Stability of an isorotating liquid bridge between equal disks under zero-gravity conditions

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The stability of the relative equilibrium of an isorotating axisymmetric liquid bridge between two equal-radius coaxial disks under zero-gravity conditions has been investigated in detail. The free surface is assumed to be pinned to the edges of the disks and in equilibrium and only perturbations compatible with this pinning are considered. In the plane of the dimensionless variables characterizing the liquid bridge length and the liquid bridge volume, the stability regions for a set of values of the Weber number have been calculated. The stability region structure and the nature of critical perturbations change when the Weber number, W, passes through the values W_0 ($2.05<W_0<2.06$) and W_1 ($2.44<W_1<2.45$). It has been found that, for $W<W_0$, the stability region is connected, and the neutral stability may take place with respect to nonaxisymmetric perturbations as well as to axisymmetric ones. In the latter case, it has been established whether the critical axisymmetric perturbations are reflectively symmetric or reflectively antisymmetric about the equatorial plane. When the increasing Weber number passes through the value W_0, the stability region breaks into two disconnected parts. The first exists for all Weber numbers larger than W_0. For the states belonging to the boundary of this part, only nonaxisymmetric perturbations are critical. The second part exists only for Weber numbers between W_0 and W_1. Its boundary is determined by the states that may be neutrally stable to nonaxisymmetric perturbations or to axisymmetric ones. The characteristics of the shape of the neutrally stable surfaces have been calculated for a wide range of the Weber number.

I. INTRODUCTION

The system analyzed consists of a liquid mass of volume v forming a liquid bridge between two circular coaxial disks of the same radius r_0, which are spaced a distance $2h$ apart (Fig. 1). The gas–liquid–solid contact lines are pinned to the edges of disks. The whole system rotates uniformly around the disk's axis with an angular velocity ω in the absence of gravity. The liquid is thus subject to a centrifugal force field and surface tension forces and is under an equilibrium with respect to a rotating reference system (such an equilibrium is called a relative equilibrium).

This equilibrium state is characterized by the following dimensionless parameters: the slenderness, Λ, the relative volume, V, and the Weber number, W, which are defined as

$$\Lambda=\frac{h}{r_0}, \quad V=\frac{v}{2\pi r_0^2 h}, \quad W=\frac{\rho\omega^2 r_0^3}{2\sigma}. \tag{1}$$

Here ρ is the liquid density and σ is the surface tension.

Let us consider the problem of the stability of an axisymmetric equilibrium state of a viscous liquid bridge, assuming that the contact lines remain still pinned to the edges under perturbations. As usual, perturbations must satisfy the condition of liquid volume conservation.

The principle of minimum potential energy is to be used as the basis for the solution of the stability problem. The matter of the validity of this principle for the equilibrium state of an isorotating capillary liquid is worthy of notice. The definition of the concept of the liquid equilibrium stability and the definition-based proof of the direct statement of the principle of minimum potential energy (serving as an analog of Lagrange's stability theorem) were first presented by Lyapunov.[1,2] Lyapunov considered an equilibrium state (absolute or relative) for an ideal liquid and a viscous liquid, however, without considering surface forces. The extension of these results to the case of a capillary liquid was carried out by Rumyantsev[3,4] and Samsonov.[5]

The reciprocal statement of the above-mentioned principle (if the second variation of potential energy can take negative values, the relative equilibrium will be unstable) for a viscous isorotating capillary liquid was proved by Rumyantsev.[3,4] It should be pointed out that, for a relative equilibrium of an isorotating liquid, the reciprocal statement is valid only in the case of a viscous liquid, unlike the rest state, where this statement is valid both for a viscous liquid and for an ideal liquid (Rumyantsev, Vladimirov, and Kopachevskii[3,4,6–11]).

For an ideal isorotating liquid, if the second variation of potential energy takes negative values, a gyroscopic stabilization may be observed. The conditions for this stabilization are well studied for isorotating mechanical systems without energy dissipation (see, for instance, Appell[12]). Examples of a gyroscopic stabilization for an isorotating ideal capillary liquid were obtained by Hocking and Michael,[13] Gillis and Suh,[14] Ross,[15] and by Chandrasekhar.[16]

Thus, the principle of minimum potential energy allows

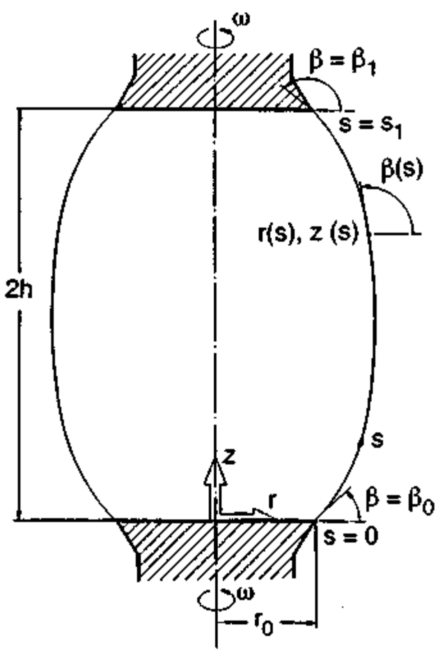

FIG. 1. Geometry and coordinate system for the isorotating liquid bridge problem.

us to find the neutrally stable equilibrium states of an isorotating liquid only when the effect of the viscosity is considered.

Additionally, it is assumed that the angular velocity of rotation of the disks remains equal to ω in a perturbed motion. In this case, the expression for the potential energy has the form

$$U=U_s-\frac{1}{2}\rho\omega^2\int_\Omega r^2\,d\Omega, \qquad (2)$$

where U_s is the potential energy of surface forces, Ω is the domain occupied by a liquid, and r is the distance from the axis of rotation. The stability criteria obtained by minimizing of the functional (2), as a rule, are more rigorous than in the case of the free evolution of a system when its angular momentum remains unchanged. For axisymmetric equilibrium surfaces, this difference appears when axisymmetric perturbations are the destabilizing ones. If nonaxisymmetric perturbations are the critical ones, the stability criteria in both cases are the same.[4,11]

In analyzing the stability problem for an isorotating liquid bridge, the data for the limiting case $W=0$ related with axisymmetric weightless liquid bridges at rest are useful. This case has been studied fairly well both experimentally[17–19] and theoretically. As a result of the theoretical analysis carried out by Gillette and Dyson[20] (stability to axisymmetric perturbations) and by Slobozhanin[21,11] (stability to arbitrary perturbations), the boundary of stability region has been constructed in the (Λ, V) plane. The boundary consists of two nonintersecting branches (upper and lower) along which $V\to\infty$ as $\Lambda\to\infty$ for both of them.

A number of particular results related with the stability of an isorotating weightless liquid bridge under the above-mentioned assumptions are known. The simplest problem concerns the stability of a cylindrical liquid bridge. Here the critical value of the Weber number is found to be

$$W=\pi^2/(8\Lambda^2)\quad (0<\Lambda\leqslant\pi\sqrt{3}/2), \qquad (3)$$

$$W=\frac{1}{2}\left(\frac{\pi^2}{\Lambda^2}-1\right)\quad\left(\frac{\pi\sqrt{3}}{2}\leqslant\Lambda\leqslant\pi\right). \qquad (4)$$

As far as we know, this result was first obtained independently by Samsonov[5] and by Slobozhanin[22] as a particular case of the solution of the more general problem on stability of an isorotating liquid cylinder. However, an English translation of the above-mentioned results was published only as late as 1987.[11] In western literature[23] a similar result obtained by Hardy and Coriell[24] (the stability to axisymmetric perturbations) and by Fowle *et al.*[25] (the stability to arbitrary perturbations), was more known.

The formulas (3) and (4) determine the critical value of W with respect to nonaxisymmetric perturbations and with respect to axisymmetric perturbations, respectively. In the indicated intervals of Λ values, the corresponding type of the perturbations is critical. It is not difficult to establish the similarity between the formulas (3) and (4) and the result of the investigation on the stability of an infinite isorotating liquid column with respect to arbitrary perturbations.[26]

The first theoretical results related to the critical states with $V\neq 1$ were obtained by Brown and Scriven.[23] For the values $\Lambda=\frac{1}{2}$ and $\Lambda=1$, they constructed the dependences $W(V)$ in the intervals $0.6\leqslant V\leqslant 1.6$ and $0.39\leqslant V\leqslant 1.4$, respectively.

