A linear analysis of g-jitter effects on viscous cylindrical liquid bridges

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This paper deals with the dynamics of isothermal, axisymmetric, cylindrical liquid columns held by capillary forces between two circular, concentric, solid disks; in particular, it deals with the dynamic response of the bridge to an excitation consisting of a small change in the value of the microgravity level. The problem has been solved by using a linearized one-dimensional Cosserat model, which includes viscosity effects, and with the axial velocity considered as constant in each section of the liquid bridge. The analysis has been performed by using the Laplace transform, and the time variation of both the axial velocity field and the liquid bridge interface have been obtained.

I. INTRODUCTION

A liquid bridge is an idealization of the fluid configuration appearing in the crystal growth technique known as floating zone melting.[1] The liquid bridge configuration analyzed in this paper consists, as sketched in Fig. 1, of an isothermal, axisymmetric mass of liquid held by surface tension forces between two parallel, coaxial, solid disks of the same diameter. Such fluid configuration can be identified by the following set of dimensionless parameters: the slenderness, $\Lambda = L/2R_0$, where R_0 is the radius of the supporting disks and L the distance between them, the Bond number, $B = \rho g R_0^2/\sigma$, and the capillary number, $C = \nu(\rho/\sigma R_0)^{1/2}$, where ρ is the liquid density, g is the axial acceleration, σ is the surface tension, and ν is the kinematic viscosity.

In this paper, a linear theoretical model of the dynamic response of viscous axisymmetric liquid bridges due to a small change in the value of the axial microgravity acting on the liquid bridge is presented. Although several studies related to the dynamics of liquid bridges have been published, they are mainly concerned with harmonic perturbations.[2–7] Nonharmonic perturbations were considered in Ref. 8, where a one-dimensional inviscid slice model was used to analyze the liquid injection or removal in the liquid bridge. On the other side, because of the complexity, viscous effects have been rarely retained.[2,4,6,7]

To solve the problem, Cosserat's one-dimensional model for continuum has been used. This model has been used for capillary jet problems by Green[9] and Bogy[10–12] and for liquid bridge problems by several investigators[2,4,6] and proved to give satisfactory results, when compared with 3-D results, provided that the slenderness of the liquid bridge is large enough ($\Lambda > 1$).

In the following, unless otherwise stated, all physical quantities are made dimensionless using the characteristic length R_0 and the characteristic time $(\rho R_0^3/\sigma)^{1/2}$.

II. PROBLEM FORMULATION

In carrying out the analysis, the following assumptions are introduced: First, the properties of both the liquid (density and viscosity) and the interface (surface tension) are assumed uniform and constant, and second, the effects of the gas surrounding the liquid bridge are negligible. In addition, since only axisymmetric configurations are considered, the problem is assumed to be independent of the azimuthal coordinate. Under such assumptions, the set of nondimensional differential equations and boundary conditions for the axisymmetric, nonrotating viscous flow, according to the Cosserat model,[4] are

$$S_t + Q_z = 0, \tag{1}$$

$$\mathscr{D}S - \frac{1}{8}\left\{S^2\left[\mathscr{D}_z - \frac{3}{2}\left(\frac{Q}{S}\right)_z^2\right]\right\}_z = -SP_z - \frac{1}{8}C\left[S^2\left(\frac{Q}{S}\right)_{zz}\right]_{zz} + 3C\left[S\left(\frac{Q}{S}\right)_z\right]_z, \tag{2}$$

where

$$\mathscr{D} = [Q_t + (Q^2/S)_z]/S, \tag{3}$$

$$P = 4(2S + S_z^2 - SS_{zz})(4S + S_z^2)^{-3/2} + B(t)z. \tag{4}$$

In these expressions, $S = F^2$ and $Q = F^2W$ represent the cross-sectional area at z, t, and the axial momentum of a slice, respectively. Here, $F(z,t)$ is the dimensionless equation of the liquid–gas interface and $W(z,t)$ is the axial velocity (uniform in each plane parallel to the disks), whereas $P(z,t)$ accounts for both hydrostatic and capillary pressure. The subscripts t and z indicate derivatives with respect to time and the axial coordinate, respectively.

