STABILITY AND BIFURCATION PROBLEMS FOR EQUILIBRIUM STATES OF A LIQUID BRIDGE

L. A. SLOBOZHANIN* AND J. M. PERALES**

* B. Verkin Institute for Low Temperature Physics and Engineering of Ukrainian National Academy of Sciences, Kharkov, Ukraine

** E.T.S.I. Aeronáuticos, Universidad Politécnica, 28040 Madrid, Spain

Abstract—The results of an investigation on the stability of a doubly connected axisymmetric equilibrium free surface pinned to the edges of two coaxial disks are presented. The general boundary of the region where the interface is stable is constructed in the plane of the parameters determining the slenderness of a liquid bridge and its relative volume. Surface forces and arbitrary (not only axisymmetric) perturbations are taken into account.

The general boundary of the stability region was calculated completely in the past only for a weightless fixed-contact-line liquid bridge between equal disks. The influence of axially directed gravity, isorotation and disks inequality on the evolution of this boundary has been analyzed successively. As a result, the families of the stability boundaries have been obtained for fixed-contact-line liquid bridges between equal disks in a wide range of Bond numbers, for isorotating weightless bridges between equal disks, in a wide range of Weber numbers and for weightless fixed-contact-line liquid bridges when the disk radii ratio is varied.

Basing on the solution of the bifurcation problem for the critical equilibrium states, the conclusion on the results of stability losing has been made for starting system of fixed-contact line weighdess bridge between equal disks.

The stability of the melt during crystal growth using the floating zone technique can be considered as a special case of the presented results.

Finally as an example, for a crystal growth system using the Stepanov's method the effect of free surface unconnectivity on the stability has been investigated under zero gravity conditions.

1. INTRODUCTION

The considered configuration is the following: an isothermal liquid mass is in equilibrium and forms an axisymmetric bridge between two coaxial disks which have radii r_1 and r_2 respectively, and are spaced a distance l apart. The free surface is pinned to the edges of the disks (Fig. 1).

In a general case the equilibrium state of the system is characterized by three geometric parameters: the slenderness, Λ, the relative volume, V, and the ratio of the disks radii, K, which are introduced as the following

$$\Lambda = \frac{l}{2r_o}\ ,\quad V = \frac{v}{\pi r_o^2 l}\ ,\quad K = \frac{r_1}{r_2} \le 1$$

Here $r_o = (r_1 + r_2)/2$ is the mean radius of the disks, and v is the liquid volume. If for a given surface tension, σ, an axial gravity, g, and an isorotation with angular velocity ω are taken into account, then two additional parameters appear—the Bond number, B, and the Weber number, W.

$$B = \frac{\varrho g r_o^2}{\sigma},\quad W = \frac{\varrho \omega^2 r_o^3}{2\sigma},$$

The stability of this system is studied with respect to arbitrary (both axisymmetric and non-axisymmetric) perturbations using the method described in [6]. As a result, the boundary of the stability region is constructed in the (Λ,V)—plane. We will call it the «general boundary» because it determines the stability or instability of a configuration for arbitrary values of Λ and V.

The stability problem arising in the floating zone method is solved considering constraints of two types. For the first type the value of V is fixed and is close to unit; for the second type the value of the growing angle, α, is fixed. Since

$$\alpha \frac{\pi}{2} - \beta_1\ (\ge 0)\ \text{ or }\ \alpha = \beta_2 - \frac{\pi}{2}\ (\ge 0),$$

the last condition is equivalent to a fixed value of the angle of inclination, β_1 or β_2, of a free surface at the smaller or at the larger disk (Fig. 1), according to which disk corresponds to the solidification front. For the above conditions, the stability results follow from an analysis of the general boundary as particular cases. Namely, they are the point of intersection of the general boundary with a horizontal line $V = \text{const}$ and the point of intersection of the boundary with a level line $\beta_1 = \text{const}$ or $\beta_2 = \text{const}$. We have constructed additionally a set of these level lines in the (Λ, V)—plane for all cases discussed below.

We proceed from a known result [6,7] on the general boundary for a weightless liquid bridge at rest between equal disks ($K = 1$, $B = W = 0$). Here the boundary consists of two non-intersecting branches (Fig. 2). On the upper branch, AT_1m, and on the left lower segment, AT_2BC, of the lower boundary the bridge loses its stability with respect to non-axisymmetric perturbations and on the right part, $CDEFn$, of the lower branch with respect to axisymmetric perturbations. Under certain conditions, depending on the value of the wetting angle [6], the instability can be governed by the detachment of contact line from the edge of the disks. From this point of view, the zero wetting angle is the most favourable case from the stability point of view. Here and in the following, we are considering that the wetting angle equals zero. For the considered system,

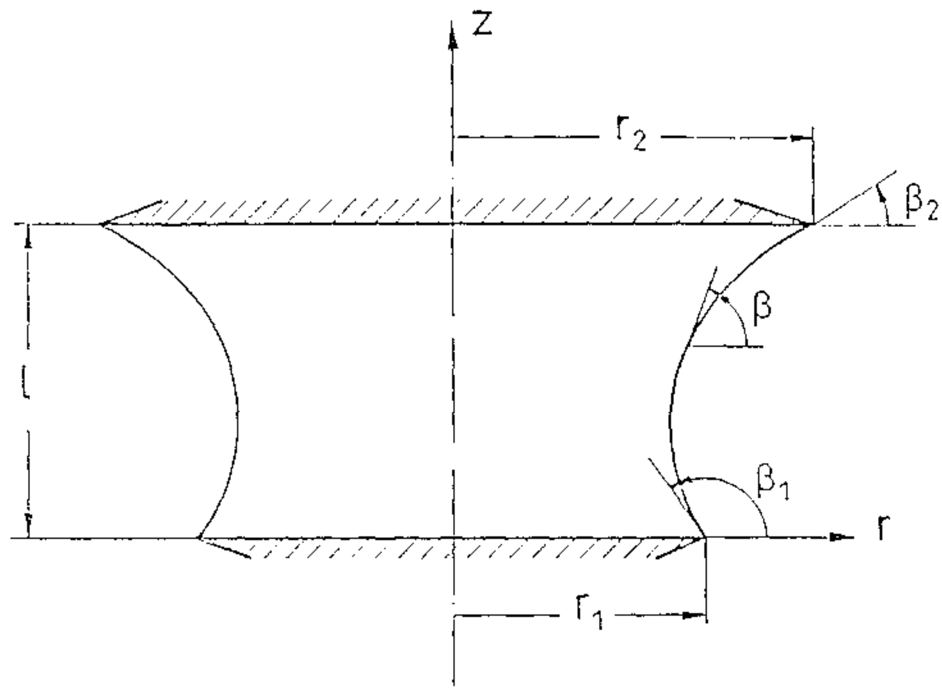

Fig. 1. Geometry and coordinate system for the liquid bridge problem.

the stability boundary due to the detachment of contact lines coincides with the left part of the lower boundary. The presented theoretical boundary agree with the known experimental results [8-10].