For $W=0.1$ and $W=0.5$, a major part of the boundary of stability region in the (Λ,V) plane was calculated by Barmin *et al.*[27]

The determination of the critical values of the parameters for the equilibrium states with a prescribed value of the angle β_0 (β_0 is the angle of inclination of the axisymmetric surface profile in the point of contact with the lower disk; see Fig. 1) is of interest for the problems of space technology. In relation to the physical properties of semiconductor materials, the values of β_0 equal to or slightly smaller than 90° are of prime interest for the study of the growing of single crystals by the floating zone method. For $\beta=90°$, it has been established that only cylindrical surfaces may be stable; any other surfaces are either critical or unstable.[28] For $\beta_0=80°$ and 75°, the dependences $\Lambda(W)$ and $V(W)$ $(0.000125\leqslant W\leqslant 343)$ for the neutrally stable states were calculated; here the nonaxisymmetric perturbations are critical always.[29,30]

In the above-mentioned papers the stability of an isorotating liquid bridge is studied with respect to arbitrary perturbations. Particular results on the stability with respect to only axisymmetric perturbations were obtained by Ungar and Brown[31] and by Martínez, Perales, and Gómez.[32] However, such perturbations are not necessarily the most dangerous.[11]

The problem of the bifurcation of the critical equilibrium states of an isorotating liquid bridge is directly connected with the stability problem. This problem was studied by Brown and Scriven,[23] Ungar and Brown,[31] and Vega and Perales.[33]

The bifurcation of the critical cylindrical states was considered in detail. This is especially true in regard to the unperturbed bifurcation, when the value of V for the bifurcating family is equal to 1. Using the asymptotic methods, it was established[23,31,33] that a subcritical bifurcation takes place when the stability is lost with respect to axisymmetric perturbations as well as to nonaxisymmetric perturbations. Thus, in this case both axisymmetric and nonaxisymmetric bifurcated equilibrium families are unstable. The results of an asymptotic analysis are valid only in the vicinity of the critical values of the parameters. A finite element method[23,31] allows us to obtain examples of a numerical solution of the unperturbed bifurcation problem for a prescribed value of Λ and values of W that differ significantly from the critical value.

The case of $V=1$ is singular.[23] The fact is that a cylindrical shape of liquid bridge exists for any values of Λ and W. The results of the asymptotic and the numerical analyses of the bifurcation problem for $V\neq 1$ show that the nature of bifurcation when the stability is lost with respect to axisymmetric perturbations depends on whether these perturbations are reflectively antisymmetric or reflectively symmetric about the equatorial plane $z=h$. In the latter case the critical state presents a limit point and, for values of W that are larger than the critical value, the axisymmetric equilibrium states corresponding to prescribed values of Λ and V does not exist.[31] Hence, it is important to define the type of the critical axisymmetric perturbations.

The examples of a numerical solution of the bifurcation problem presented by Brown and Scriven[23] show that, when the loss of stability of an axisymmetric liquid bridge is due to nonaxisymmetric perturbations, a subcritical bifurcation or a supercritical bifurcation takes place depending on the value of V.

In summary, in spite of the existence of particular results on stability and in-depth investigations of the more complicated bifurcation problem, it should be concluded that the full picture of the stability of equilibrium states of an isorotating liquid bridge remains vague. The aim of the present paper is to clarify this picture and to carry out a systematic investigation of the stability of an axisymmetric isorotating liquid bridge with respect to arbitrary perturbations.

II. LIQUID BRIDGE EQUILIBRIUM SHAPES

Let us place the origin of the cylindrical system of coordinates (r,θ,z), rigidly bound to the disks, at the center of the lower disk and let us point the z axis upward (Fig. 1). The shape of an axisymmetric equilibrium surface of a liquid bridge in parametric form $r(s),z(s)$ is described by the solutions of the following system of differential equations:[11]

$$r''=-z'(pr^2+q-z'/r),\quad z''=r'(pr^2+q-z'/r). \quad (5)$$

Here s is the arclength of a profile (a meridian section θ=const) of an equilibrium free surface and primes denote derivatives with respect to s; it has been assumed that the fluid domain Ω remains to the left as we move along a profile in the direction of increasing s. The quantity p is equal to $\rho\omega^2/(2\sigma)$ and has the dimensions of $(\text{length})^{-3}$, and $q=(P-P_0)/\sigma$ is a constant proportional to the pressure difference between the reference pressure, P, inside the liquid bridge at the axis $r=0$ and that of the surrounding gas, P_0. The system (5) is equivalent to the Young–Laplace equation.

Let us introduce the dimensionless variables

$$R=rp^{1/3},\quad Z=zp^{1/3},\quad \tau=sp^{1/3},\quad Q=qp^{-1/3}. \quad (6)$$

Then the system (5) assumes the following form:

$$R''=-Z'\beta',\quad Z''=R'\beta',\quad \beta'=R^2+Q-Z'/R. \quad (7)$$

Here $\beta=\beta(\tau)$ is the angle measured from the R axis to the tangent to the equilibrium profile that is directed in the sense of increasing τ and primes denote derivatives with respect to τ.

Integrating the second equation of the set (7) and taking the third equation into account, we obtain

$$Z'R=\tfrac{1}{4}R^4+\tfrac{1}{2}QR^2+A_0, \quad (8)$$

where

$$A_0=-\tfrac{1}{4}R_0^4-\tfrac{1}{2}QR_0^2+R_0Z_0',$$

and the subindex "0" refers to the values at the initial point.

The analysis of the relation (8) together with the (obvious) definition of the arclength $R'^2+Z'^2=1$ allows us to prove[34] that the solution $R(\tau)$ always has at least one stationary point and that the solution of the system (7) in the (R,Z) plane is reflectively symmetric about the horizontal straight line passing through this point. Generally the functions $R(\tau)$ and $Z'(\tau)$ are periodic. However, for any R_e (R_e is the value of R in the stationary point) when $Q=-R_e^2+R_e^{-1}$ the vertical straight line $R(\tau)=R_e$ is the solution of the equilibrium equations and, furthermore, there exist a value of Q for which, after self-intersection, the integral line tends monotonically on both sides to a vertical straight line as an asymptote. If $R_e < 2(\sqrt[3]{2})$ and $Q=2R_e^{-1}-\tfrac{1}{2}R_e^2$, then the integral line crosses orthogonally the Z axis. Results of the investigation of the integral line shape depending on the values of R_e and Q with full details were presented by Slobozhanin.[34]

Let us choose the profile point located at the edge of the lower disk as the initial point $\tau=0$. Then the equilibrium liquid bridge shape is determined by solutions of the equations (7) with the initial conditions

$$\begin{aligned}&R(0)=R_0,\quad R'(0)=\cos\beta_0,\quad Z(0)=0,\\&Z'(0)=\sin\beta_0,\quad \beta(0)=\beta_0.\end{aligned} \quad (9)$$

These solutions must satisfy the conditions

$$R(\tau_1)=R_0,\quad Z(\tau_1)=2H,\quad \int_0^{\tau_1}R^2Z'\,d\tau=2HR_0^2V. \quad (10)$$

Here τ_1 is the value of τ at the end point of the profile, and the quantities R_0 and H, according to (1) and (6), are equal to

$$R_0=r_0p^{1/3}=W^{1/3},\quad H=hp^{1/3}=\Lambda W^{1/3}. \quad (11)$$

The conditions (10) define the values τ_1, β_0, and Q in terms of the parameters (1). Of course, the obtained liquid bridge shape should not cross the solid disk.

III. SOLUTION METHOD FOR THE STABILITY PROBLEM

According to the principle of minimum potential energy, the stability problem for the relative equilibrium of a liquid bridge under the allowed perturbations (satisfying the condition of volume conservation and the condition of fixed contact lines) can be reduced to the determination of the sign of the smallest eigenvalue ν_* of a known spectral problem:[11] an equilibrium is stable if $\nu_*>0$ and is unstable if $\nu_*<0$. The method of solution of the stability problem for an axisymmetric equilibrium state, which does not require the calculation of ν_*, is also described by Myshkis *et al.*[11]

For an axisymmetric isorotating liquid bridge, the abovementioned spectral problem can be reduced to a one-dimensional boundary-value problem,

$$\begin{gathered}\mathscr{L}\varphi_0-\mu=\nu\varphi_0 \quad (0\leqslant\tau\leqslant\tau_1), \quad \varphi_0(0)=0,\\ \varphi_0(\tau_1)=0, \quad \int_0^{\tau_1} R\varphi_0(\tau)d\tau=0,\end{gathered} \tag{12}$$

associated with the stability to axisymmetric perturbations, and to the one-dimensional boundary-value problem,

$$\begin{gathered}\mathscr{L}\varphi_1-(1/R^2)\varphi_1=\nu\varphi_1 \quad (0\leqslant\tau\leqslant\tau_1),\\ \varphi_1(0)=0, \quad \varphi_1(\tau_1)=0,\end{gathered} \tag{13}$$

associated with the stability to nonaxisymmetric perturbations corresponding to the first harmonic in the polar angle θ. Here

$$\mathscr{L}\varphi\equiv\varphi''+\frac{R'}{R}\varphi'+a(\tau)\varphi, \quad a(\tau)=2RZ'+\beta'^2+\frac{Z'^2}{R^2}.$$

The reason for this reduction is the validity of the equality $\nu_*=\min\{\nu_{01},\nu_{11}\}$, where ν_{01} and ν_{11} are the smallest of eigenvalues of the problems (12) and (13), respectively.

