Triangular Bézier Developable Patches

Leonardo Fernández-Jambrina

Matemática Aplicada
E.T.S.I. Navales
Universidad Politécnica de Madrid, Spain
leonardo.fernandez@upm.es

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Sketch

Motivation

- Developable surfaces are defined as zero gaussian curvature surfaces (intrinsically flat).
- That is, plane patches that are curved by just folding, rolling or cutting, but without stretching or combing.
- Useful for depicting steel plates in naval industry, cloth in textile industry. . .
- But they are difficult to include in the NURBS formulation for the zero curvature requirement.

Description

Developable surfaces are a special case of ruled surfaces,

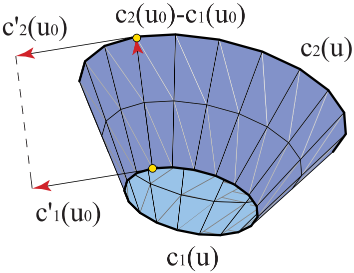
$$c(u,v) = (1-v)c(u) + vd(u),$$

with null gaussian curvature,

$$d'(u) \cdot c'(u) \times \big(d(u) - c(u)\big) = 0.$$

This amounts to requiring that vectors $\mathbf{w}(u) = d(u) - c(u)$, $c'(u)$, $d'(u)$ are coplanary.

And the tangent plane is the same for points on the same ruling.



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Or put in another way,

$$c'(u) = \lambda(u)\,\mathbf{w}(u) + \mu(u)\,\mathbf{w}'(u),$$

except in the case $\mathbf{w} \parallel \mathbf{w}'$ (cylinders).

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Or put in another way,

$$c'(u) = \lambda(u)\,\mathbf{w}(u) + \mu(u)\,\mathbf{w}'(u), \left\{\begin{array}{l} \tilde{c}(u) = c(u) - \mu(u)\,\mathbf{w}(u), \\ \tilde{c}'(u) = \tilde{\lambda}(u)\,\mathbf{w}(u), \\ \tilde{\lambda}(u) = \lambda(u) - \mu'(u). \end{array}\right.$$

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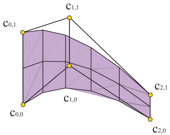
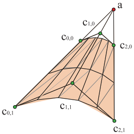
Classification of developable surfaces

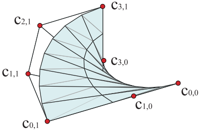
- Planar surfaces: pieces of planes.
- Cylindrical surfaces: The rulings are parallel.



- Conical surfaces: The rulings have a common point, the vertex.



- Tangent surfaces: Tangent lines to a curve (edge of regression).



Developability condition in polar form

Let us consider two n-degree rational curves $c(u)$, $d(u)$ with control polygons $\{c_0, \ldots, c_n\}$, $\{d_0, \ldots, d_n\}$ and weights $\{w_0, \ldots, w_n\}$, $\{\omega_0, \ldots, \omega_n\}$.
The derivative of a rational curve $c(u)$ is

$$c'(u) = \frac{n w_0^{n-1)}(u) w_1^{n-1)}(u)}{w_0^{n)}(u)^2} \left(c_1^{n-1)}(u) - c_0^{n-1)}(u) \right),$$

as a difference between the two last-but-one points in the de Casteljau algorithm.

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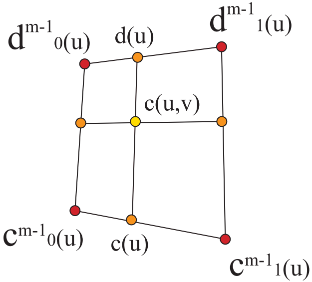
The surface is then developable iff $c_0^{n-1)}(u)$, $c_1^{n-1)}(u)$, $d_0^{n-1)}(u)$, $d_1^{n-1)}(u)$ are coplanary.



