u that is unique up to translations, and such that*

$$u' > 0, \qquad u'^2 = 1 - (1 + u) \exp(-u) \quad \text{in } -\infty < x < \infty. \tag{A.6}$$

If $m = 0$ and $0 < \theta < \infty$, $\theta \neq 1$, then (A.1)-(A.4) *has no solution.*

Proof. Equation (A.6) is obtained after multiplication of (A.1) by u' and integration from $-\infty$ to x, when taking into account (A.5). A further integration of (A.6) easily yields the desired result by phase-plane arguments.

The case $m<0$ is considered next. We first prove the following uniqueness result, which is used in the proof of Theorem 3.1 in § 3.

LEMMA A.3. *If $m<0$ and $0<\theta<\infty$, then there are not two distinct solutions of* (A.1), (A.5), *u_1 and u_2, such that $u_1(x)\leqq u_2(x)$ in $-\infty<x<\infty$.*

Proof. Suppose, on the contrary, that $u_1\neq u_2$, and define $U_i(x)=u_i'(x)^2/2-F(u_i(x))\exp(mx)$, for $i=1$ and 2, where $F(u)=[1-(1+u)\exp(-u)]/2$. Then $U_i'(x)=-mF(u_i(x))\exp(mx)$, and

$$U_i(x)=-m\int_{-\infty}^{x}F(u_i(x))\exp(mx)\,dx$$

(the improper integral is seen to exist). Since the function F is strictly increasing, the function $x\to U_2(x)-U_1(x)$, which does not vanish identically, is nonnegative and increasing. Therefore, $\lim U_1(x)<\lim U_2(x)$ as $x\to\infty$, and this is not possible since $\lim U_1(x)=\lim U_2(x)=\theta^2/2$ as $x\to\infty$, from condition (A.5).

THEOREM A.4. *If $m<0$ and $0<\theta<\infty$, then* (A.1), (A.4) *have a unique solution u such that $u'(x)>0$ in $-\infty<x<\infty$, $u(x)\to 0$ as $x\to-\infty$, $u(x)-\theta x\to c$ as $x\to\infty$, for some finite constant c.*

Proof. If $\theta=1$, the result is readily obtained from Corollary 3.2 and Lemma A.1; if $\theta\neq 1$, the result is obtained by means of the transformation (A.3).

Remark A.5. Theorem A.4 contains the results by Hastings and Poore [18], who proved existence and uniqueness of solution of (A.1) for $m<0$, with boundary conditions

$$u'(x)\to 0 \quad \text{as } x\to-\infty, \qquad u'(x)\to\theta \quad \text{as } x\to\infty, \tag{A.7}$$

since, as we will see now, conditions (A.4) and (A.7) are equivalent when applied to (A.1). In fact, we will see that both (A.4) and (A.7) are equivalent to the following boundary condition:

$$u(x)\to 0 \quad \text{as } x\to-\infty, \qquad u(x)-\theta x\to c \quad \text{as } x\to\infty \tag{A.8}$$

for some finite constant c. That (A.4) implies (A.7) and (A.8) comes from Lemma A.1. Formulae (A.7) imply (A.4) since, by the argument in the proof of Lemma A.1, any solution u of (A.1), (A.7) satisfies (A.5). Furthermore, the function $x\to u(x)-\theta x$ is decreasing and thus it is bounded above as $x\to\infty$. Then, $u(x)\leqq w_2(x)$ in $-\infty<x<\infty$, where w_2 is the supersolution of (3.2), if b is sufficiently large. By the argument of the proof of Theorem 3.1, $u(x)\leqq\tilde{u}(x)$ in $-\infty<x<\infty$, where $\tilde{u}$ is the unique solution of (A.1), (A.4). Since $\tilde{u}$ satisfies (A.5), $u=\tilde{u}$ (Lemma A.3) and satisfies (A.4). Finally, any solution of (A.1), (A.8) clearly satisfies (A.4).

Now we consider the case $m>0$.

LEMMA A.6. *If $m>0$, for each $u_0\geqq 0$, there is a unique solution, $u(u_0;x)$, defined in $-\infty<x<\infty$, of the initial value problem*

$$\partial^2u/\partial x^2=(u/2)\exp(mx-u), \quad u\to u_0, \quad \partial u/\partial x\to 0 \quad \text{as } x\to-\infty, \tag{A.9}$$

and it is such that

(a) $u(0;x)=0$ *for all* $x\in\mathbb{R}$; $\partial u/\partial x>0$ *for all* $u_0>0$ *and all* $x\in\mathbb{R}$.