The formulation must be completed with suitable boundary conditions, which state that the interface must remain anchored to the disk edges and that the axial velocity at each one of the disks must be zero,

$$S(\pm\Lambda,t) = 1, \tag{5}$$

$$Q(\pm\Lambda,t) = 0. \tag{6}$$

The initial conditions are assumed to be

$$S(z,0) = 1, \tag{7}$$

$$Q(z,0) = 0, \tag{8}$$

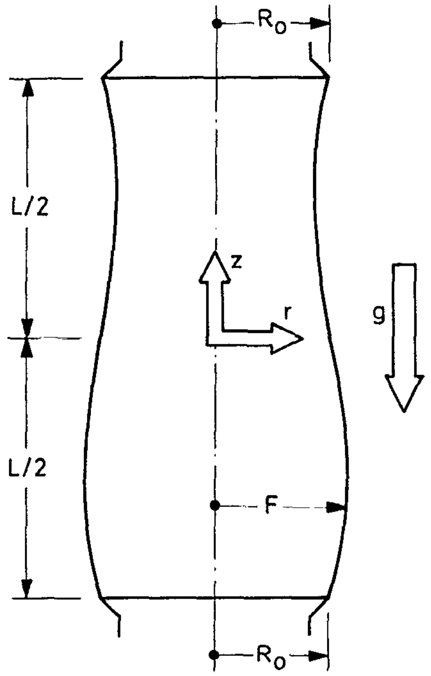

FIG. 1. Geometry and coordinate system for the liquid bridge problem.

that is, the liquid bridge is at rest at $t = 0$ [its shape being that of a cylinder, which implies $B(t) = 0$ for $t \leqslant 0$].

In addition, one more condition could be introduced stating the overall mass conservation during the evolution, that is

$$\int_{-\Lambda}^{\Lambda} S(z,t)dz = 2\Lambda, \tag{9}$$

although this condition can be deduced by integrating Eq. (1) with respect to z, and using the boundary conditon (6) and initial condition (7).

In the following, it is assumed that the value of the Bond number suddenly changes from the initial value $B = 0$ to a new one $B = \epsilon$, $\epsilon \ll 1$, where ϵ stands for the magnitude of the change, then

$$B(t) = \epsilon \mathscr{H}(t), \tag{10}$$

$\mathscr{H}(t)$ being the Heaviside function [$\mathscr{H}(t) = 0$ for $t \leqslant 0$ and $\mathscr{H}(t) = 1$ for $t > 0$].

Such dependence with time of the Bond number will affect both the interface shape and the velocity field in such a way that the variables could be expanded as

$$S(z,t) = 1 + \epsilon s(z,t) + o(\epsilon),$$

$$Q(z,t) = \epsilon q(z,t) + o(\epsilon), \tag{11}$$

$$P(z,t) = 1 + \epsilon p(z,t) + o(\epsilon).$$

Note that, since $S(z,t) = 1 + o(1)$, $\epsilon q(z,t)$ also represents the first-order approximation to the axial velocity. The introduction of these expressions in (1)–(8), neglecting second-order terms, gives the following first-order problem:

$$s_t + q_z = 0, \tag{12}$$

$$q_t - \tfrac{1}{8} q_{tzz} = -p_z - \tfrac{1}{8} C q_{zzzz} + 3 C q_{zz}, \tag{13}$$