The vanishing of any eigenvalues of the problems (12) and (13) is equivalent to the equalities $\mathscr{D}(\tau_1)=0$ and $\varphi_{11}(\tau_1)=0$, where

$$\mathscr{D}(\tau)=\varphi_{01}(\tau)\int_0^{\tau}R\varphi_{02}(\tau)d\tau-\varphi_{02}(\tau)\int_0^{\tau}R\varphi_{01}(\tau)d\tau, \tag{14}$$

$$\mathscr{L}\varphi_{01}=0, \quad \varphi_{01}(0)=0, \quad \varphi_{01}'(0)=1, \tag{15}$$

$$\mathscr{L}\varphi_{02}-1=0, \quad \varphi_{02}(0)=0, \quad \varphi_{02}'(0)=1, \tag{16}$$

$$\mathscr{L}\varphi_{11}-\frac{1}{R^2}\varphi_{11}=0, \quad \varphi_{11}(0)=0, \quad \varphi_{11}'(0)=1. \tag{17}$$

From the properties of ν_{01} and ν_{11} (see Ref. 11), it follows that an isorotating liquid bridge is stable with respect to axisymmetric (nonaxisymmetric) perturbations if the function $\mathscr{D}(\tau)$ [the function $\varphi_{11}(\tau)$] does not vanish for $0\leqslant\tau\leqslant\tau_1$. The first (with τ increasing) point $\tau=\tau_*$, where $\mathscr{D}(\tau)$ or $\varphi_{11}(\tau)$ vanishes (in practice, changes sign) is critical: if $\tau_*>\tau_1$ then an axisymmetric equilibrium state is stable, if $0<\tau_*<\tau_1$ is unstable. The profile for which $\tau_*=\tau_1$ corresponds to a neutrally stable equilibrium state.

Avoiding the solution of the complicated equilibrium problem given by the equations (7), (9), and (10), we shall find only the neutrally stable states. To do this, for a given R_0 (given W) and a chosen value of the profile slope $\beta_0=\beta_0^1$, the numerical integration of the initial value problems (7), (9), and (15)–(17) is performed up to the point $\tau=\tau_*$. Since the quantity $R(\tau_*)$ depends on Q, the quantity Q must be changed until we obtain, for some $Q=Q_1$, a value $\tau_*^{(1)}=\tau_*(R_0,\beta_0^{(1)},Q_1)$, such that the condition $R(\tau_*^{(1)};R_0,\beta_0^{(1)},Q_1)=R_0$ is satisfied within the required accuracy.

The solutions $R(\tau;R_0,\beta_0^{(1)},Q_1)$ and $Z(\tau;R_0,\beta_0^{(1)},Q_1)$ of the problem (7), (9) on the interval $0\leqslant\tau\leqslant\tau_*^{(1)}$ determine the shape of the profile of the neutrally stable equilibrium surface. This surface is critical with respect to axisymmetric perturbations if the function that vanishes in the point $\tau=\tau_*^{(1)}$ is $\mathscr{D}(\tau)$ and is critical with respect to nonaxisymmetric perturbations if the function that vanishes is $\varphi_{11}(\tau)$. The component of the critical nonaxisymmetric perturbations normal to the axisymmetric equilibrium surface is proportional to $\varphi_{11}(\tau)\cos\theta$.

The neutrally stable surface constructed in this way determines the coordinates,

$$\Lambda=\frac{Z(\tau_*^{(1)})}{2R_0}, \quad V=\frac{1}{R_0^2Z(\tau_*^{(1)})}\int_0^{\tau_*^{(1)}}R^2Z'\,d\tau, \tag{18}$$

of one point of the stability boundary in the (Λ,V) plane for a given value of the Weber number, $W=R_0^3$. Other points can be determined in a similar way by changing the value of β_0.

In an analogous way, the neutrally stable surface for a given W might also be constructed by the fixing the value of Q and the changing value of β_0.

IV. PRELIMINARY ANALYSIS

Since the contact lines are circles of equal radius, the profile of an equilibrium liquid bridge surface either is reflectively symmetric about the horizontal straight line passing through the "equatorial" point [which coincides with one of the stationary points of the function $R(\tau)$] or has an arclength τ_1 that is a multiple of the period T of the function $R(\tau)$.

Let us assume that the neutrally stable liquid bridge has an equatorial symmetry plane. If the nonaxisymmetric perturbations are critical, then, choosing the equatorial point on the profile of this surface as the initial point, taking into account the form of the equation (17), and the well-known Sturm theorem of the separation of zeros for the solutions of a second-order differential equation, it can be proved that the function $\varphi_{11}(\tau)$ is symmetric about the equatorial point $\tau=\tau_*/2$. If the axisymmetric perturbations are critical, then they are either reflectively symmetric or reflectively antisymmetric with respect to the equatorial plane (the abbreviations used by Brown and Scriven[23] and Ungar and Brown[31] are ax.-r.s. perturbations and ax.-r.a. perturbations, respectively).

In addition, the following statements are useful.

(a) If $\beta_0=\pi/2$ and the liquid bridge shape is noncylindrical, then $\varphi_{01}=R'(\tau)$ and the function $\mathscr{D}(\tau)$ vanishes at $\tau=T$ for the first time.

(b) If a given integral line of the system (7) has two portions that correspond to neutrally stable surfaces, none of them can fully contain the other.[11]

These statements lead to the following conclusions.

For the neutrally stable surfaces not being cylindrical and having an equatorial symmetry plane, the profile arclength τ_1^* is not larger than T. The equality $\tau_1^* = T$ holds for $\beta_0 = \pi/2$. If $\tau_1 > T$, an equilibrium surface with an equatorial symmetry plane is unstable.

Furthermore, it can be proved that ax.-r.a. perturbations are critical for neutrally stable noncylindrical equilibrium surfaces if and only if $\beta_0 = \pi/2$. In this case, the component of the critical perturbations normal to the equilibrium surface is proportional to $R'(\tau)$.

It is known[20] that in the absence of rotation a weightless liquid bridge without an equatorial symmetry plane is always unstable with respect to axisymmetric perturbations. For regards an isorotating liquid bridge without an equatorial symmetry plane, the same conclusion for surface with $\tau_1 = nT$, where $n > 1$, follows from the statements (a) and (b). There is no full proof of a similar result for the case $\tau_1 = T$. However, it is possible to study this case for almost cylindrical isorotating liquid bridges. To do this, let us represent

$$R(\tau) = R_0 + \sum_{i=1}^{\infty} \epsilon^i R_i(\tau), \quad Z(\tau) = \tau + \sum_{i=1}^{\infty} \epsilon^i Z_i(\tau),$$

$$\beta(\tau) = \frac{\pi}{2} + \sum_{i=1}^{\infty} \epsilon^i \beta_i(\tau), \tag{19}$$

$$\varphi_{01}(\tau) = \sum_{i=0}^{\infty} \epsilon^i \psi_i(\tau), \quad \varphi_{02}(\tau) = \sum_{i=0}^{\infty} \epsilon^i f_i(\tau),$$

$$Q = Q_0 + \sum_{i=1}^{\infty} \epsilon^i Q_i, \quad \beta(0) = \frac{\pi}{2} + \sum_{i=1}^{\infty} \epsilon^i \beta_{i0},$$

where ϵ is a small parameter, and the expansions for Q and $\beta(0)$ are considered as given. The quantities R_0 and Q_0 are connected by the relation

$$Q_0 = R_0^{-1} - R_0^2. \tag{20}$$

Then, the period T and the first positive root μ of the equation $\mathscr{R}(\tau) = 0$ are of the form

$$T = \sum_{i=0}^{\infty} \epsilon^i T_i, \quad \mu = \sum_{i=0}^{\infty} \epsilon^i \mu_i. \tag{21}$$