For the considered system in equilibrium, the effect of gravity, isorotation and the disks radii inequality on the presented boundary is crucial. The first results on the separate effect of gravity and isorotation on the whole boundary were obtained in [11]. We have carried out a comprehensive analysis in the case of isolated effects of all stated factors.

Besides, the nature of the axisymmetric bridge stability losing is of interest. To obtain this behaviour, the bifurcation problem for the critical states should be solved. For the system considered initially ($K = 1$, $B = W = 0$), bifurcation patterns which are plausible for an explanation of the Plateau's experimental results [8] were suggested in [12]. By now an exact solution of bifurcation problem is known [13, 14] only for a critical cylinder (the point F in Fig. 2).

2. EFFECT OF GRAVITY ($K = 1$, $W = 0$, $B > 0$)

When gravity is considered, the stability regions have been analyzed [1] for Bond numbers in the range $0.005 \leq B \leq 7$. In Fig. 3 the boundary for $B = 0.1$ is shown together with the boundary for $B = 0$. The boundary becomes now closed. The shape of the represented boundary is typical for Bond numbers in the interval $0 < B < 3.06$. It consists of three parts. On the segment $OABC$ the non-axisymmetric perturbations are critical. This segment always contains the point B where the maximum slenderness for a given Bond number is achieved. On the segment CD the axisymmetric perturbations are critical. This segment was well studied earlier for different not-too-large Bond numbers [15-17]. In the liter-

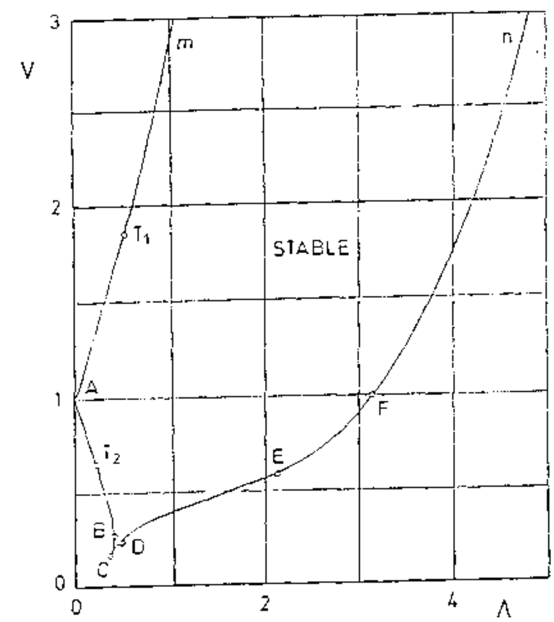

Fig. 2. Stability limits of a liquid bridge at rest between equal disks under zero-gravity conditions ($K = 1$, $B = W = 0$). Meaning of the different points is explained in the text.

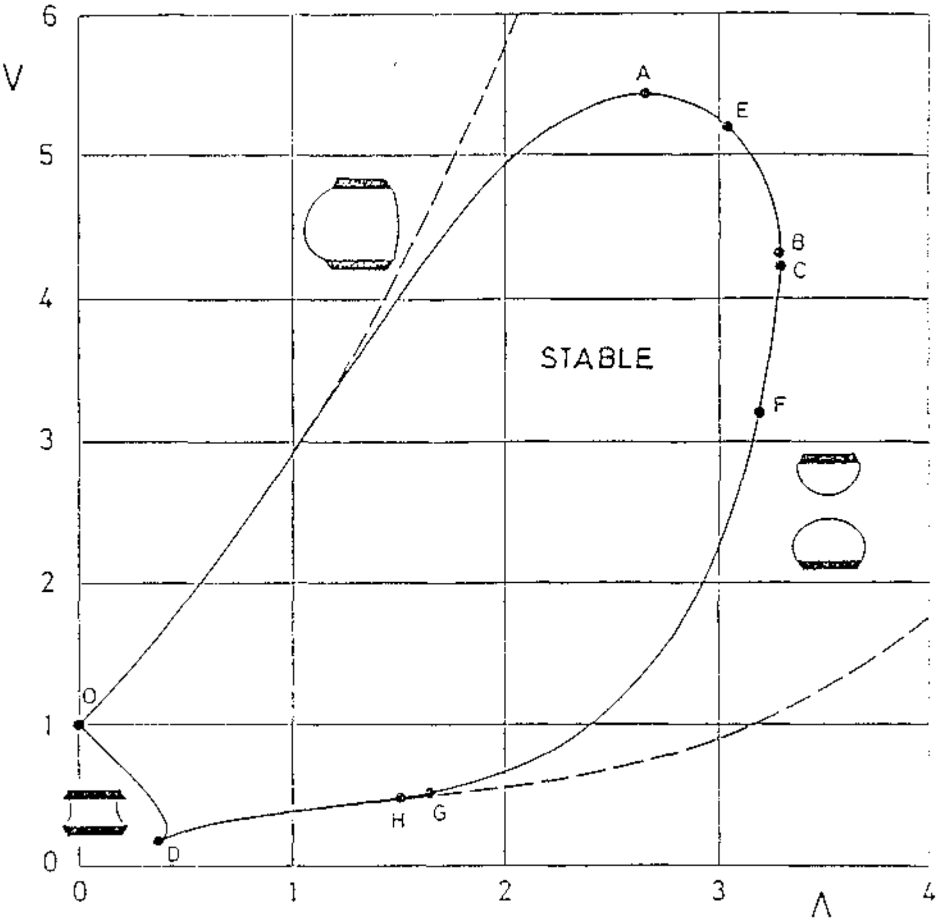

Fig. 3. Typical stability diagram of liquid bridges between equal disks subjected to a small constant axial acceleration (measured by Bond number): $B = 0.1$ (solid line), $B = 0$ (dashed line). The sketches indicate the different types of instability appearing on the different parts of the stability curve. A, maximum of volume; B, maximum of slenderness; C, transition between axisymmetric and non-axisymmetric instabilities; D, zero angle at the top disk; E, local minimum in pressure; F, local minimum of the angle at the top disk; G, local maximum of the angle at the top disk; H, local maximum of the pressure.

ature it is known as a minimum volume stability limit. Finally, on the segment OD the instability occurs due to the detachment of free surface from the edges of the top disk (gravity is downward-directed).

In Fig. 4a the boundaries are plotted for rather small Bond numbers. Curve labeled A shows the locus of the points with a maximum volume for a given Bond number, curve B shows the locus of the points with a maximum slenderness and curve C the transition points between destabilizing non-axisymmetric and destabilizing axisymmetric perturbations.

A similar situation appears for not-too-small Bond numbers (Fig. 4b). However, for $B > 3.06$ the boundary segment where the axisymmetric perturbations are the destabilizing ones disappears. The boundary consists now of two parts. If $V > 1$, the non-axisymmetric instability takes place, and if $V < 1$ the detachment of the top contact line from the disk edges.