$$p_z = -\tfrac{1}{2} s_z - \tfrac{1}{2} s_{zzz} + \mathscr{H}, \tag{14}$$

$$s(\pm\Lambda, t) = 0, \tag{15}$$

$$q(\pm\Lambda, t) = 0, \tag{16}$$

$$s(z,0) = 0, \tag{17}$$

$$q(z,0) = 0. \tag{18}$$

The introduction of (14) into (13) allows the elimination of p. To eliminate $s(z,t)$ from the formulation, Eqs. (12) and (13) can be used simultaneously to formulate the problem only in terms of $q(z,t)$, the resulting equation being

$$q_{tt} - \tfrac{1}{8} q_{ttzz} + \tfrac{1}{2} q_{zz} + \tfrac{1}{2} q_{zzzz} + \tfrac{1}{8} C q_{tzzzz} - 3 C q_{tzz} = -\mathscr{H}_t, \tag{19}$$

which must be solved with the boundary and initial conditions:

$$q(\pm\Lambda, t) = 0, \tag{20}$$

$$q_z(\pm\Lambda, t) = 0, \tag{21}$$

$$q(z,0) = 0, \tag{22}$$

$$q_t(z,0) = 0, \tag{23}$$

where boundary conditions (21) are obtained from Eq. (12) and boundary conditions (15), whereas initial condition (23) results from (13) and (17).

III. PROBLEM SOLUTION

Let $\mathbf{Q}(z,h)$ be the Laplace transform of $q(z,t)$; taking into account the initial conditions (22) and (23), Eq. (19) and boundary conditions (20) and (21) in the Laplace plane result in

$$(1 + \tfrac{1}{4} h C) \mathbf{Q}_{zzzz} + (1 - \tfrac{1}{4} h^2 - 6 h C) \mathbf{Q}_{zz} + 2 h^2 \mathbf{Q} = -2, \tag{24}$$

$$\mathbf{Q}(\pm\Lambda, h) = 0, \tag{25}$$

$$\mathbf{Q}_z(\pm\Lambda, h) = 0. \tag{26}$$

The solution of the above problem is of the form

$$\mathbf{Q}(z,h) = -1/h^2 + A_1 e^{\theta_1 z} + A_2 e^{-\theta_1 z} + A_3 e^{\theta_2 z} + A_4 e^{-\theta_2 z}, \tag{27}$$

where $\pm\theta_1$, $\pm\theta_2$ are the roots of the characteristic equation

$$(1 + \tfrac{1}{4} h C) \theta^4 + (1 - \tfrac{1}{4} h^2 - 6 h C) \theta^2 + 2 h^2 = 0, \tag{28}$$

that is,

$$\theta_i^2 = \frac{-1 + \frac{1}{4}h^2 + 6hC}{2 + \frac{1}{2}hC} \pm \frac{[1 - 12hC - h^2(\frac{17}{2} - 36C^2) + h^3C + \frac{1}{16}h^4]^{1/2}}{2 + \frac{1}{2}hC}. \tag{29}$$

The four constants, A_i, that appear in Eq. (27) are determined by solving the four equations resulting from the fulfillment of boundary conditions (25) and (26):

$$\begin{bmatrix} \exp(\theta_1\Lambda) & \exp(-\theta_1\Lambda) & \exp(\theta_2\Lambda) & \exp(-\theta_2\Lambda) \\ \exp(-\theta_1\Lambda) & \exp(\theta_1\Lambda) & \exp(-\theta_2\Lambda) & \exp(\theta_2\Lambda) \\ \theta_1\exp(\theta_1\Lambda) & -\theta_1\exp(-\theta_1\Lambda) & \theta_2\exp(\theta_2\Lambda) & -\theta_2\exp(-\theta_2\Lambda) \\ \theta_1\exp(-\theta_1\Lambda) & -\theta_1\exp(\theta_1\Lambda) & \theta_2\exp(-\theta_2\Lambda) & -\theta_2\exp(\theta_2\Lambda) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \frac{1}{h^2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \tag{30}$$

the final solution being

$$\mathbf{Q}(z,h) = (1/h^2)\{[D(z,h) - D(\Lambda,h)]/D(\Lambda,h)\}, \tag{31}$$

where

$$D(z,h) = \theta_2 \sinh\theta_2\Lambda \cosh\theta_1 z - \theta_1 \sinh\theta_1\Lambda \cosh\theta_2 z. \tag{32}$$

