Computing Hessenberg Matrix associated to self-similar measures

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Happy sixties Guillermo!

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Let μ be a **positive measure in $\mathbb{C}$ with compact support Ω**.

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2. Let D be the infinite **upper Hessenberg matrix** of the multiplication by z operator in the basis of ONPS $\widehat{P}_n(z)$ in the closure of the polynomials.
3. The Hessenberg matrix D is the natural generalization to the hermitian case of Jacobi matrix.

Self-similar Measures

An Iterated Functions System (IFS) (M. Barnsley 1988) is a family of contractive maps $\{\varphi_s\}_{s=1}^{k}$ on a complete metric space.
In all this work, assume that φ_s $(s = 1, \ldots, k)$ are contractive similarities ($|\varphi(x) - \varphi(y)| = r|x - y|, 0 \leq r < 1,$ for all x, y). The family $\{\varphi_s\}_{s=1}^{k}$ then, will be called an **Iterated Functions System of Similarities (IFSS)**.

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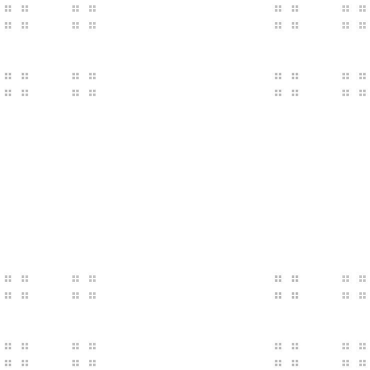
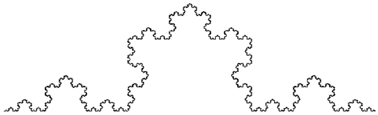
Given an IFSS $\{\varphi_s\}_{s=1}^{k}$ on a complete metric space, there exists a unique compactum K (**self-similar set**) satisfying

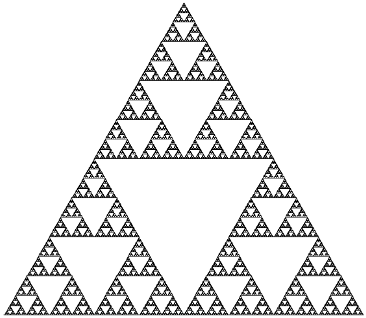
$$K = \bigcup_{s=1}^{k} \varphi_s(K).$$

Self-similar Measures

Examples of self similar sets







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We call μ the **self-similar measure** associated to the IFSS with probabilities $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_k; p_1, p_2, \ldots p_k\}$.

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The support of μ is the self-similar set K and satisfies (Hutchinson, 1981,Mandelbrot, 1977)

$$\mu = \sum_{s=1}^{k} p_s \mu \varphi_s^{-1}, \quad \int_{\mathrm{Supp}(\mu)} f d\mu = \sum_{s=1}^{k} p_s \int_{\mathrm{Supp}(\mu)} f \circ \varphi_s d\mu,$$

for any continuous function f on K.

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Then,

$$M_\varphi = A_\varphi^H M A_\varphi$$

where A_φ^H denotes the conjugated transposed matrix of A_φ given by

$$A_\varphi = \begin{pmatrix} \binom{0}{0}\alpha^0\beta^0 & \binom{1}{0}\alpha^0\beta^1 & \binom{2}{0}\alpha^0\beta^2 & \binom{3}{0}\alpha^0\beta^3 & \dots \\ 0 & \binom{1}{1}\alpha^1\beta^0 & \binom{2}{1}\alpha^1\beta^1 & \binom{3}{1}\alpha^1\beta^2 & \dots \\ 0 & 0 & \binom{2}{2}\alpha^2\beta^0 & \binom{3}{2}\alpha^2\beta^1 & \dots \\ 0 & 0 & 0 & \binom{3}{3}\alpha^3\beta^0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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Then, the sections the moment matrix M of μ satisfy the following matricial relation

$$M = \sum_{s=1}^{k} p_s A_{\varphi_s}^{H} M A_{\varphi_s}$$

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Then the set $f(K)$ and the measure $\mu_\Phi \circ f^{-1}$ are self-similar for the IFSS

$$f\Phi f^{-1} = \{f \circ \varphi_1 \circ f^{-1}, f \circ \varphi_2 \circ f^{-1}, \ldots, f \circ \varphi_k \circ f^{-1}; p_1, p_2, \ldots p_k\},$$

and

$$\mu_{f\Phi f^{-1}} = \mu_\Phi \circ f^{-1}.$$

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When the measure is supported in the unit ball, the moments are bounded. In any other case they are not bounded.