(b) *If $u_0>0$, then the derivative $\partial u(u_0;x)/\partial u_0=z(u_0;x)$ exists in $-\infty<x<\infty$ and is twice continuously differentiable with respect to x.*

(c) *If* $u_0 > 0$, *then the limits* $\lim \partial u(u_0; x)/\partial x = \psi(u_0)$ *and* $\lim \partial z(u_0; x)/\partial x = h(u_0)$, *as* $x \to \infty$, *exist. In addition, the function* $u_0 \to \psi(u_0)$ *is continuously differentiable and satisfies* $\psi(u_0) > m$, $\psi'(u_0) = h(u_0) \neq 0$ *in* $0 < u_0 < \infty$.

Proof. See Hastings and Poore [19], where this result is used to obtain uniqueness for (A.1), (A.7) when $m > 0$ and $\theta = 1$.

LEMMA A.7. *If* $m > 0$, *the function* ψ *of Lemma* A.6 *is such that*

(a) $\psi(u_0) \to 2m$ *as* $u_0 \to \infty$;

(b) $\psi(u_0) \to \infty$ *as* $u_0 \to 0$;

(c) $\psi'(u_0) < 0$ *for all* $u_0 > 0$.

Proof. (a) We multiply (A.9) by $\partial u/\partial x$, integrate from zero to $+\infty$, and integrate by parts twice, to obtain (recall that $\partial u(u_0; \infty)/\partial x > m$),

$$\begin{aligned} &[\psi(u_0) + \partial u(u_0; 0)/\partial x - 2m][\psi(u_0) - \partial u(u_0; 0)/\partial x] \\ &\quad = [1 + u(u_0; 0)] \exp[-u(u_0; 0)] \\ &\qquad + m \int_0^\infty \exp[mx - u(u_0; x)]\, dx. \end{aligned} \tag{A.10}$$

(Equation (A.10) was used by Ludford, Yanitell, and Buckmaster [5] to prove that (A.1), (A.7) has no solution if $\theta = 1$ and $\frac{1}{2} \leq m < 1$.)

Now, since the function $x \to u(u_0; x)$ is strictly increasing, we have

$$\begin{aligned} \int_0^\infty \exp(mx - u)\, dx &< \int_0^\infty (u/u_0) \exp(mx - u)\, dx \\ &= (2/u_0)[\psi(u_0) - \partial u(u_0; 0)/\partial x] \end{aligned} \tag{A.11}$$

(use (A.9) to obtain the last equality). In addition, we multiply (A.9) by $\partial u/\partial x$, and integrate from $-\infty$ to zero, to obtain

$$\begin{aligned} [\partial u(u_0; 0)/\partial x]^2 &= \int_{-\infty}^0 u(\partial u/\partial x) \exp(mx - u)\, dx \\ &< \int_{-\infty}^0 u(\partial u/\partial x) \exp(-u)\, dx \\ &< \int_{u_0}^\infty u \exp(-u)\, du. \end{aligned}$$

Then

$$\partial u(u_0; 0)/\partial x \to 0 \quad \text{as } u_0 \to \infty, \tag{A.12}$$

and the desired result is easily obtained from (A.12), when we take into account (A.11), (A.12).

(b) For each $u_0 < 1$, let $x_1 \in \mathbb{R}$ be (uniquely) defined by $u(u_0; x_1) = 1$. Then $u(u_0; x) < 1$ for $x < x_1$ and integration of (A.9) from $-\infty$ to x yields

$$\begin{aligned} \partial u(u_0; x)/\partial x &= \int_{-\infty}^x (u/2) \exp(mx - u)\, dx \\ &< (u/2) \exp(-u) \int_{-\infty}^x \exp(mx)\, dx \\ &= (u/2m) \exp(mx - u) \end{aligned}$$

for all $x \in]-\infty, x_1[$ (the function $u \to u \exp(-u)$ is strictly increasing in $0 \leqq u \leqq 1$). Then $(2m/u)(\partial u/\partial x) \exp(u) < \exp(mx)$ in $-\infty < x < x_1$, and integration of this inequality from $-\infty$ to x_1 leads to

$$2m^2 \int_{u_0}^{1} u^{-1} \exp(u)\, du < \exp(mx_1).$$

Thus $x_1 \to \infty$ as $u_0 \to 0$, and the conclusion follows from the next equation, which is obtained by multiplying (A.9) by $\partial u/\partial x$ and integrating from x_1 to ∞

$$\psi(u_0)^2 - [\partial u(u_0; x_1)/\partial x]^2 = \int_{x_1}^{\infty} u(\partial u/\partial x) \exp(mx - u)\, dx$$

$$> \exp(mx_1) \int_{1}^{\infty} u \exp(-u)\, du.$$

(c) Since $\psi'(u_0) \neq 0$ in $0 < u_0 < \infty$ (Lemma A.6), parts (a) and (b) above yield the result.