Since Eq. (31) has no poles at $h = 0$, the inverse transform of $\mathbf{Q}(z,h)$ will be formally[13]

$$q(z,t) = \sum_{n=1}^{\infty} \frac{D(z,h_n)}{h_n^2 D_h(\Lambda,h_n)} \exp(h_n t), \tag{33}$$

h_n being the poles of $\mathbf{Q}(z,h)$, i.e., the roots of $D(\Lambda,h) = 0$, and

$$D_h(\Lambda,h_n) = \left.\frac{dD(\Lambda,h)}{dh}\right|_{h=h_n}. \tag{34}$$

For each value of the root index n, these roots are either of the form $h_n = \gamma_n \pm i\omega_n$ if C is smaller than a critical value C_n^* (C_n^* depending on Λ and n) or of the form $h_n = \gamma_{n_1}, \gamma_{n_2}$, where both γ_{n_1} and γ_{n_2} are real numbers, if $C > C_n^*$ (γ_{n_1} and γ_{n_2} are both negative real numbers provided that $\Lambda < \pi$; if not, one of the corresponding to $n = 1$ becomes positive).

Observe that Eq. (24) becomes singular when $hC = -4$ [in this case the coefficient of the highest derivative term in Eq. (24) vanishes]. Such a particular case would require a singular perturbation analysis, which is out of the scope of this paper: This case only applies, for the values of Λ here considered, to very damped high-oscillation modes, which are not of importance in the liquid bridge dynamics, as explained in the following.

Concerning the liquid bridge interface, if $\mathbf{S}(z,h)$ is the Laplace transform of $s(z,t)$, from Eq. (12) and initial condition (17) this results in

$$h\mathbf{S}(z,h) = -\mathbf{Q}_z(z,h) \tag{35}$$

so that, taking into account (31), $\mathbf{S}(z,h)$ becomes

$$\mathbf{S}(z,h) = -(1/h^3)[D_z(z,h)/D(\Lambda,h)]. \tag{36}$$

In this case, $h = 0$ is a simple pole of $\mathbf{S}(z,h)$ and the behavior of the interface for long time is given by

$$\lim_{t\to\infty} s(z,t) = \lim_{h\to 0} h\mathbf{S}(z,h) = 2[z - (\Lambda/\sin\Lambda)\sin z] \tag{37}$$

thence, the following expression for $s(z,t)$ is obtained

$$s(z,t) = 2\left(z - \frac{\Lambda}{\sin\Lambda}\sin z\right) - \sum_{n=1}^{\infty} \frac{D_z(z,h_n)}{h_n^3 D_h(\Lambda,h_n)} \exp(h_n t), \tag{38}$$

where

$$D_z(z,h) = \theta_1\theta_2(\sinh\theta_2\Lambda \sinh\theta_1 z - \sinh\theta_1\Lambda \sinh\theta_2 z). \tag{39}$$

Observe that, as h_n appear as pairs of conjugate values, expression (38) is real, and that, for $t\to\infty$, the value of the perturbation in the liquid bridge interface [Eq. (37)] coincides with the static linearized solution calculated by Meseguer,[14] which can be easily deduced from Eq. (14) assuming that the Bond number does not change with time.

Note that $D(z,h)$ is a symmetric function of z, $D(z,h) = D(-z,h)$, whereas $D_z(z,h)$ is antisymmetric, $D_z(z,h) = -D_z(-z,h)$. Therefore the liquid bridge deformation is antisymmetric with respect to the middle plane parallel to the supporting disk or, in other words, only odd oscillation modes will appear in the liquid bridge evolution, according to the kind of perturbation considered.