In Fig. 5, as an example, the level lines $\beta_1 = \text{const}$ for the critical states are plotted with solid lines; the dashed lines are the stability boundaries. A similar diagram is obtained for the angle β_2 on the top disk. All these results allow us to estimate the validity of the particular results corresponding to the case $V = 1$ [6, 14, 18-21] or to a given value of the growing angle [6, 20-23].

The fundamental difference between our boundaries (the solid lines in Fig. 6) and the boundaries found in [16] (the solid lines continued by the dashed lines) is explained by the fact that the authors of [16] studied the stability only with respect to axisymmetric perturbations. Our results agree with the experimental results published in [24] (see Fig. 7).

3. EFFECT OF ISOROTATION ($K = 1$, $B = 0$, $W > 0$)

To study the effect of rotation we calculated [2] the stability boundaries for Weber numbers in the range $0.01 \le W \le 10$. Let first $W < W_0$, where the value W_0 is some value between 2.05 and 2.06. Then the boundary for $W = 1$ plotted in Fig. 8 with a solid line

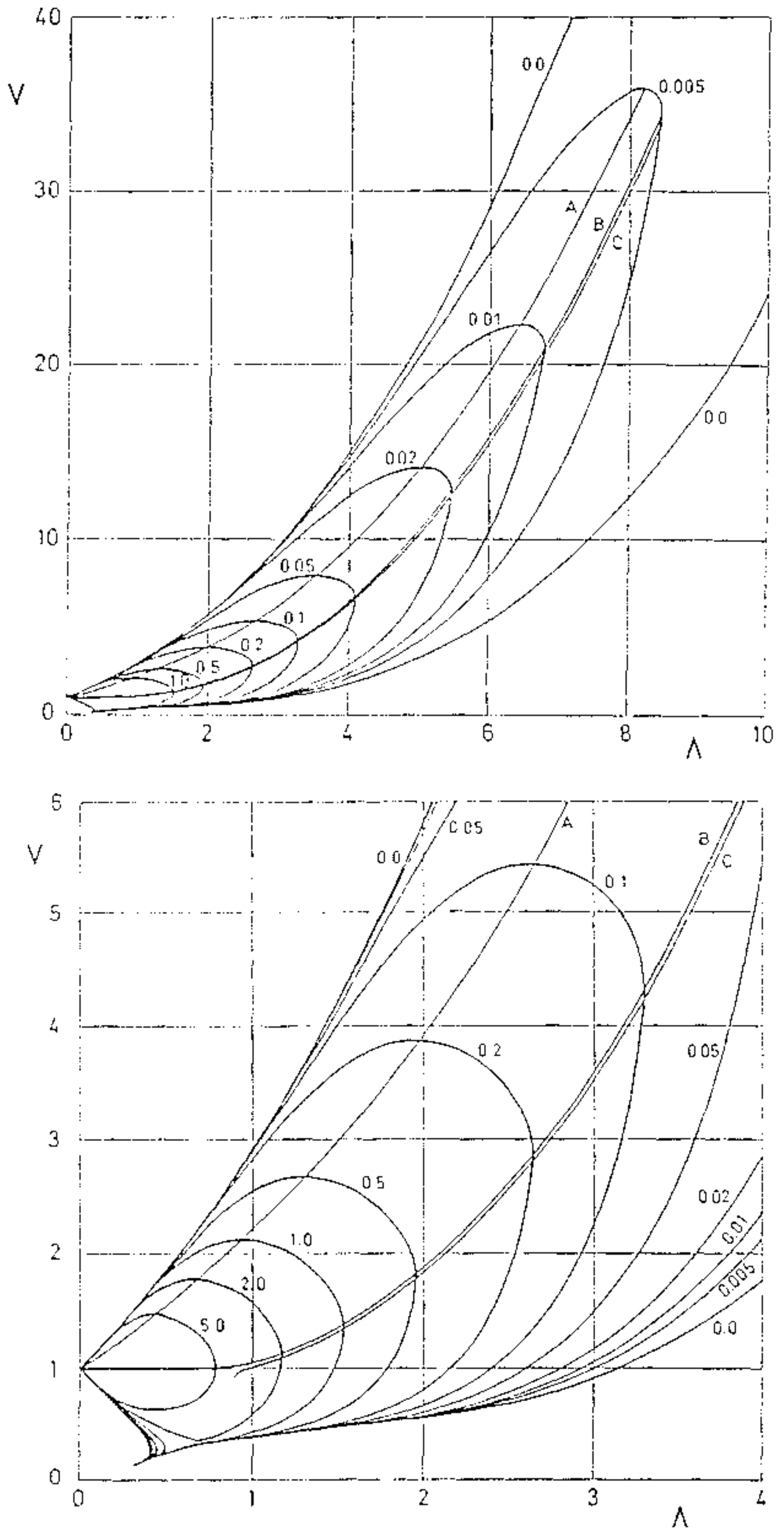

Fig. 4. Influence of the Bond number on the stability limits of liquid bridges between equal disks. Numbers on the curves indicate the value of the Bond number. Curve labeled *A* shows the locus of the points with maximum volume for a given value of the Bond number, curve *B* shows the locus of the points with maximum slenderness for a given value of the Bond number and curve *C* the transition between axisymmetrical breakage and non-axisymmetric deformation

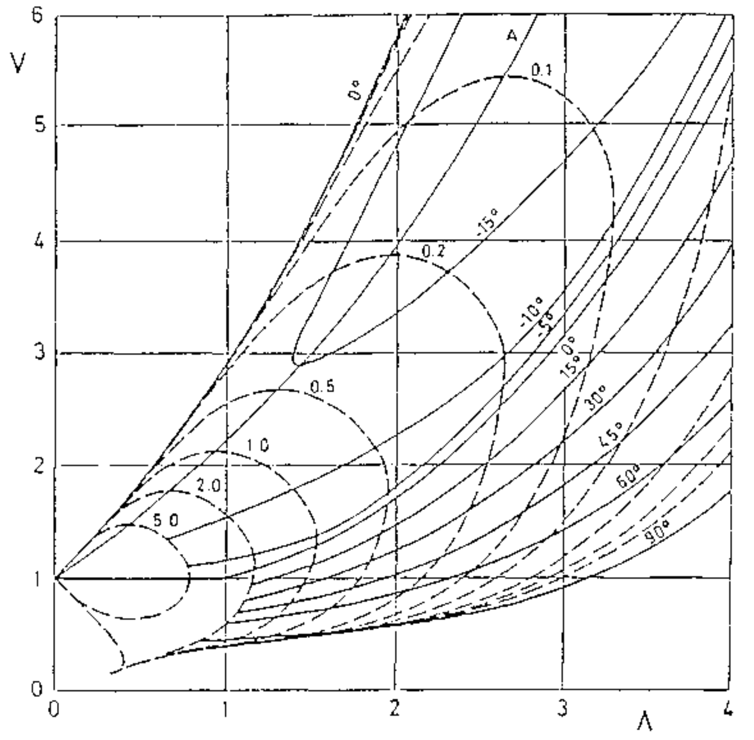

Fig. 5. Values of the angle in the bottom disk β_1 at the stability limit. Numbers on the dashed curves indicate the value of the Bond number. Curve A is the locus of minimum value of β_1 for each Bond number.