THEOREM A.8. (a) *If $m > 0$ and $2m < \theta < \infty$, then* (A.1), (A.4) *have a unique solution u, and $u'(x) > 0$ in $-\infty < x < \infty$, $u(x) \to u_0$ as $x \to -\infty$ and $u(x) - \theta x \to c$ as $x \to \infty$, for some finite constants u_0 and c such that $u_0 > 0$. If $0 < \theta \leqq 2m$, then* (A.1), (A.4) *have no solution.*

(b) *If $m > 0$, let u_1 and u_2 be the solutions of* (A.1), (A.4) *for $\theta = \theta_1$ and $\theta = \theta_2$, with $2m < \theta_1 < \theta_2 < \infty$. Then $u_2 < u_1$ as $x \to -\infty$.*

Proof. Apply Lemmas A.1 and A.7.

Remarks A.9. (a) Part (b) of Theorem A.8 is needed in § 3 to analyze the asymptotic behavior of some solutions of (1.1)–(1.4) as $t \to \infty$ when $m > 0$.

(b) Part (a) of Theorem A.8 contains the results by Hastings and Poore [19], who proved existence and uniqueness of the solution of (A.1), (A.7) for $\theta = 1$ if $0 < m < \frac{1}{2}$ and nonexistence if $m \geqq \frac{1}{2}$, since, as in Remark A.5, conditions (A.4), (A.7), and

$$\text{(A.13)} \qquad u \to u_0 \quad \text{as } x \to -\infty, \qquad u - \theta x \to c \quad \text{as } x \to \infty,$$

are equivalent when applied to (A.1). The equivalence of conditions (A.4), (A.7), and (A.13) can be proved by the same argument in Remark A.5 using Lemmas A.6 and A.7 and Theorem A.8.

The following result is needed in the proof of Theorem 3.4.

LEMMA A.10. *If $0 < m < \frac{1}{2}$, let u be the unique solution of* (A.1), (A.2). *If a solution $\bar{u}$ of* (A.1) *is such that*

$$0 \leqq \bar{u}(x) < u(x) \quad \textit{for all } x \in \mathbf{R},$$

then $\bar{u}(x) = 0$ for all $x \in \mathbf{R}$.

Proof. Since the function $x \to \bar{u}'(x)$ is strictly increasing, there exist the limits of $\bar{u}'$ as $x \to -\infty$ and as $x \to \infty$, and $\lim \bar{u}'(x) = 0$ as $x \to -\infty$, $\lim \bar{u}'(x) = \bar{\theta}$ as $x \to \infty$, for some $\bar{\theta} \in [0, 1]$. Then $\bar{u}$ satisfies (A.1), (A.4) (Remark A.9b), the limit of $\bar{u}$ as $x \to -\infty$, $\bar{u}_0$, exists and is finite (Lemma A.1), and $\bar{\theta}$ cannot be strictly positive (Theorem A.8). Therefore, $\bar{u}_0 = 0$ (Lemma A.6c) and $\bar{u}(x) = 0$ for all $x \in \mathbf{R}$ (Lemma A.6a).

REFERENCES

[1] E. CONWAY, D. HOFF, AND J. SMOLLER, *Large time behavior of solutions of systems of nonlinear reaction-diffusion equations*, SIAM J. Appl. Math., 35 (1978), pp. 1–16.

[2] A. LIÑÁN, *The asymptotic structure of counterflow diffusion flames for large activation energy*, Acta Astronaut., 1 (1974), pp. 1007–1039.

[3] ———, *Monopropellant droplet decomposition for large activation energies*, Acta Astronaut., 2 (1975), pp. 1009–1029.