Before proceeding further, it would be convenient to introduce some comments on the values of the roots of the equation $D(z,h) = 0$. As already stated, corresponding to each oscillation mode, provided the viscosity parameter C is small enough, there are two conjugate roots $h_n = \gamma_n \pm i\omega_n$. The loci of the roots corresponding to the different oscillation modes for a typical liquid bridge configuration ($\Lambda = 2$) has been represented in Fig. 2. According to this plot, if $C = 0$ all the roots are imaginary ($\gamma_n = 0$) and, consequently, the liquid bridge response will be oscillatory, without damping, as one could expect from an inviscid movement, and all the odd modes will be present in the liquid bridge response.

As C increases, the value of γ_n becomes more and more negative, whereas the absolute value of ω_n decreases. That means that all oscillation modes are damped, their oscilla-

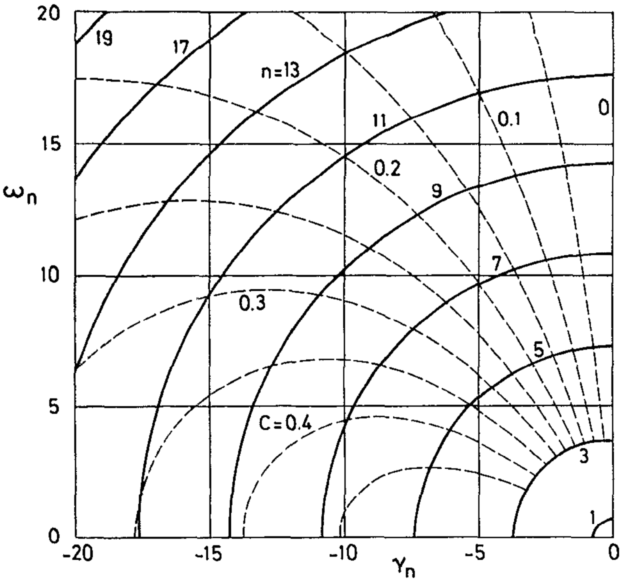

FIG. 2. Variation with the dimensionless viscosity parameter C of the real and the imaginary parts of the roots of $D(\Lambda,h)=0$, $h_n = \gamma_n \pm i\omega_n$. Solid lines correspond to the different oscillation modes identified by the subscript n. Numbers on dashed lines indicate the value of the viscosity parameter C. These results correspond to a cylindrical liquid bridge with slenderness $\Lambda = 2$.

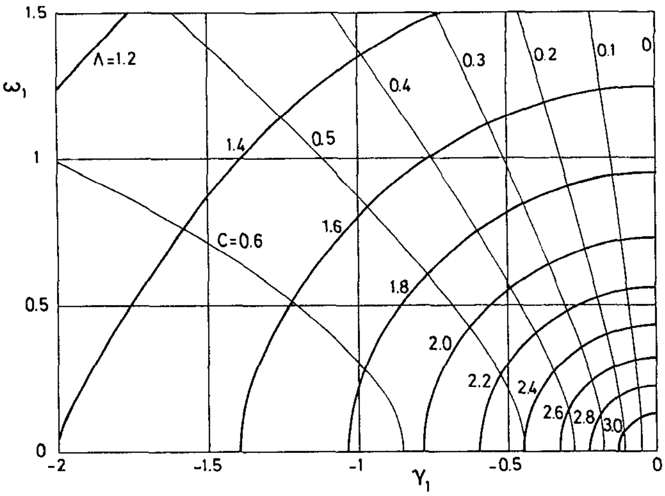

FIG. 3. Variation with the slenderness Λ and the dimensionless viscosity parameter C of the real and imaginary part of the root of $D(\Lambda,h)=0$ corresponding to the first oscillation mode $h_1 = \gamma_1 \pm i\omega_1$. Numbers on thick lines indicate the value of Λ, whereas numbers on thin lines indicate the value of C.

tion frequencies being smaller as C grows. For each oscillation mode, there is a critical value $C = C_n^*$ beyond which the associated movement is only damped, without oscillation ($\omega_n = 0$).