As a result of rather cumbersome calculations, we obtain

$$T_0 = \mu_0 = 2\pi K^{-1}, \quad T_1 = \mu_1 = 2\pi K^{-3} A Q_1,$$

$$\mu_2 = 2\pi K^{-3}\{M_1 K^{-2} Q_1^2 + M_2 \beta_{10}^2 + A Q_2\}, \tag{22}$$

$$T_2 = \mu_2 + 2\pi K^{-3} M_3 \beta_{10}^2,$$

where

$$K = (2R_0 + R_0^{-2})^{1/2}, \quad A = (1 - R_0^{-3}) K^{-2},$$

$$M_1 = \tfrac{1}{48}(10 A R_0^{-1} + 140 A^2 + 5 R_0^{-2} + 3K^2 - 90 R_0^{-4} K^{-2}),$$

$$M_2 = \tfrac{1}{16}(10 A R_0^{-1} + 4A^2 + 5R_0^{-2} - 5K^2 - 18 R_0^{-4} K^{-2}),$$

$$M_3 = \tfrac{1}{24}(-10 A R_0^{-1} + 4A^2 - 5R_0^{-2} + 9K^2 + 18 R_0^{-4} K^{-2}).$$

Since $M_3 > 0$, then $\mu_2 < T_2$ and $\mu < T$. Consequently, the almost cylindrical isorotating liquid bridges without an equatorial symmetry plane are unstable.

This statement is valid not only for isorotating liquid bridges with $V = 1$. It includes a result earlier obtained by Brown and Scriven,[23] which proved using an asymptotic analysis of the bifurcation problem for a cylindrical state of an isorotating liquid bridge that the almost cylindrical liquid bridges with $V = 1$ and without an equatorial symmetry plane are unstable. Furthermore, according to the numerical results of Brown and Scriven,[23] all other axisymmetric shapes, which are not necessarily close to a cylinder, with $V = 1$ and without an equatorial symmetry plane are unstable.

Consequently, only cylindrical states can be roughly stable in the cases $\beta_0 = \pi/2$ or $V = 1$. For $\beta_0 = \pi/2$, other equilibrium states are either neutrally stable or unstable, and, for $V = 1$, unstable.

V. RESULTS

Prior to presenting the numerical results, let us make a remark directly related to the subject under consideration. For some values of the equilibrium parameters, an inaccurate numerical determination of the shape of a neutrally stable liquid bridge by means of changing the value of β_0 with given R_0 and Q may lead to the wrong result that $\beta_0 = \beta_1 = \pi/2 \pm \delta$ [$\beta_1 = \beta(\tau_1)$], where δ is small.

The equality $\beta_0 = \beta_1$ is sufficient to show that this liquid bridge does not have an equatorial symmetry plane. However, with a decreasing integration step, we obtained $\beta_0 = \pi/2 \pm \epsilon$, $\beta_1 = \pi/2 \mp \epsilon$, where $\epsilon \to 0$, and arrived at the known result on the neutral stability of a liquid bridge with an equatorial symmetry plane, and $\beta_0 = \pi/2$. If calculations are not accurate enough, a mistake may arise from the fact that for β_0 values close to $\pi/2$ the quantities T and μ are closely related [that is why the first two terms in the both expansions (21) coincide]. In general, as a result of sufficiently accurate calculations, in no case was detected any neutrally stable liquid bridge without the equatorial symmetry plane.

A. Typical form of the stability region boundary for not-too-large Weber numbers (the case $W = 1$)

As an illustration, let us first consider the stability region boundary in the case $W = 1$. The general boundary, obtained if arbitrary (both axisymmetric and nonaxisymmetric) perturbations are accounted for, consists of four parts [the solid line in Fig. 2(a)].

The upper segment ABC of the boundary determines the maximum possible volume V of a stable axisymmetric liquid bridge for a given Λ. The points of this segment correspond

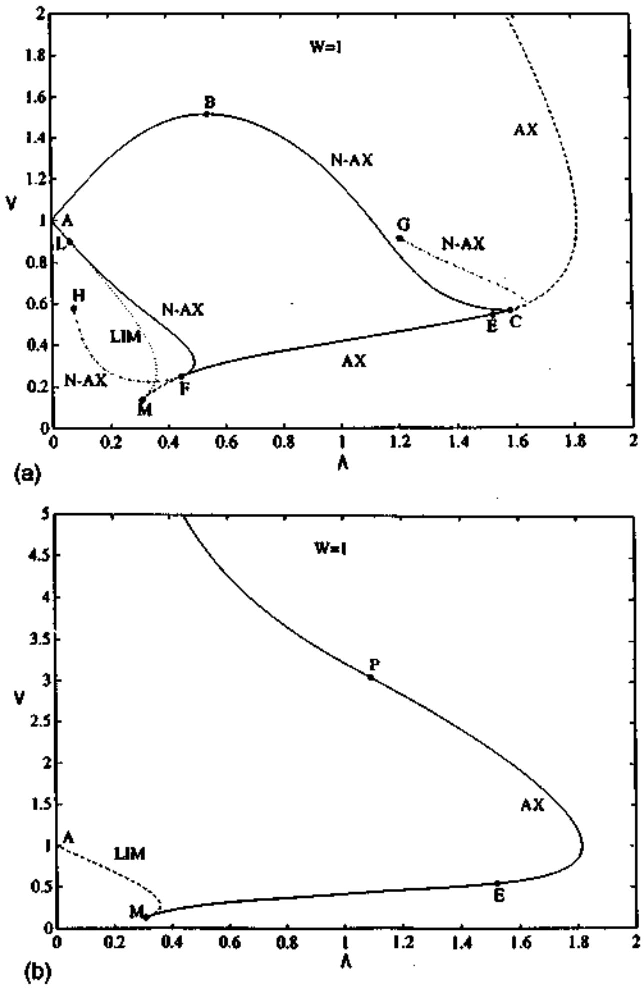

FIG. 2. Boundary of the region of stable isorotating liquid bridges for $W=1$. Axisymmetric and nonaxisymmetric perturbations have been considered in (a) while only axisymmetric perturbations have been considered in (b). The relevant part of the boundary is plotted with a solid line. The meaning of the different labels is explained in the text.

to the neutrally stable states that are critical with respect to nonaxisymmetric perturbations. The dependence $V(\Lambda)$ reaches the maximum at the point B.

The right-hand part FEC of the lower boundary is determined by the neutrally stable states, which are critical with respect to axisymmetric perturbations. These perturbations are ax.-r.s. perturbations on the segment FE and ax.-r.a. perturbations on the segment EC. At the point C, which is common for the upper boundary and this segment, the quantity Λ reaches its maximum value along the whole boundary.

The points belonging to the mid-part LF of the lower boundary correspond to the neutrally stable states, which are critical to nonaxisymmetric perturbations. The segments LF and FC touch each other at the point F, where the quantity V reaches its minimum value along the whole boundary. When approaching along this segment to the point L the value of β_0 for the corresponding critical surfaces is increasing up to the value $\beta_0=\pi$ in the point L. On further decreasing of Λ, the value of β_0 for the neutrally stable surfaces becomes larger than π. However, such surfaces cannot satisfy the geometric condition of fitting between flat solid disks.

Therefore, the left-hand segment AL of the lower boundary is determined by the so-called limiting surfaces. These surfaces are characterized by the equality $\beta_0=\pi$ and are stable.

Using the case $W=1$ as an example, it is interesting to follow the behavior of the boundary of a stability region when only axisymmetric perturbations or only nonaxisymmetric perturbations are taken into account. The region boundary of the stability with respect to axisymmetric perturbations is shown in Fig. 2(b). Along its upper part, $V\rightarrow\infty$ as $\Lambda\rightarrow 0$. When moving along the upper part in the direction of decreasing V, the value of β_0 increases, starting from negative values up to a value $\pi/2$ in the point P. The points in the segment PE correspond to surfaces with $\beta_0=\pi/2$. The maximum Λ value takes place for $V=1$ and corresponds to the cylindrical surface that is critical to axisymmetric perturbations. According to (4), here $\Lambda = \pi/\sqrt{1+2W}$. When moving from the point E to the point M, the value of β_0 increases monotonically from $\pi/2$ to π. On the segment PE the ax.-r.a. perturbations are critical. In all other points of the boundary shown by the solid line the ax.-r.s. perturbations are critical. Since the surfaces, which are neutrally stable with respect to axisymmetric perturbations and have $\beta_0>\pi$ and $V<1$ do not satisfy the condition of fitting between disks, then part of the boundary must be replaced by the dashed line MA corresponding to the limiting surfaces (with $\beta_0=\pi$).