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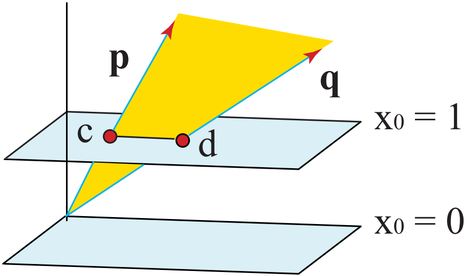
Or using the corresponding $4D$ vectors,



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Equivalently, iff the $4D$ vectors $\mathbf{p}_0^{n-1)}(u)$, $\mathbf{p}_1^{n-1)}(u)$, $\mathbf{q}_0^{n-1)}(u)$, $\mathbf{q}_1^{n-1)}(u)$ are linearly dependent,

$$\begin{aligned}
\mathbf{p}_i^{n-1)}(u) &= \left(w_i^{n-1)}(u), w_i^{n-1)}(u) c_i^{n-1)}(u)\right), \\
\mathbf{q}_i^{n-1)}(u) &= \left(\omega_i^{n-1)}(u), \omega_i^{n-1)}(u) d_i^{n-1)}(u)\right).
\end{aligned}$$

Developability and linear dependence

The linear dependence condition for $\mathbf{p}_0^{n-1)}(u)$, $\mathbf{p}_1^{n-1)}(u)$, $\mathbf{q}_0^{n-1)}(u)$, $\mathbf{q}_1^{n-1)}(u)$ may be written as

$$(1-\Lambda)\mathbf{p}_0^{n-1)}(u)+\Lambda\mathbf{p}_1^{n-1)}(u)=\sigma\left((1-M)\mathbf{q}_0^{n-1)}(u)+M\mathbf{q}_1^{n-1)}(u)\right),$$

with rational Λ, M, σ.
Or using the de Casteljau algorithm in terms of blossoms,

$$\mathbf{p}[u^{<n-1>},\Lambda]=\sigma\mathbf{q}[u^{<n-1>},M],$$

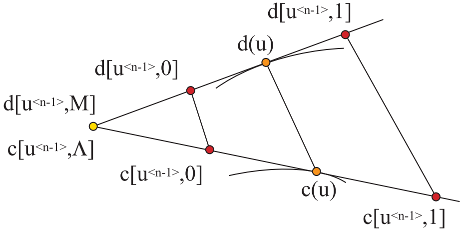
taking into account that

$$\mathbf{p}_0^{n-1)}(u)=\mathbf{p}[u^{<n-1>},0],\qquad \mathbf{p}_1^{n-1)}(u)=\mathbf{p}[u^{<n-1>},1].$$

Theorem

Two rational curves $c(u)$, $d(u)$ with control polygons $\{c_0, \ldots, c_n\}$, $\{d_0, \ldots, d_n\}$ and weights $\{w_0, \ldots, w_n\}$, $\{\omega_0, \ldots, \omega_n\}$ define a generic developable surface iff their respective 4D polar forms are related by

$$\mathbf{p}[u^{<n-1>}, \Lambda] = \sigma \mathbf{q}[u^{<n-1>}, M].$$



Interpretation of Λ, M, σ

We know that $c(u)$, $\mathbf{w}(u) = d(u) - c(u)$ satisfy

$$c'(u) = \lambda(u)\,\mathbf{w}(u) + \mu(u)\,\mathbf{w}'(u).$$

In the polinomial and $\sigma = 1$ case,

$$\lambda = \frac{n}{\Lambda - M}, \qquad \mu = \frac{M - u}{\Lambda - M}.$$

$\mathbf{p}[u^{<n-1>}, \Lambda] = \mathbf{q}[u^{<n-1>}, M]$ is then $c' = \lambda\,\mathbf{w} + \mu\,\mathbf{w}'$ in blossom form!