Concerning the importance of the different oscillation modes in the liquid bridge dynamics, it must be pointed out that, for the perturbation considered here, the only significant oscillation mode is the first one for two reasons. The first one concerns the amplitude of the different oscillation modes, much larger in the first oscillation mode than in the following ones. For instance, in the case $C = 0$, $\Lambda = 2$, the interface deformation at $z = \Lambda/2$ caused by the third oscillation mode is only about 1.5% of the interface deformation due to the first oscillation mode, this ratio being even smaller as the slenderness increases. Even more, when $C \neq 0$ (damped oscillations), the absolute value of γ_n increases as the index of the oscillation mode n grows, as shown in Fig. 2, and every oscillation mode is damped much quicker than the first one (oscillation modes that are different from the first one are only important for a short time when all of them appear to fulfill the initial conditions).

The dependence on Λ and C of the roots corresponding to the first oscillation mode, which is the only significant one for $C \neq 0$, is shown in Fig. 3. Note that, roughly speaking, the modulus of h_1 decreases as Λ increases so that the amplitude of the oscillation grows as Λ increases (Fig. 4). Note also that $h_1 \to 0$ (the amplitude of the interface deformation becomes infinite) as $\Lambda \to \pi$, which is the well-known stability limit for cylindrical liquid bridges. Beyond this stability limit, h_1 becomes a real positive number or, in other words, the liquid bridge breaks.

The dynamic response of a liquid bridge of slenderness $\Lambda = 2$ is shown in Figs. 5 and 6. In these plots, the variation with time of $q(0,t)$ and $s(\Lambda/2,t)$ has been represented. These particular values of the axial coordinate z have been chosen because, within the approximation presented here, the axial velocity is maximum at $z = 0$ (as explained above, to the first order of approximation, the perturbation in the axial momentum and the perturbation in axial velocity coincide), whereas the largest interface deformation appears for z close to $\pm \Lambda/2$.

Finally, the interface deformation of the liquid bridge interface for small values of time is shown in Fig. 7. As can be observed, if t is small enough, the liquid bridge interface is only perturbed in small regions (whose size depends on

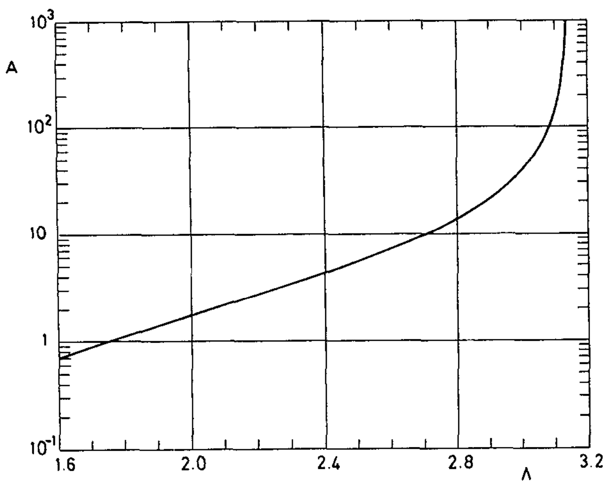

FIG. 4. Variation with the slenderness Λ of the amplitude A of the interface deformation at $z = \Lambda/2$ corresponding to the first oscillation mode $A = D_z(\Lambda/2,h_1)/[h_1^3 D_h(\Lambda,h_1)]$, see Eq. (38). These results correspond to a value of the viscosity parameter $C = 0$.