The upper ABCG and the lower LFH branches of the stability boundary with respect to nonaxisymmetric perturbations are presented in Fig. 2(a). As approaching to the end points G and H along these branches, the equatorial radius of a liquid bridge tends to zero [the quantity A_0 appearing in (8) vanishes at the end points]. It is intriguing that the liquid bridges having a very small equatorial radius may be neutrally stable with respect to nonaxisymmetric perturbations, although they are clearly unstable with respect to axisymmetric perturbations. Furthermore, the surfaces, which are neutrally stable with respect to nonaxisymmetric perturbations, may have a profile arclength τ_1 that is larger than T. The surfaces of this type correspond to the inner points of the section CG. Here $0.32\pi<\beta_0=\pi-\beta_1<\pi/2$, in spite of the fact that $V<1$.

B. General case. Evolution of a stability region

Let us follow the evolution of the general boundary of the stability region when changing the Weber number. The results of a numerical construction of this boundary for a set values of W are presented in Fig. 3.

Unlike the case $W=0$,[21,11] the stability region for an isorotating liquid bridge becomes closed. When increasing the W number the stability region narrows. If $0<W<W_0$, where $2.05<W_0<2.06$, the stability region boundary consists of the four parts described above. The largest among the analyzed values of W having this structure of the boundary is equal to 2.05 [Fig. 3(b)]. According to Eqs. (3) and (4), the maximum value of Λ for $W=1/6$ is equal to $\pi\sqrt{3}/2$ and takes place for a critical cylindrical surface ($V=1$). Thus, for $W<\frac{1}{6}$, the

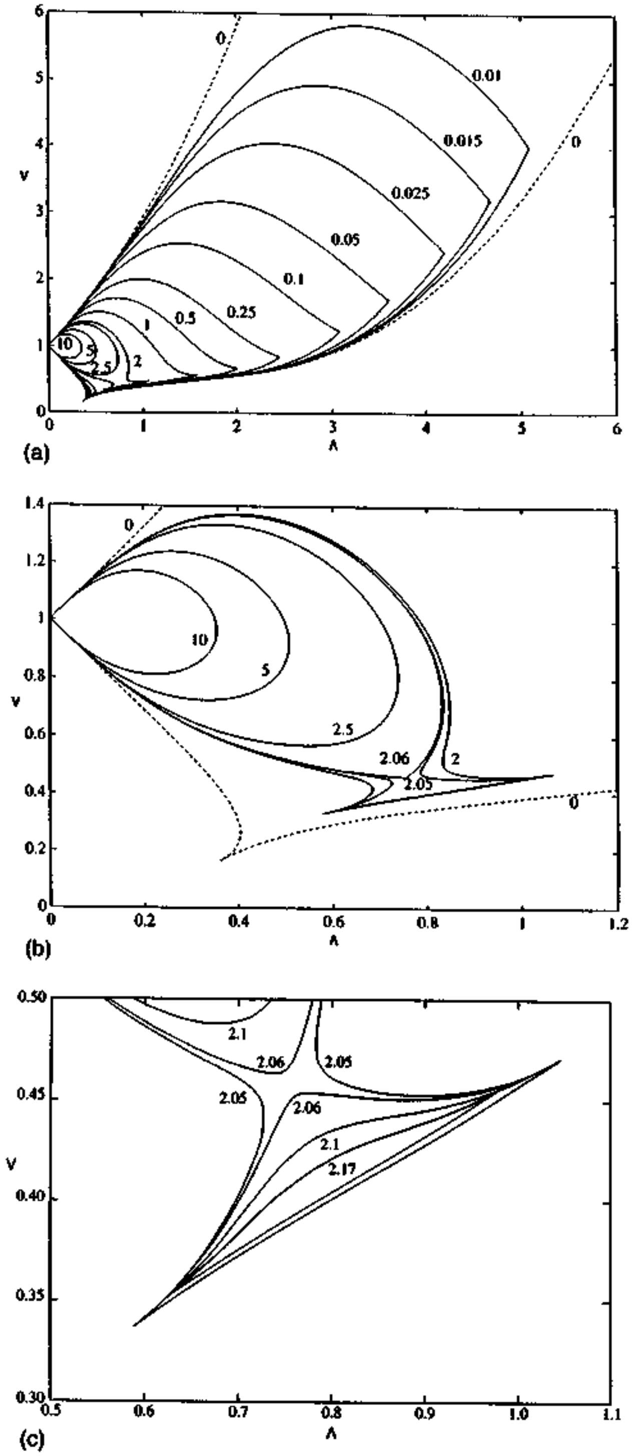

FIG. 3. Boundaries of the region of stable isorotating liquid bridges. Numbers on the curves indicate the value of the Weber number. The limit for no rotating ($W=0$) liquid bridges has also been plotted with a dashed line.

value of V is larger than 1 at the points of the upper boundary and, for $\frac{1}{6}<W<W_0$, this value is smaller than 1 along the lower boundary.

The evolution of the stability boundary as the value of W increases near W_0 is clear from Figs. 3(b) and 3(c). The upper and the lower left boundary branches, related to the states that are critical with respect to nonaxisymmetric per-

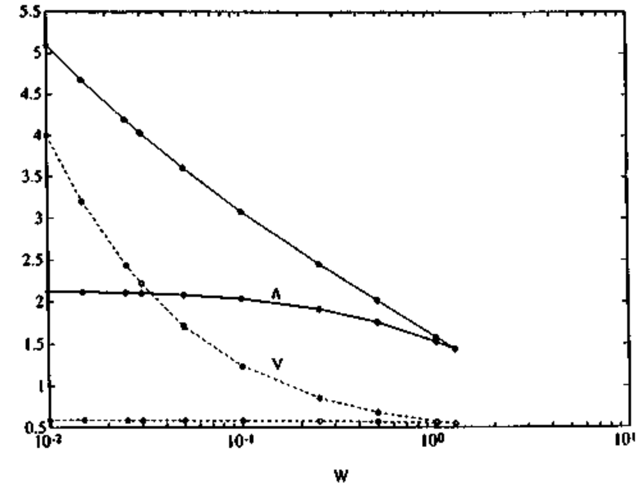

FIG. 4. Interval of slendernesses, Λ, and volumes, V, where ax.-r.a. perturbations are critical in the stability limit as a function of the Weber number, W.

turbations, form a neck when the W value is somewhat greater than 1. As the W number increases the neck size decreases until, for $W=W_0$, a saddle point arises on the stability boundary. On further increase of the Weber number, the stability region breaks into two disconnected parts. The upper (basic) part exists for any value $W>W_0$. There is no neutrally stable equilibrium state on its general boundary, which is critical with respect to axisymmetric perturbations. Here the boundary points correspond either to neutrally stable surfaces that are critical with respect to nonaxisymmetric perturbations or to limiting surfaces.

As regards the lower part of the stability region, its boundary consists of two segments. For the states belonging to the upper (lower) segment, nonaxisymmetric (axisymmetric) perturbations are critical. [The lower segment for $W=2.1$ is not shown in Fig. 3(c). For $W=2.06$, it practically coincides with the similar boundary segment related to $W=2.05$.] The lower part is considerably smaller in size than the upper part of the stability region. Moreover, it exists on the limited interval of the Weber number $W_0<W<W_1$, and disappears when $W>W_1$. According to our calculations, $2.44<W_1<2.45$. Thus, there exist two disconnected parts of the stability region when $W_0<W<W_1$.

The boundaries shown in Fig. 3 contain all the necessary information to follow the evolution of the representative points of the stability curve, except for the points that are analogous to the points E and L in Fig. 2(a).

If $0\leq W<1.25$, the boundary segment corresponding to critical axisymmetric perturbations consists of two parts: the lower part, where ax.-r.s. perturbations are dangerous, and the upper part, where ax.-r.a. perturbations are dangerous [the parts FE and EC in Fig. 2(a)]. If $1.25<W<W_1$ the ax.-r.s. perturbations are critical along the whole indicated segment of the general boundary of the stability region.