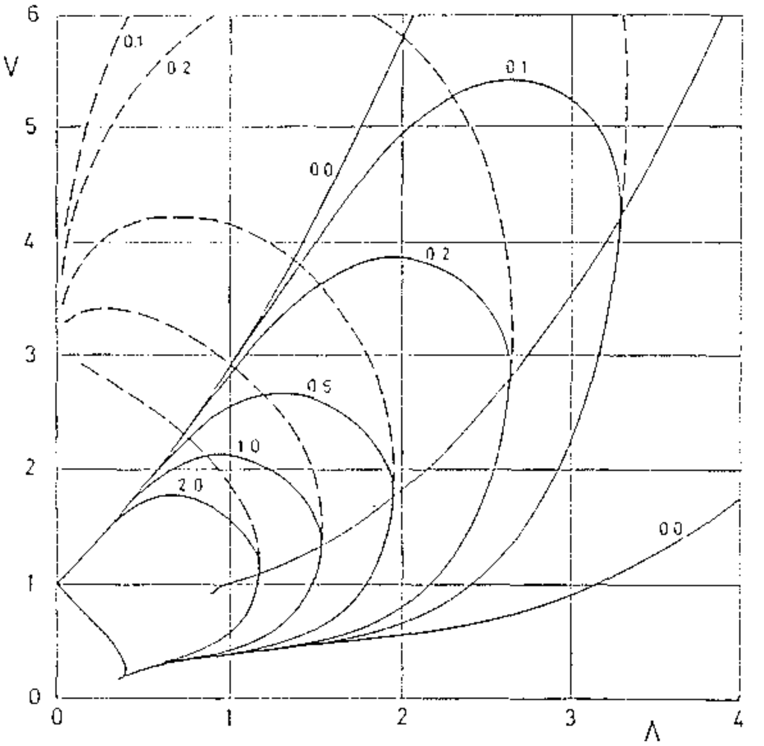

Fig. 6. Influence of the Bond number on the stability limits of liquid bridges between equal disks according to Martinez, Haynes and Langbein [16] (dashed lines) and according to the present method. Numbers on the curves indicate the value of the Bond number.

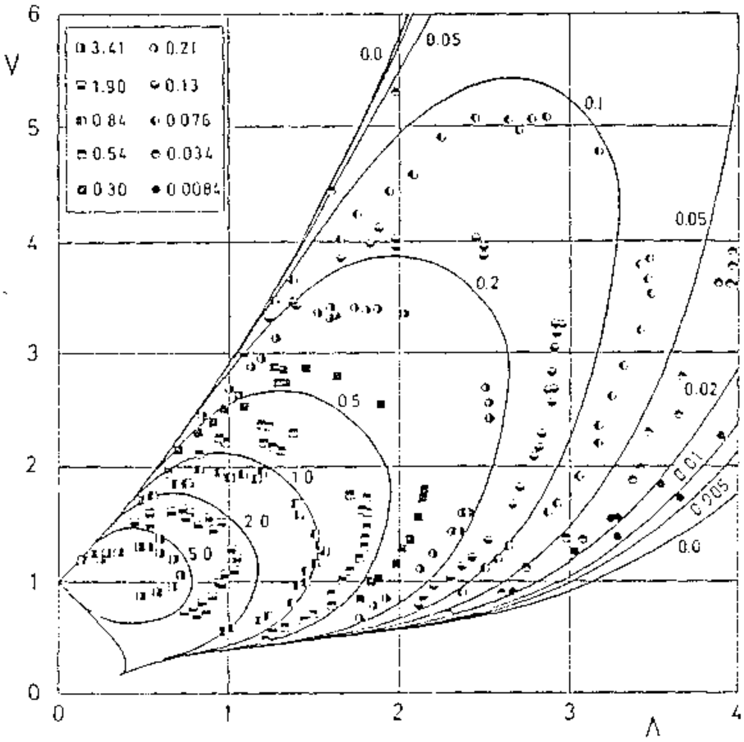

Fig. 7. Comparison with the experimental stability limits of liquid bridges between equal disks reported by Bezdenejnykh, Meseguer and Perales [24]. The symbols indicate experimental results for the Bond numbers quoted in the legend. Numbers on the curves indicate the value of Bond number.

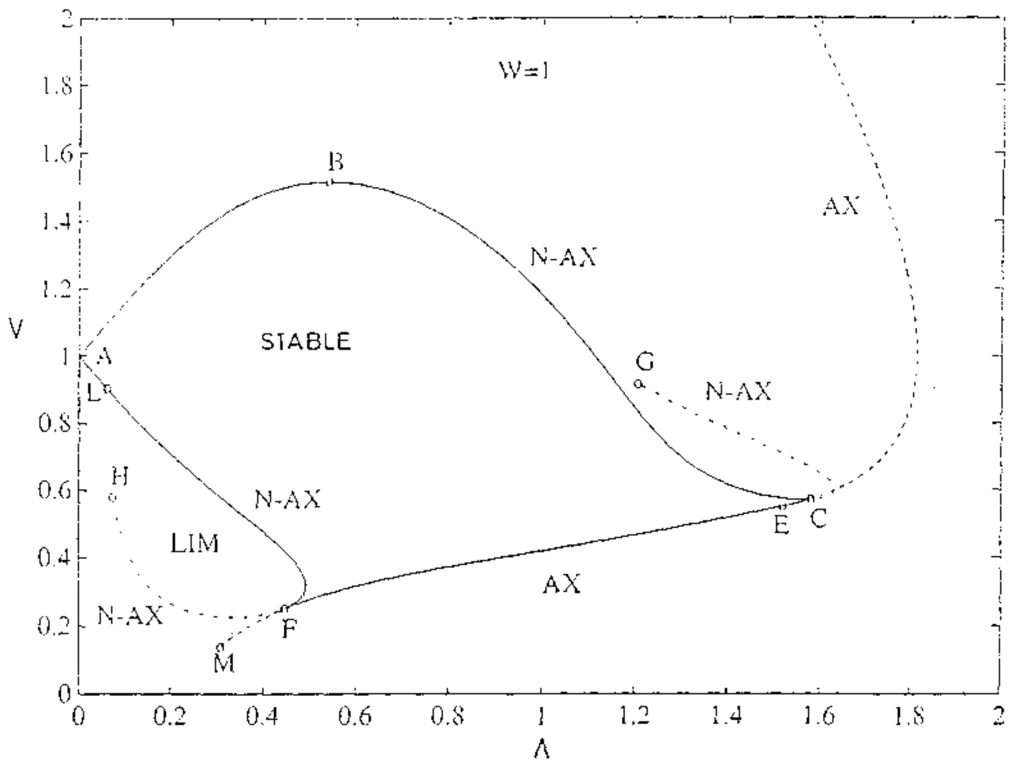

Fig. 8. Boundary of the region of stable isorotating liquid bridges for $W = 1$. Axisymmetric and non-axisymmetric perturbations have been considered. The relevant part of the boundary is plotted with solid line. Meaning of the different labels is explained in the text.

is typical. It consists of four parts. On the segments ABC and LF the bridge loses its stability with respect to non-axisymmetric perturbations, on the segment FEC with respect to axisymmetric perturbations and on the segment LA due to detachment of the contact lines from the edges.