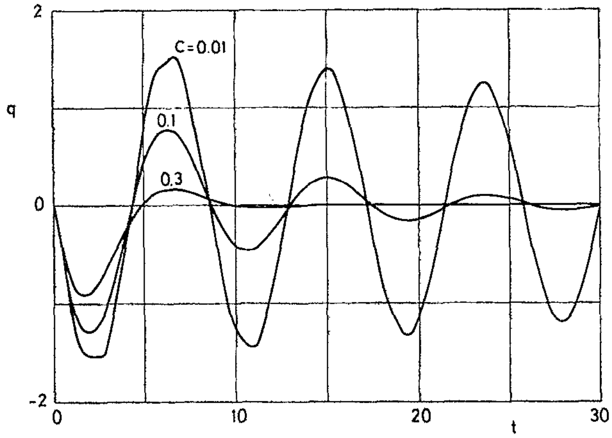

FIG. 5. Variation with dimensionless time t of the axial velocity $q(z,t)$ at $z = 0$ of a liquid bridge with slenderness $\Lambda = 2$. Numbers on the curves indicate the value of the viscosity parameter C.

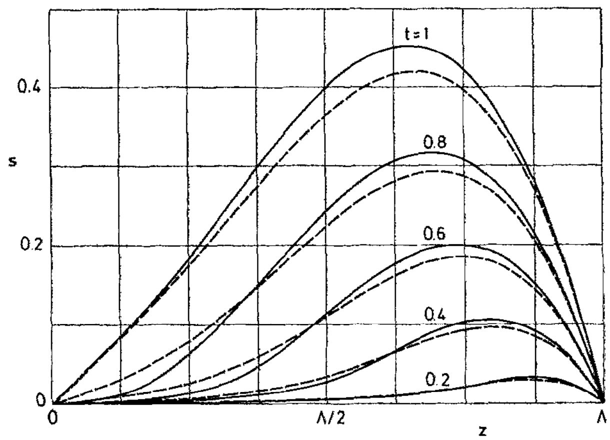

FIG. 7. Variation with dimensionless time t of the interface deformation $s(z,t)$ of a liquid bridge with $\Lambda = 2$. Line types indicate the value of the viscosity parameter: $C = 0$ (solid lines) and $C = 0.1$ (dashed lines).

the value of C) close to the supporting disks, as t increases the disturbed region grows until the entire interface becomes perturbed.

IV. CONCLUSIONS

A simple linearized method to calculate the dynamic response of long liquid columns to a sudden change in the value of the axial microgravity level has been presented. This method is only valid for cylindrical liquid bridges with not-too-small slenderness (the Cosserat model gives acceptable results up to $\Lambda \approx 1$) but it allows us to analyze the influence of viscosity on the liquid bridge dynamics.

Obviously, once the liquid bridge response to the step function has been calculated, the response to any time variation of the microgravity level can be easily calculated by using the well-known Duhamel's formula, which could be of great interest in the analysis of other g-jitter problems.

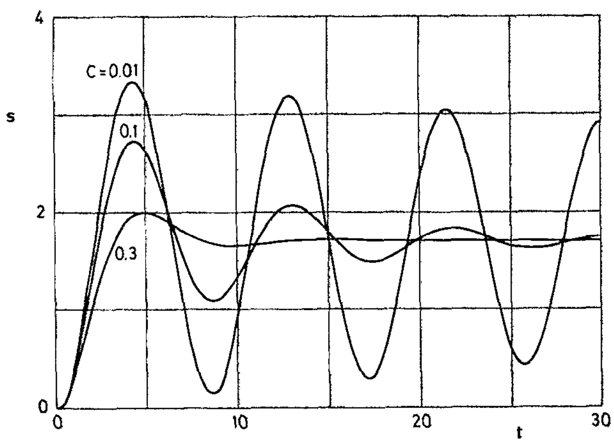

FIG. 6. Variation with dimensionless time t of the interface deformation $s(z,t)$ at $z = \Lambda/2$ of a liquid bridge with slenderness $\Lambda = 2$. Numbers on the curves indicate the value of the viscosity parameter C.

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