The extent of the boundary segment, where ax.-r.a. perturbations are critical, can be determined for $W<1.25$ from Fig. 4. Either of the presented dependences $V(W)$ and $\Lambda(W)$ consists of two branches. The lower branch corresponds to

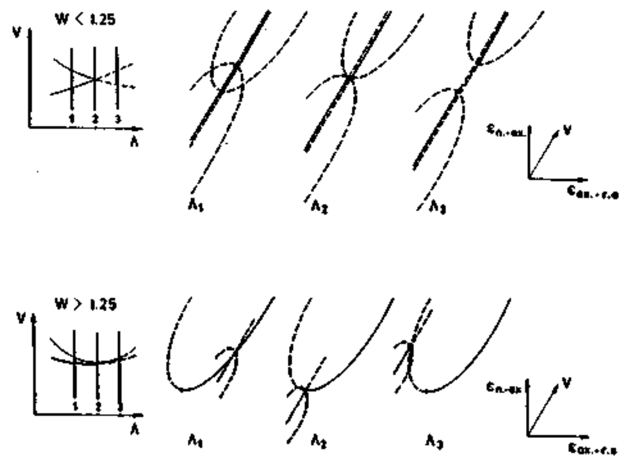

FIG. 5. Sketch of the possible situations appearing in the neighborhood of the point where the change between axisymmetric and nonaxisymmetric perturbations appear: (a) $W<1.25$; (b) $W>1.25$.

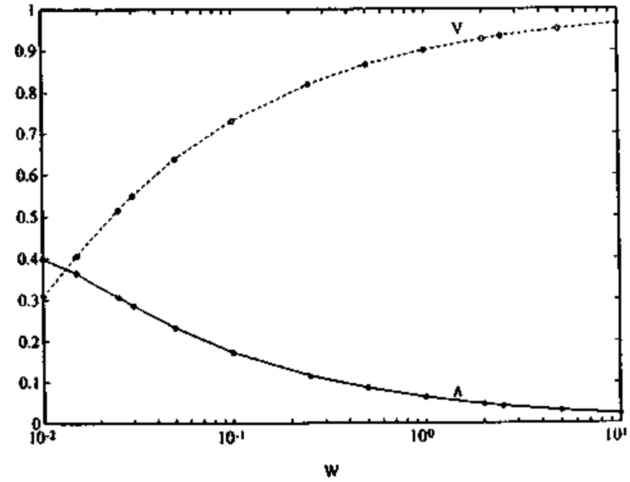

FIG. 6. Coordinates of the transition point between liquid bridges neutrally stable with respect to nonaxisymmetric perturbations and limiting liquid bridges.

the points of transition from the states critical with respect to ax.-r.s. perturbations to the states critical with respect to ax.-r.a. perturbations. These points are similar to the point E in Fig. 2(a). The upper branch corresponds to the points that are similar to the point C. Along the lower branch the values of V and Λ tend to 0.5909 and 2.1283, respectively, as $W\rightarrow 0$.[21,11] For $W=1.25$, the values of V and Λ are equal to 0.545 and 1.441, respectively.

The change from critical ax.-r.s. perturbations to critical nonaxisymmetric perturbations is characterized by a common tangent of the corresponding boundary segments. Consequently, the section of the lower boundary related to the states critical to axisymmetric perturbations has a common tangent with the upper boundary if $1.25<W<W_0$, and has a common tangent with the segment of the lower boundary related to the states critical with respect to nonaxisymmetric perturbations if $0\leqslant W<W_0$. For $W_0<W<W_1$, the upper and lower boundary segments of the lower part of the stability region have common tangents at both common points.

The reason for having this common tangent may be explained as follows (see Fig. 5). The ax.-r.s. perturbations are associated with the existence of a limit point in the $V-\epsilon_{\text{ax.-r.s.}}$ diagram (where $\epsilon_{\text{ax.-r.s.}}$ represents the amplitude of such a perturbation), whereas the ax.-r.a. perturbations (the nonaxisymmetric perturbations) are associated with a pitchfork bifurcation in the $V-\epsilon_{\text{ax.-r.a.}}$ map (the $V-\epsilon_{\text{n.-ax.}}$ map). Then, for $W<1.25$ the points where the pitchfork bifurcations occur (those corresponding to nonaxisymmetric perturbations and ax.-r.a. perturbations) simply cross as Λ increases [Fig. 5(a)], while for $W>1.25$ the point where the pitchfork bifurcation corresponding to nonaxisymmetric perturbations occurs moves along the curve where the limit point (corresponding to ax.-r.s. perturbations) occurs [Fig. 5(b)]. In this latter case $V^{*\neg}_{\text{n.-ax}} \gtrless V^{*\neg}_{\text{ax.-r.s.}}$ for slendernesses larger, equal to, or smaller than that of the common tangent point.

C. Limiting surfaces

The coordinates of the point of transition from critical equilibrium surfaces to limiting surfaces [like the point L in Fig. 2(a)] are presented for different values of W in Fig. 6. Here, $\Lambda\rightarrow 0.3611$ and $V\rightarrow 0.1643$ as $W\rightarrow 0$.[21,11] For $W=0$, the point of transition to limiting surfaces (which, simultaneously, are the critical surfaces with respect to nonaxisymmetric perturbations) coincides with the boundary point corresponding to the minimum permissible value of V.

The limiting surfaces, satisfying the condition of fitting between flat solid disks, exist only if $-\infty<Q\leqslant -W^{2/3}$. For $-\infty<Q<-W^{2/3}$, the curvature $\beta'(\tau)$ of the profile, bounded by the initial ($\beta_0=\pi$) and the end ($\beta_1=0$) points, is everywhere negative, and, for $Q=-W^{2/3}$, it vanishes at the terminal points. If $Q>-W^{2/3}$, the curvature is positive in the terminal points and the corresponding surface intersects the disk planes in some interior points.

Figure 7 shows two examples of the line whose points correspond to the limiting surfaces with $Q\leqslant -W^{2/3}$ for a given W value. The larger is W, the larger is the difference

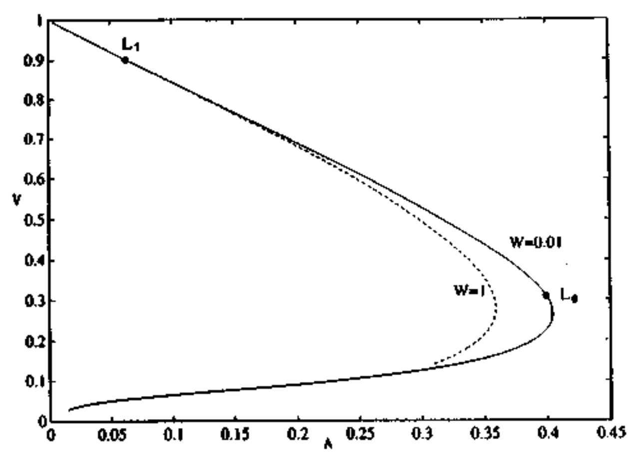

FIG. 7. Lines corresponding to the limiting surfaces for two values of the Weber number ($W=0.01,1$). Points L_0 and L_1 indicate the transition from limiting surfaces to critical surfaces, as explained in the text.

between this line and the corresponding line for $W=0$. However, only the segment of this line, bounded by the point (0,1) and the point of transition from the limiting to the critical surfaces, belongs to the boundary of the stability region. Owing to the position of the transition points (the points L_0 and L_1, for $W=0.01$ and $W=1$, respectively; see Fig. 7), the indicated segments, independently of the Weber number, practically lie on a common line, which corresponds to limiting surfaces for $W=0$. This line represents the left-hand part of the lower boundary of the stability region for $W=0$ [see Fig. 3(b)]. Here the following relation holds for small values of Λ:

$$V=1-\Lambda\,\frac{\pi}{2}+\Lambda^2\left(\frac{8}{3}-\frac{1}{4}\,\pi^2\right)+O(\Lambda^3). \tag{23}$$

From the above observations it appears that this relation should be valid for any Weber number.

D. Slope of the profiles of critical surfaces at the disks

In this section we will present the values of β_0 and β_1 in degrees. The values of the angles $\beta=\beta_0$ and $\beta=\beta_1$ at the lower and at the upper disk are important features of the shape of a liquid bridge profile. As remarked in the Introduction, of particular interest for the crystal growing are the values of β_0 close to 90°.