[4] G. S. S. LUDFORD, D. W. YANNITELL, AND J. D. BUCKMASTER, *The decomposition of a hot monopropellant in an inert atmosphere*, Combust. Sci. Tech., 14 (1976), pp. 125–131.
[5] ———, *The decomposition of a cold monopropellant in an inert atmosphere*, Combust. Sci. Tech., 14 (1976), pp. 133–145.
[6] M. MATALON, G. S. S. LUDFORD, AND J. D. BUCKMASTER, *Diffusion flames in a chamber*, Acta Astronaut., 6 (1979), pp. 943–959.
[7] A. K. KAPILA AND G. S. S. LUDFORD, *Two step sequential reaction for large activation energy*, Combust. Flame, 29 (1977), pp. 167–176.
[8] J. PELÁEZ, *Stability of premixed flames with two thin reaction layers*, SIAM J. Appl. Math., 47 (1987), pp. 781–799.
[9] A. K. KAPILA AND A. B. POORE, *The steady response of a nonadiabatic tubular reactor: new multiplicities*, Chem. Engrg. Sci., 37 (1982), pp. 57–68.
[10] F. A. WILLIAMS, *Combustion Theory*, Benjamin–Cummings, Menlo Park, CA, 1985.
[11] G. I. SIVASHINSKY, *Nonlinear analysis of hydrodynamic instability in laminar flames*. I: *Derivation of basic equations*, Acta Astronaut., 4 (1977), pp. 1177–1206.
[12] B. J. MATKOWSKY AND G. I. SIVASHINSKY, *An asymptotic derivation of two models in flame theory associated with the constant density approximation*, SIAM J. Appl. Math., 37 (1979), pp. 686–699.
[13] G. JOULIN AND P. CLAVIN, *Linear stability analysis of non-adiabatic flames: Diffusional-thermal model*, Combust. Flame, 35 (1979), pp. 139–153.
[14] A. K. KAPILA AND B. J. MATKOWSKY, *Reactive–diffusive systems with Arrhenius kinetics: Multiple solutions, ignition and extinction*, SIAM J. Appl. Math., 36 (1979), pp. 373–389.
[15] A. K. KAPILA, B. J. MATKOWSKY, AND J. M. VEGA, *Reactive-diffusive systems with Arrhenius kinetics: Peculiarities of the spherical geometry*, SIAM J. Appl. Math., 38 (1980), pp. 382–341.
[16] J. M. VEGA AND A. LIÑÁN, *Singular Langmuir–Hinshelwood reaction–diffusion problems: Strong adsorption under quasi-isothermal conditions*, SIAM J. Appl. Math., 42 (1982), pp. 1047–1068.
[17] J. M. VEGA, *Singular Langmuir–Hinshelwood reaction–diffusion problems: Strongly nonisothermal conditions*, SIAM J. Appl. Math., 43 (1983), pp. 1367–1389.
[18] S. P. HASTINGS AND A. B. POORE, *A nonlinear problem arising from combustion theory: Liñán's problem*, SIAM J. Math. Anal., 14 (1983), pp. 425–430.
[19] ———, *Liñán's problem from combustion theory, part* II, SIAM J. Math. Anal., 16 (1985), pp. 331–340.
[20] N. PETERS, *On the stability of Liñán's premixed flame regime*, Combust. Flame, 33 (1978), pp. 315–318.
[21] D. S. STEWART AND J. D. BUCKMASTER, *The stability of Liñán's premixed flame regime revisited*, SIAM J. Appl. Math., 46 (1986), pp. 582–587.
[22] J. D. BUCKMASTER, A. NACHMAN, AND S. D. TALIAFERRO, *The fast time instability of diffusion flames*, Phys. D, 9 (1983), pp. 408–424.
[23] S. D. TALIAFERRO, *Stability and bifurcation of equilibrium solutions of reaction–diffusion equations on an infinite space interval*, J. Differential Equations, 54 (1984), pp. 19–59.
[24] O. A. LADYZENSKAJA, V. A. SOLONNIKOV, AND N. N. URAL'CEVA, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, RI, 1968.
[25] S. D. EIDEL'MAN, *Parabolic Systems*, North-Holland, Amsterdam, New York, 1969.
[26] M. H. PROTTER AND H. F. WEINBERGER, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1967.
[27] A. FRIEDMAN, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1964.
[28] R. A. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.
[29] C. ALVAREZ PEREIRA AND J. M. VEGA, *On the large-time behavior of a premixed, non-adiabatic reaction zone*, in preparation.
[30] J. SMOLLER, *Shock Waves and Reaction–Diffusion Equations*, Springer-Verlag, Berlin, New York, 1983.
[31] M. W. HIRSCH, *The dynamical systems approach to differential equations*, Bull. Amer. Math. Soc. (N.S.), 11 (1984), pp. 1–64.
[32] E. A. CODDINGTON AND N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
[33] D. HENRY, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. 840, Springer-Verlag, Berlin, New York, 1981.
[34] P. C. FIFE, *Asymptotic states for equations of reaction and diffusion*, Bull. Amer. Math. Soc., 84 (1978), pp. 693–726.
[35] M. W. HIRSCH, *Stability and convergence in strongly monotone flows*, preprint, Center for Pure and Applied Mathematics, University of California, Berkeley, CA, 1984.
[36] N. D. ALIKAKOS AND P. BATES, *Stabilization of solutions for a class of degenerate equations in divergence form in one space dimension*, preprint, 1988.