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The rational case can be done either, but introducing third function ν,

$$\lambda = \frac{n\sigma}{\Lambda - \sigma M + u(\sigma - 1)}, \mu = \frac{\sigma(M - u)}{\Lambda - \sigma M + u(\sigma - 1)}, \nu = \frac{n(\sigma - 1)}{\Lambda - \sigma M + u(\sigma - 1)}$$

Edge of regression

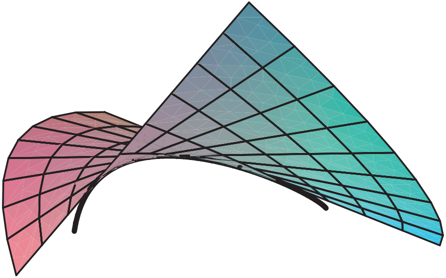
We may obtain the curve of singular points of the developable surface, just requiring

$$c_u(u,v) \parallel c_v(u,v).$$

Singular points are located along the edge of regression

$$v = \frac{\sigma(u-M)}{\sigma(u-M)+\Lambda-u} \qquad (u = (\Lambda - M)v + M \text{ if } \sigma = 1),$$

which is a rational curve on the developable surface.



Tangent surfaces

In particular we may put the developable surface in the form of a surface tangent to its edge of regression,

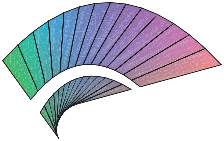
$$c(u,v) = c(u) + v\,c'(u).$$

Functions Λ, M and σ are remarkably simple in this case,

$$\Lambda = u + n, \quad M = u, \quad \sigma = 1.$$

Since all generic rational developable surfaces have a rational edge of regression $c(u)$, this means we may construct all rational developable surfaces by enlarging these singular patches.

$$\tilde{c}(u,v) = c(u) + c'(u)\left\{v\,a(u) + (1-v)\,b(u)\right\}.$$



Constant Λ, M, σ

For instance, extending the tangent surface with

$$a(u) = l - \frac{u}{n}, \qquad b(u) = m - \frac{u}{n},$$

we get the constant functions case,

$$\Lambda = (n+1)l, \qquad M = (n+1)m, \qquad \sigma = 1,$$

which is a totally equivalent representation to singular Bézier patches for all generic developable surfaces.
This construction has been used for generating developable surfaces due to its simplicity, since it provides planar control cells,

$$(1-\Lambda)\mathbf{p}_j + \Lambda\mathbf{p}_{j+1} = (1-M)\mathbf{q}_j + M\mathbf{q}_{j+1}, \quad j = 0, \ldots, n-1,$$

for both rational and polynomial developable surfaces (and even spline).

Changes of parametrizations

- $u = f(\tilde u)$, f of degree m: $\Lambda(f(\tilde u)) = \tilde f\left(\tilde u, \tilde\Lambda\right)$.
  e.g. $u = \tilde u^2$ implies $\tilde\Lambda \tilde u = \Lambda$.
- Change of weights: $\{w_0, \dots, w_n\} \to \{\alpha w_0, \dots, \alpha w_n\}$,
  $\{\omega_0, \dots, \omega_n\} \to \{\beta w_0, \dots, \beta w_n\}$ implies $\sigma \to \frac{\alpha}{\beta}\sigma$.
- Reparametrization: $u = \frac{\tilde u}{(1-\rho)\tilde u + \rho} \Leftrightarrow$
  $\{w_0, \dots, w_n\} \to \{\rho^n w_0, \dots, \rho^0 w_n\}$,
  $\{\omega_0, \dots, \omega_n\} \to \{\rho^n \omega_0, \dots, \rho^0 \omega_n\}$
  implies $\Lambda = \frac{\tilde\Lambda}{(1-\rho)\tilde\Lambda + \rho}$, $M = \frac{\tilde M}{(1-\rho)\tilde M + \rho}$, $\sigma = \tilde\sigma \frac{(1-\rho)\tilde M + \rho}{(1-\rho)\tilde\Lambda + \rho}$.
- ...

Summary

- We have shown that every generic rational developable surface may be generated from a singular Bézier patch (explicit tangent surface).
- Or equivalently from a constant Λ, M, σ patch.
- By extending the patch along the rulings of the surface.
- Usual transformations of weights and coordinates are easily implemented.

Some references

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