For the stable and the neutrally stable liquid bridges, the equality $\beta_0+\beta_1=180°$ holds. The level lines $\beta_0=\text{const}$ for the neutrally stable liquid bridges, together with the stability region boundaries for different W numbers, are shown in Fig. 8. The change of β_0 along the stability boundary, excluding the boundary of the lower part of a stability region for $W_0<W<W_1$, may follow one of four different patterns depending on the value of W. The respective examples are presented in Fig. 9.

It is known[21,11] that $\beta_0=0$ along the upper boundary corresponding to $W=0$. When rotation is considered, there exist critical states with $\beta_0<0$. When moving from the point A ($\Lambda=0$, $V=1$) along the upper boundary corresponding to $W>0$, the value of β_0 increases from negative values and crosses zero in the point whose coordinates are shown in Fig. 10. If $0<W\leq\frac{1}{6}$, then the further rise of β_0 along the whole upper boundary is also monotonic up to the value $\beta_0=90°$ at the point of intersection with the lower boundary (see the curve for $W=0.1$ in Fig. 9). For $\frac{1}{6}<W\leq1.25$, the value of β_0 increases up to a maximum that is larger than 90° and then decreases down to 90° at the end point of the upper boundary (see the curve for $W=1$ in Fig. 9). The value of this maximum increases with the Weber number and is equal to 106.0° for $W=1.25$. The value $\beta_0=90°$ at some interior point of the upper boundary corresponds to a critical cylindrical state, and values $\beta_0>90°$ to critical states with $V<1$ [Fig. 8(a)]. The case $1.25<W<W_0$ differs from the previous one in that the value of β_0 at the end point of the upper boundary is larger than 90° (see the curve for $W=2$ in Fig. 9). For $W=2.05$, we obtain that the maximum value of β_0 is equal to 128.0° and the value at the end point is equal to 102.7°. Finally, if $W>W_0$, the value of β_0 increases monotonically

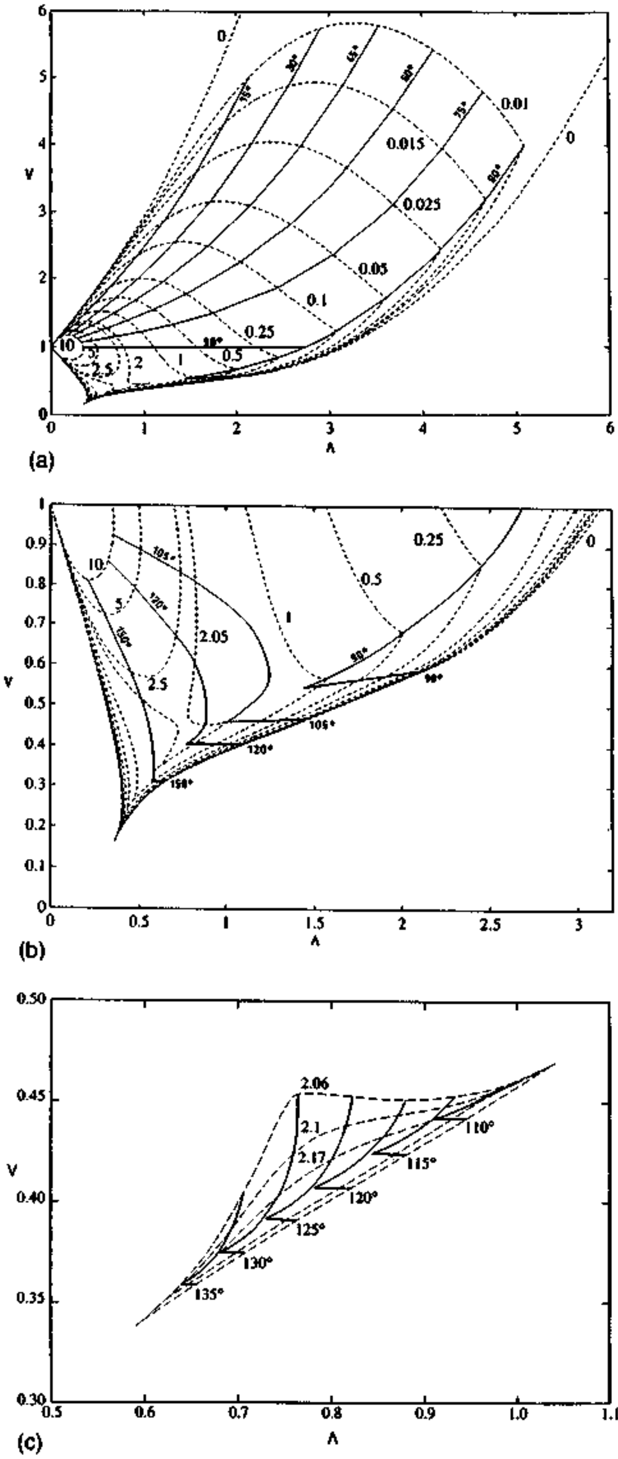

FIG. 8. Values of the slenderness and volume of neutrally stable liquid bridges with constant angle at the lower disk, β_0 (solid lines). Dashed lines represent the stability limits for different Weber numbers, W.

along a major part of the whole boundary up to 180° at the point of transition to limiting surfaces (see the curve for $W=5$ in Fig. 9).

From the above results it follows that, for an arbitrary Weber number, when moving along the upper boundary segment, where $V\geq1$, the monotonic increase of β_0 takes place up to a value equal to 90° at the point where $V=1$ or at the

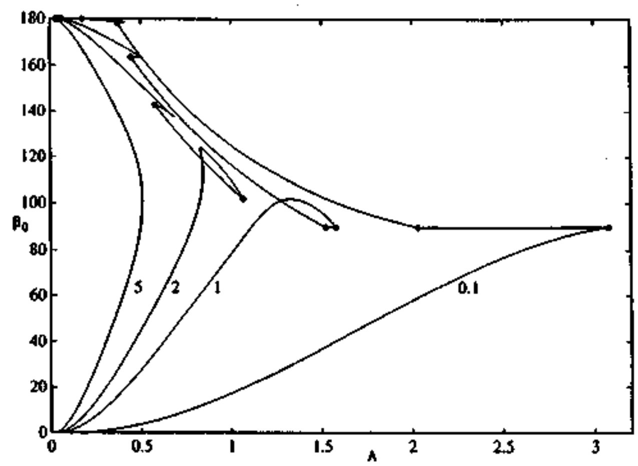

FIG. 9. Typical evolutions of the slope of the liquid bridge surface at the lower disk, β_0, along the stability boundary. Numbers on the curves indicate the values of the Weber number.

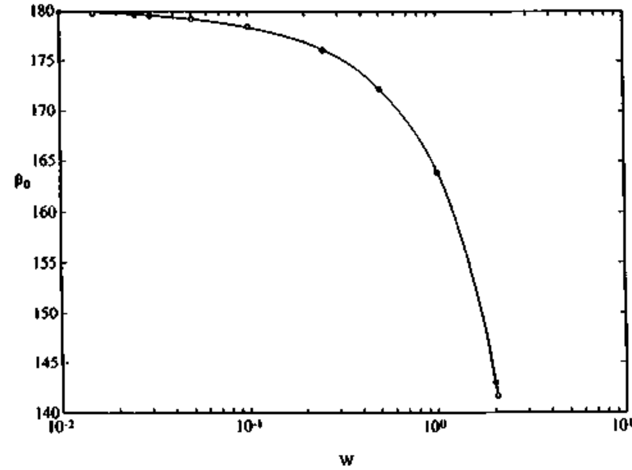

FIG. 11. Values of the slope of the liquid bridge surface at the lower disk, β_0, at the point of the lower stability boundary, where transition from axisymmetric to nonaxisymmetric perturbations appears.

end point of the upper boundary if $V>1$ in this point [see Fig. 8(a)].

Consider the change of β_0 when moving along the right-hand part of the lower boundary ($W<W_0$) from the point of its intersection with the upper boundary. For $W<1.25$, β_0 is equal to 90° on the segment corresponding to states critical with respect to ax.-r.a. perturbations (see the curves for $W=0.1$ and $W=1$ in Fig. 9). Along the segment corresponding to states critical with respect to ax.-r.s. perturbations, β_0 increases monotonically from 90° up to the value corresponding to the lower point of transition to the states critical with respect to nonaxisymmetric perturbations. A similar increase, however, from a value that is larger than 90°, takes place along the whole right-hand part of the lower boundary when $1.25<W<W_0$, that is, when there is not any state critical with respect to ax.-r.a. perturbations (see the curve for $W=2$ in Fig. 9). The values of β_0 corresponding to the lower point of transition from the critical axisymmetric perturbations to the critical nonaxisymmetric perturbations are shown for different Weber numbers in Fig. 11. These values tend to 180° as $W\rightarrow 0$. The change of β_0 along the right-hand part of the lower boundary can also be deduced after the lines $\beta_0=$const presented in Fig. 8(b).