In Fig. 9 a set of the boundaries for different Weber numbers is presented. When the Weber number becomes slightly larger than one, the upper and lower boundary segments where the destabilizing perturbations are non-axisymmetric form a neck. As the Weber number increases the neck size decreases until the indicated segments touch each other for $W = W_o$. The neck collapse leads to a jumplike contraction of the stability region. For $W > W_o$, the boundary consists of two segments: the basic segment where non-axisymmetric perturbations are the destabilizing ones and a small segment of the lower boundary where the detachment from the disk edges occurs. The obtained results agree with the earlier published particular results for $V = 1$ and for typical values of the growing angle [21, 25, 26].

4. EFFECT OF DISK RADII INEQUALITY ($B = W = 0, K < 1$)

The effect of the disk radii inequality was studied [3] for the values of K in the interval $0.1 \le K \le 0.95$. Fig. 10 shows the stability boundaries for $K = 1$ (dot-dashed line) and for $K = 0.7$ (solid line). The latter boundary is typical for $K < 1$. It consists of two non-intersecting branches (upper and lower). On the right segment of lower boundary the bridge loses its stability with respect to axisymmetric perturbations. The basic part of this segment for different values of K was constructed earlier in [27, 28]. We completely found this segment and, besides, determined the left segment of the lower boundary where the detachment of the contact line from the edges of the larger disk takes place. The disk inequality radically changes the upper boundary. The critical volume tends to infinity not only as Λ tends to infinity but also as Λ tends to zero.

A set of the lower boundaries for different K is shown in Fig. 11. Fig. 12 presents a set of the upper boundaries. If $K > 0.307$, the non-axisymmetric perturbations are critical along whole upper boundary. The smaller the value of K, the higher the position of upper boundary. This is not necessarily so if $K < 0.307$. The reason is that, for these values of K, the axisymmetric perturbations become critical on the left part of the upper boundary. This part of the upper boundary expends as K decreases.

Besides, we have analyzed the stability problems for values of V close to one ($V = 0.9, 1, 1.1$) and for typical values of the growing angle ($\alpha = 0, 10^o, 15^o$).

5. BIFURCATION PROBLEM

By using Lyapunov-Schmidt method, we considered [4] the bifurcation problem for all critical states of the initial system ($K = 1$, $B = W = 0$) except those corresponding to the points T_1, T_2, T_3, C and E on the boundary of stability region (Fig. 13). The bifurcation was studied in the neighbourhood of this boundary.

As a result, the first approximation for the shapes of the bifurcating equilibrium surfaces is found. The bifurcating surfaces have various shapes depending on the values of the parameters Λ and V and depending on the types of the perturbations causing the loss of stability of the axisymmetric states.

Besides, the bifurcation structures are determined and used to conclude on the stability or instability of the bifurcated equilibrium states. The obtained bifurcation diagrams, typical for various segments of the stability boundary, have been plotted schematically in Fig. 13.

For the every interior point of the segment EFn, the bifurcated solutions are axisymmetric (but antisymmetric with respect to equatorial plane), lie in the subcritical region and are unstable.

There are two branches of axisymmetric solutions with an equatorial symmetry plane that bifurcate from the critical equilibrium state corresponding to any point within the CDE segment. Both branches lie in the subcritical region. One of the branches consists of stable equilibrium states, and the other of unstable ones. For the critical state, the corresponding point on the bifurcation diagram is a fold (or limiting) point.

Thus, we can conclude: if the exit from the stability region takes place through the boundary segment $CDEFn$ where the axisymmetric perturbations are critical, the bridge breaks. The same conclusion was early reached by the authors of [29] who studied the dynamic behaviour of an axisymmetric bridge when crossing this boundary segment.

With regard to the boundary segment where non-axisymmetric perturbations are critical, the bifurcation may be either supercritical (the points within the segments T_1m and T_2BT_3) or subcritical (the interior points of the segments T_1AT_2 and T_3C). The bifurcated non-axisymmetric equilibrium states are either stable or unstable, respectively.

Thus, the loss of stability results in a continuous transition of critical axisymmetric bridge shapes to stable non-axisymmetric states if the exit from stability region takes place through the boundary segments T_1m and T_2BT_3. The possibility of appearance of stable non-axisymmetric bridge shapes in the course of experiments was already described by Plateau [8].

If the exit occurs through the boundary segments T_1AT_2 and T_3C, the loss of stability leads to a jump. Some experimental results described in [10] suggests that this jump is finite within the segment AT_1. Here a critical bridge jumplikely goes over to a finite-distance-apart stable non-axisymmetric state.

6. EFFECT OF THE UNCONNECTIVITY OF FREE SURFACE

So far, we considered only the connected free surfaces. In [5] we have considered two systems that appear during the growth of crystals by the Stepanov's method under zero-gravity conditions (Fig. 14). Here the free surface consists of two unconnected parts: an axisymmetric surface Γ_1 of a bridge pinned to the edges of the crystal and the shaper, and a spherical segment Γ_2 resting at the lateral wall (system I) or at the edge (system II) of a cylindrical container. The following general conclusions have been reached:

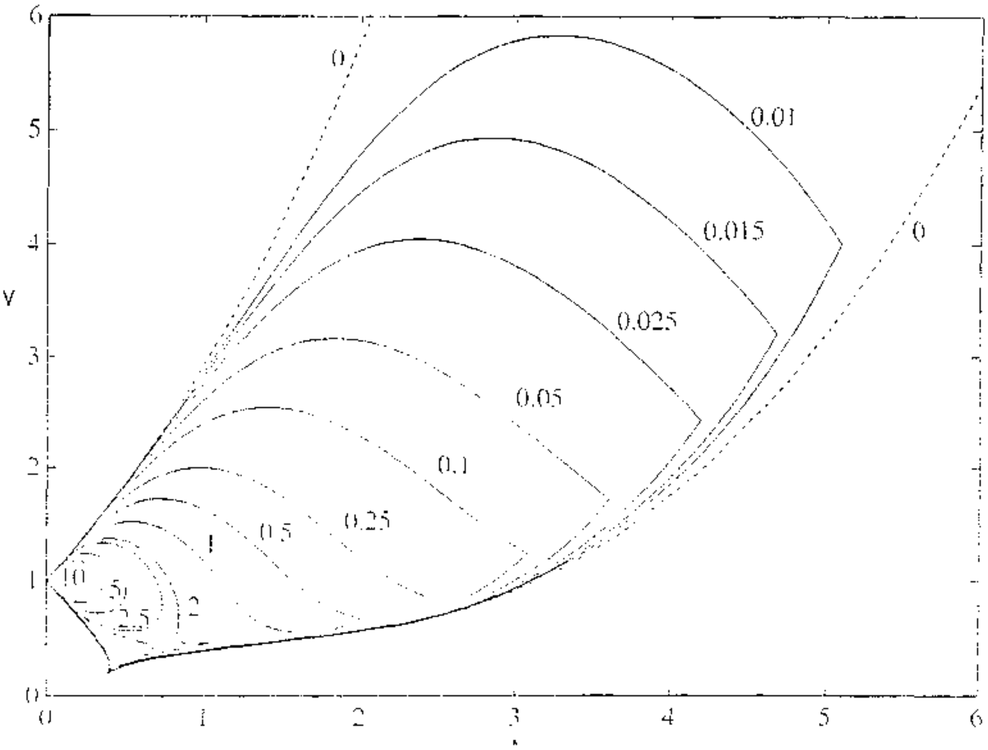

Fig. 9. Boundaries of the region of stable isorotating liquid bridges. Numbers on the curves indicate the value of the Weber number. The limit for no rotating ($W = 0$) liquid bridges has also been plotted in dashed line.

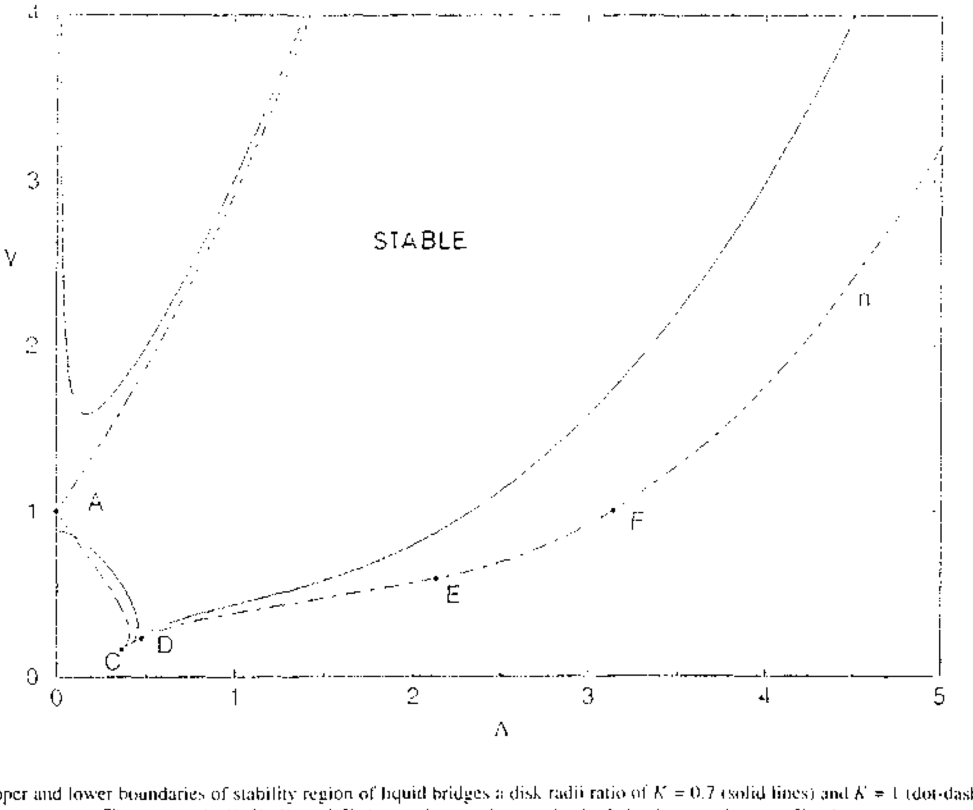

Fig. 10. Upper and lower boundaries of stability region of liquid bridges a disk radii ratio of $K = 0.7$ (solid lines) and $K = 1$ (dot-dash lines). The points A, C, D, E, and F show relevant changes in the behaviour in the case $K = 1$.

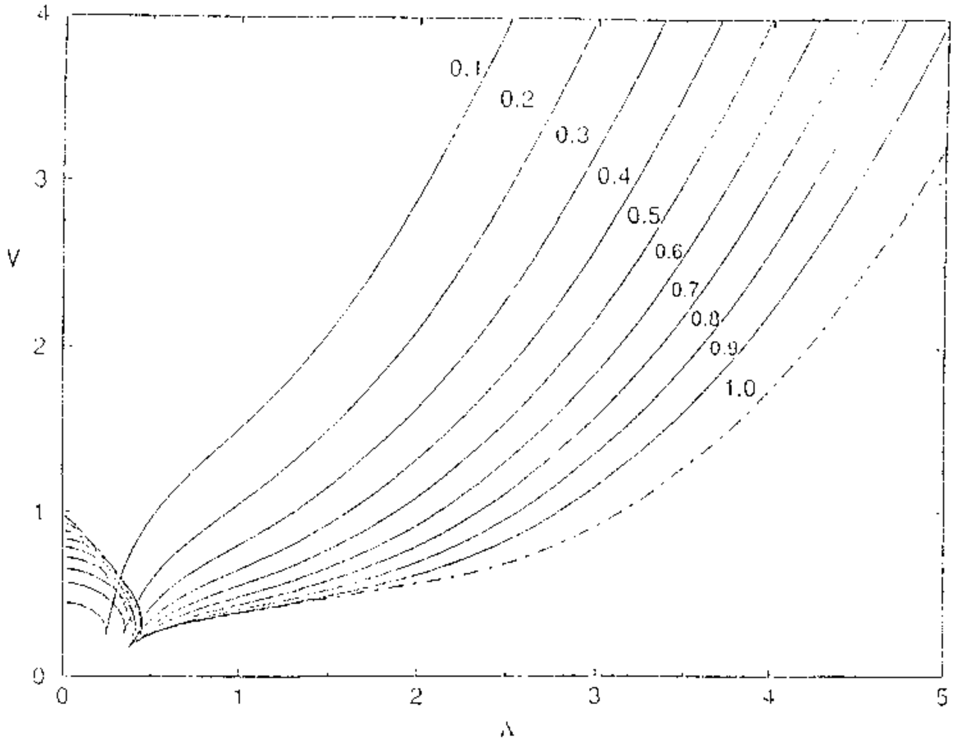

Fig. 11. Lower boundaries of stability region for various values of disk radii ratio, K, indicated by the numbers on the curves.