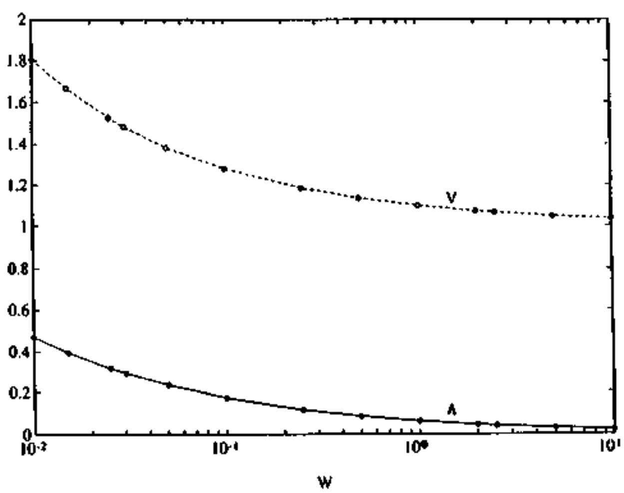

FIG. 10. Values of the slenderness, Λ, and the volume, V, in the point where $\beta_0=0$ for the upper boundary.

It has already been noted that all states, except the cylindrical, corresponding to $\beta_0=90°$ are either neutrally stable (if $\tau_1=T$) or unstable (if $\tau_1>T$). In Fig. 8(a) the horizontal segment $\beta_0=90°$ within the line $V=1$ corresponds to cylindrical liquid bridges that are critical with respect to nonaxisymmetric perturbations. The other part of the line $\beta_0=90°$ does not have a continuous slope. It passes through the terminal points of the boundary segments corresponding to neutrally stable states that are critical with respect to ax.-r.a. perturbations. The points, belonging to the unclosed region that is bounded by this line and by the lower boundary of stability for $W=0$, correspond to the above-mentioned neutrally stable surfaces with $\beta_0=90°$.

For $W<W_0$, along the lower boundary segment corresponding to the states critical with respect to nonaxisymmetric perturbations, β_0 decreases initially from the value presented in Fig. 11 and then increases up to 180° at the point of transition to limiting surfaces. Finally, along the boundary segment relating to limiting surfaces, $\beta_0=180°$.

As regards the boundary of the lower part of the stability region for $W_0<W<W_1$, the angle β_0 changes monotonically along the upper and lower segments, and takes limiting values at common points of the segments [see Fig. 8(c)]. For $W=2.06$, these values are equal to 102.8° and 141.4°.

VI. CONCLUDING REMARKS

Thus, only a rigorous analysis of arbitrary (both axisymmetric and nonaxisymmetric) perturbations allows one to construct the true region of stability for axisymmetric equilibrium states. Such an analysis has been performed for an isorotating liquid bridge between equal disks. As a result the stability boundaries in the (Λ,V) plane have been calculated for a wide range of the Weber number, and basic characteristics of related neutrally stable axisymmetric states have been determined. It has been found that there exists the interval of the Weber number values for which the stability region consists of two disconnected parts. A similar peculiarity escaped detection in studies of the influence of an axial gravity and unequal radii disks on the stability region.[35,36]

The following remarks have a direct relationship to the problem under consideration.

(a) First we consider the critical states of an isorotating liquid bridge for a given Λ and a decreasing set of values of V. Brown and Scriven[23] made the quite right conclusion that, for $\Lambda=1$, the loss of stability occurs initially with respect to nonaxisymmetric perturbations and then, up to the V value for which the critical W number is equal to zero, with respect to ax.-r.s. perturbations. However, the conclusion that in the case $\Lambda=\frac{1}{2}$ the stability is always lost with respect to nonaxisymmetric perturbations is not fully right. The confusion was due to the fact that not all critical states were defined. It is followed from our results that the loss of stability occurs always with respect to nonaxisymmetric perturbations only if $\Lambda<0.3611$.

(b) Based on the principle of minimum potential energy, the analysis of the conditions for stable equilibrium of a liquid when the contact line coincides with the edge of a solid was performed by Slobozhanin and Tyuptsov.[37,11] It was found that the perturbations, under which the contact line remains unmoved, are the most dangerous ones if the inequalities

$$\psi>\alpha \quad \text{and} \quad \psi_0>\pi-\alpha \qquad (24)$$

hold. Here ψ and ψ_0 are the dihedral angles formed by the liquid and the gas at the contact line and α is the wetting angle, i.e., the boundary angle formed by the liquid at the contact with the smooth solid surface. If the first (the second) of the inequalities (24) transforms in an equality, then the perturbations displacing the contact line along the solid surface toward the liquid (the gas) should be considered as well. If at least one of inequalities (24) is replaced by the opposite then the equilibrium of a free surface contacting with the edge of a solid is unstable.

Results presented in this paper are obtained if only the perturbations under which the contact lines remain immobile are accounted for. These results still stand for the perturbations that allow for the contact line displacement along the flat surfaces of the disks, if the angle α is equal to zero. If $\alpha>0$, then, according to the first inequality of (24), the boundary segment, corresponding to limiting equilibrium surfaces with $\beta_0=\pi$, should be replaced by the segment that corresponds to the equilibrium surfaces with $\beta_0=\pi-\alpha$ (and a given value of W). In a similar way, taking into account the geometric fitting condition and the first condition (24), the stability boundary should be reconstructed in the case when the wetted solid surfaces are not flat but convex toward the liquid. This case is characteristic for the real shape of the fronts of solidification and melting during growth of crystals by the floating zone method. To avoid the possibility of contact lines displacement along the lateral solid surfaces, the upper boundary should be reconstructed considering the shape of these surfaces and the second inequality of (24). The boundary reconstruction method considering arbitrary perturbations and the shape of solid body having an edge has been described elsewhere.[29]

(c) It should be noted that the simulation on Earth of the equilibrium and stability of an isorotating zero-gravity liquid is extremely complicated. The commonly used for simulation of zero-gravity Plateau's method of neutral buoyancy is suitable only for an experimental study of the equilibrium and stability of a liquid at rest (for an investigation of a weightless liquid bridge at rest this method was used successfully by Plateau, and later by Sanz and Martínez[18] and Russo and Steen[19]). For an isorotating liquid, it may lead to qualitatively different results. A convincing critical analysis, explaining the unsuitability of the neutral buoyancy method for an experimental study of a relative equilibrium of a weightless isorotating liquid, can be found in the works of Wang, Saffren, and Elleman,[38] Brown and Scriven,[23] and Myshkis *et al.*[11] An additional argument in favor of this criticism is the fact that the deduction[12] based on the known experimental Plateau's result[17] on an existence of stable annular equilibrium shapes of an isorotating free mass of a capillary liquid was wrong. By now, this deduction is contradicted by both a theoretical analysis[39] and an experiment performed in space.[38]

As regards the problem under consideration on the stability of a weightless isorotating liquid bridge, the relative experimental Plateau's results[17] (reproduced later by Tagg and Wang[40]) and the results of Carruthers and Grasso[41,42] obtained by the neutral buoyancy method are different from the experimental results obtained in space and from known theoretical results.[38,23]

Another method of simulation of zero gravity on Earth is based on a reduction of the linear size of the system. It does not eliminate fully but only diminishes the influence of gravity. It should be expected that the stability region boundary of an isorotating liquid will be modified only slightly if the Bond number is small as compared to the Weber number. Therefore this method is satisfactory for the study of the equilibrium and stability of a weightless capillary liquid isorotating with a relatively high velocity. For an isorotating liquid bridge, this method was used by Fowle *et al.*[25] Of course, the existence of even a weak gravity may essentially change the theoretical results connected with the existence of the equatorial symmetry plane for a weightless liquid bridge. In particular, the bifurcation of the critical states that were critical with respect to ax.-r.a. perturbations will become unrecognizable.[31,33]

ACKNOWLEDGMENTS

This work has been supported by the Spanish Comisión Interministerial de Ciencia y Tecnología (CICYT) and is part of a more general endeavor for the study of fluid physics and materials processing under microgravity (Project No. ESP92-0001-CP). One of the authors, L. A. S., has been supported by the Spanish Dirección General de Investigación Científica y Técnica (DGICYT) Project No. SAB93-0008.

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