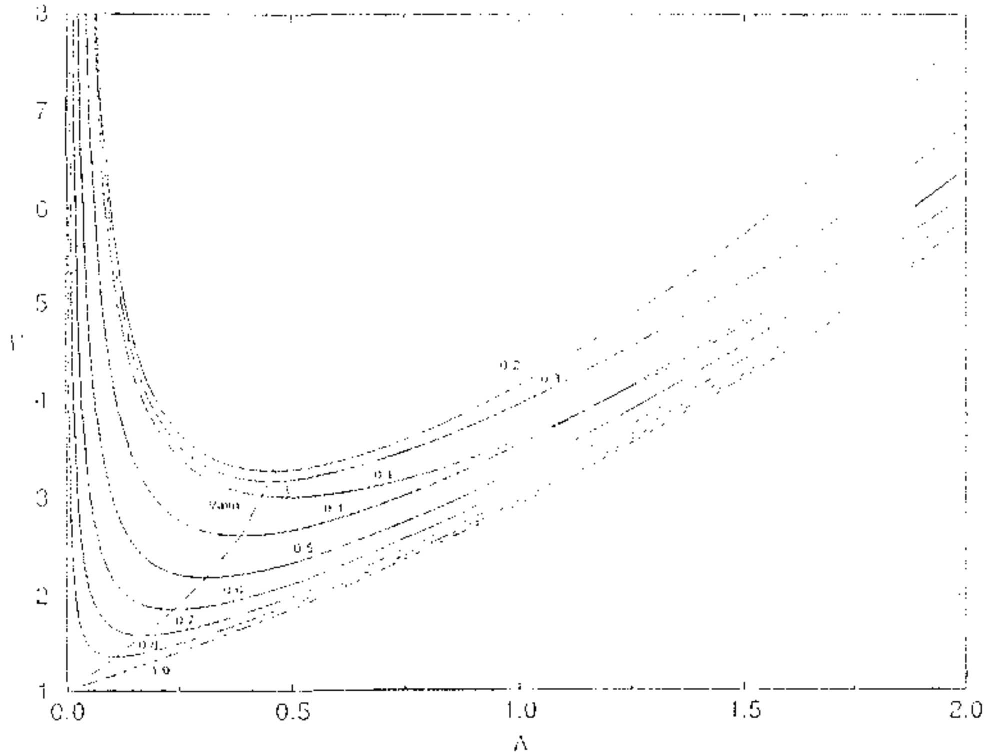

Fig. 12. Upper boundaries of stability region for different values of disk radii ratio, K. Numbers on the curves indicate the values of K. The dotted line joins the points of minimum V for given K.

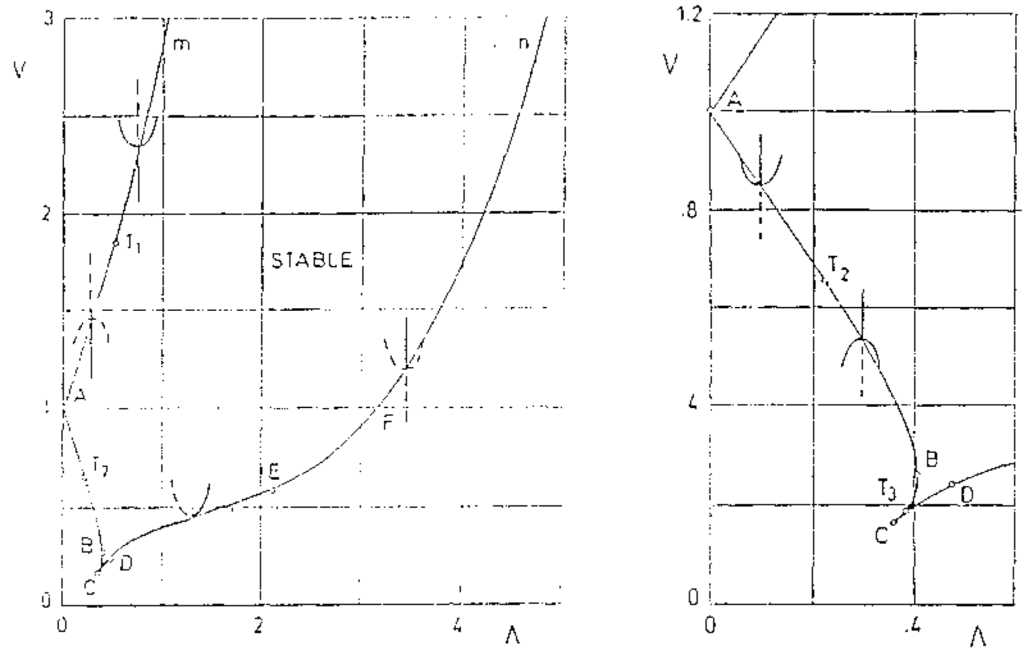

Fig. 13. Bifurcation structure in the stability limit for $B = W = 0$, $K = 1$. Stable branches have been plotted with continuous line whereas unstable ones with dashed line.

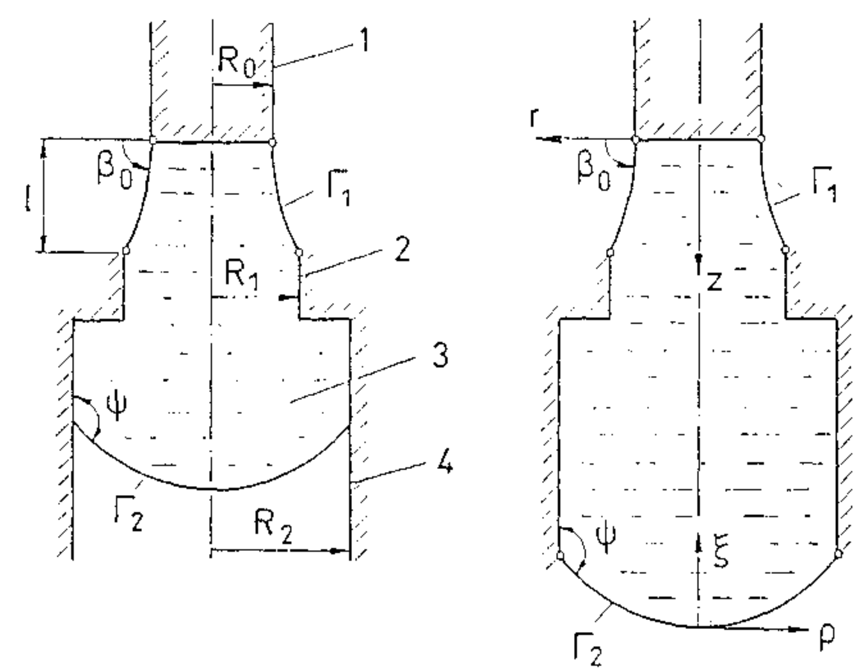

Fig. 14. Geometry and parameters appearing in the fluid configuration modelling the crystal growth in the Stepanov's method.

i) The axisymmetric pertubations are critical for both systems.

ii) For the same values of the parameters, the system II is more stable than the system I owing to the different boundary conditions for the surface Γ_2.

iii) Due to the property (i) the effect of the unconnectivity is as follows. Compared with system presented in Fig. 1 and having only one connected surface of a bridge, the system I is always less stable for the same values of the bridge parameters. Usually the system II is also less stable. However, there are cases when the stability of the system II and system presented in Fig. 1 is the same.

For the systems I and II, we obtained the stability conditions depending on geometric parameters, the values of growing angle and boundary angle ψ (Fig. 14).

Let, as an example, the bridge be cylindrical. It is well known that in this case, for the system corresponding to Fig. 1, the critical values of slenderness is $\Lambda_* = \pi$. However, for the system I, $\Lambda_* = \pi/2$ and, for the system II, $\Lambda_* = \pi$ if $90° < \psi < 162°$, and $\frac{\pi}{2} < \Lambda_* < \pi$ if $162° < \psi < 180°$.

REFERENCES

[1] L. A. Slobozhanin, J. M. Perales. «Stability of liquid bridges between equal disks in an axial gravity field». *Phys. Fluids* **A5**, 1305-1314 (1993).

[2] L. A. Slobozhanin, J. M. Perales. «Stability of an isorotating liquid bridge between equal disks». *Phys. Fluids* (to appear).

[3] L. A. Slobozhanin, M. Gómez, J. M. Perales. «Stability of liquid bridges between unequal disks under zero-gravity conditions». *Microgravity Sci. Technol.* (in press).

[4] L. A. Slobozhanin. «Bifurcation of the equilibrium states of a weightless liquid bridge». (in preparation).

[5] L. A. Slobozhanin, M.A. Srechkareva, V.A. Tatarchenko. «Stability of the melt meniscus during growth of crystals by the technique of pulling from shaper (TPS) under zero-gravity conditions». *J. Crystal Growth* **133**, 273-280 (1993).

[6] A. D. Myshkis, V. G. Babskii, N. D. Kopachevskii, L. A. Slobozhanin, A. D. Tyuptsov. *Low-Gravity Fluid Mechanics* (Springer-Verlag, Berlin, 1987).

[7] L. A. Slobozhanin. «Problems of stability of liquid in equilibrium, appearing in space technology researches». In *Hydromechanics and Heat and Mass Transfer in Zero-Gravity* (Nauka, Moscow, 1982), pp. 9-24 [in Russian].

[8] J. Plateau. *Statique Expérimentale et Théoretique des Liquides* (Gautier-Villars, Paris, 1873).

[9] A. Sanz, I. Martinez. «Minimum volume for a liquid bridge between equal disks». *J. Coll. Interface Sci.* **93**, 235-240 (1983).

[10] M. J. Russo, P. H. Steen. «Instability of rotund capillary bridges to general disturbances: experiment and theory». *J Coll. Interface Sci.* **113**, 154-163 (1986).

[11] I. V. Barmin, B. E. Vershinin, I. G. Levitina, A. S. Senchenkov. «Stability of liquid rotating zone». *Izv. Akad. Nauk SSSR, Ser. Fiz.* 49, 661-666 (1985) [in Russian]. English transl.: *Bull. Acad. Sci. USSR, Phys. Ser.* **49** (1985).

[12] D. H. Michael. «Meniscus stability». *Ann. Rev. Fluid Mech.* **13**, 189-215 (1981).

[13] R. A. Brown, L.E. Scriven. «The shapes and stability of captive rotating drops». *Phil. Trans. R. Soc. London* A 297, 51-79 (1980).

[14] J. M. Vega, J. M. Perales. «Almost cylindrical isorotating liquid bridges for small Bond numbers». In *ESA SP-191* (European Space Agency, Paris, 1983), pp. 247-252.

[15] J. Meseguer, A. Sanz. «Numerical and experimental study of the dynamics of axisymmetric slender liquid bridges». *J. Fluid Mech.* **153**, 83-101 (1985).

[16] I. Martínez, J. M. Haynes, D. Langbein. «Fluid statics and capillarity». In *Fluid Sciences and Material Science in Space* (Springer-Verlag, Berlin, 1987), pp. 53-81.

[17] J. M. Perales, J. Meseguer, I. Martinez. «Minimum volume stability limits for axisymmetric liquid bridges subject to steady axial acceleration». *J. Crystal Growth* **110**, 855-861 (1991).

[18] S. R. Coriell, S. C. Hardy, M. R. Cordes. «Stability of liquid zones». *J. Coll. Interface Sci.* **60**, 126-136 (1977).

[19] J. Meseguer. «The influence of axial microgravity on the breakage of axisymmetric slender liquid bridge». *J. Crystal Growth* **62**, 577-586 (1983).

[20] L. A. Slobozhanin. «Hydrostatic problems arising during simulation of the process of purification of materials and growth of single crystals by the floating zone method. Part 3: Equilibrium and stability of zone in gravitational field and the field of centrifugal forces». Preprint 25-84. *Fiz. Tekh. Inst. Nizk. Temp. Akad Nauk Ukr. SSR, Kharkov* (1984) [in Russian].

[21] L. A. Slobozhanin. «Investigation of hydrostatic problems simulating the zone melting process». Izv. Akad. Nauk SSSR, Ser. Fiz. 49, 652-660 (1985) [in Russian]. English transl.: *Bull. Acad. Sci. USSR. Phys. Ser.* 49 (1985).

[22] W. Heywang. «Zur Stabilit»t senkrechter Schmelzzonen». *Z. Naturforschg.* **11a**, 238-243 (1956).

[23] S. R. Coriell, M. R. Cordes. «Theory of molten zone shape and stability». *J. Crystal Growth* **42**, 466-472 (1977).

[24] N. A. Bezdenejnykh, J. Meseguer, J. M. Perales. «Experimental analysis of stability limits of capillary liquid bridges». *Phys. Fluids* **A4**, 677-680 (1992).

[25] L. A. Slobozhanin. «Some problems of equilibrium stability of zero-g liquid bridges». In *Hydromechanics and Heat/Mass Transfer in Microgravity* (Gordon and Breach, Philadelphia, 1992), pp. 185-192.

[26] M. A. Srechkareva, L. A. Slobozhanin. «Stability of a rotating weightless liquid phase in zone melting». *Prikl. Mekh. Tekh. Fiz., No.* 6, 47-49 (1992) [in Russian]. English transl: *J Appl. Mech. Tech. Fiz.* 33, 811-813 (1992).

[27] I. Martinez. «Stability of axisymmetric liquid bridges» In *ESA SP-191* (European Space Agency, Paris, 1983), pp. 267-273.

[28] I. Martínez, J. M. Perales. «Liquid bridge stability data». *J. Crystal Growth* 78, 369-378 (1986).

[29] J. Meseguer, A. Sanz, D. Rivas. «The breaking of axisymmetric non-cylindrical liquid bridges». In *ESA SP-191* (European Space Agency, Paris, 1983), pp. 261-265.