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If $\text{Supp}(\mu) = K \not\subset B_1(0)$, there exists a contractive map $f(z) = \alpha z$ such that $f(K) \subset B_1(0)$. Then,

$$\mathcal{M}_f = \{M | A_f^H M A_f \in \mathcal{M}_1\} \text{ with } ||M||_f = ||A_f^H M A_f||_{\sup},$$

is a complete metric space.

Fixed point theorem for moment matrix of self-similar measures

Theorem

Let $\Phi = \{\varphi_s; p_s\}_{s=1}^{k}$ be an IFSS with probabilities. Let K_Φ and μ_Φ be the self-similar set and measure, respectively. Let $f(z) = \alpha z$ be a contractive central dilation such that $f(K) \in B_1(0)$. Let $\mathcal{T}_{f\Phi f^{-1}} : (\mathcal{M}_f, ||\cdot||_f) \to (\mathcal{M}_f, ||\cdot||_f)$ be the transformation defined as

$$\mathcal{T}_{f\Phi f^{-1}}(M) = \sum_{s=1}^{k} p_s A^H_{f\varphi_s f^{-1}} M A_{f\varphi_s f^{-1}}.$$

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Then $\mathcal{T}_{f\Phi f^{-1}}$ is a contractive map with the moment matrix of the self-similar measure μ_Φ as unique fixed point.
Moreover, the ratio of this contractive map is

$$r = \sup\{|\alpha_s|, s = 1, 2, \dots k\}$$

Hessenberg Matrix. Cholesky Factoritation

Then we have the following algorithm

$$\begin{array}{ccccccccccc}
\nu & \longrightarrow & \mathcal{T}(\nu) & \longrightarrow & \mathcal{T}^2(\nu) & \longrightarrow & \cdots & \mathcal{T}^n(\nu) & \longrightarrow & \mu \\
\updownarrow & & \updownarrow & & \updownarrow & & & \updownarrow & & \updownarrow \\
M_\nu & \longrightarrow & \mathcal{T}_\Phi(M_\nu) & \longrightarrow & \mathcal{T}_\Phi^2(M_\nu) & \longrightarrow & \cdots & \mathcal{T}_\Phi^n(M_\nu) & \longrightarrow & M_\mu
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Since M and D are related (even for every PDH matrix M) by the formula

$$D = T^H S_R T^{-H}$$

where $M = TT^H$ is the **Cholesky factorization** and S_R is the shift-right matrix; we can approximate the n-section of D_μ

$$M_{\mu,n} \;\to\; M_{\mu,n} = T_n T_n^H \;\to\; D_{\mu,n} = T_n^{-1} M'_{\mu,n} T_n^{-H}$$

Hessenberg Matrix associated to a sum of measures

From now on, we use the following notation.

1. μ sum of measures, i.e., $d\mu = \sum_{i=1}^{m} p_i d\mu_i$, where $\sum_{i=1}^{m} p_i = 1$.
2. every measure μ_i has compact support on the complex plane.
3. $D = (d_{ij})_{i,j=1}^{\infty}$ the Hessenberg matrix associated to μ.
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Remark

First note that the matrices $D^{(i)}$ are bounded in ℓ^2 because the support of every μ_i is compact; second, remark that every matrix defines a subnormal operator in ℓ^2 (Atzmon, 1975, Torrano-Guadalupe, 1993, and Tomeo, 2003), due to the fact that the matrix of the inner produtc is a moment matrix. These two properties allow us to extend the spectral Mantica's techniques (2000).

Large recurrence formula

$D = (d_{jk})_{j,k=1}^{\infty}$ upper Hessenberg matrix. The ONPS satisfy

$$z\widehat{P}_{n-1}(z) = \sum_{k=1}^{n+1} d_{k,n}\widehat{P}_{k-1}(z), \quad n > 1.$$

with $\widehat{P}_1(z) = 0$ and $\widehat{P}_1(z) = 1$ when $c_{00} = 1$. Then

$$d_{n+1,n}\widehat{P}_n(z) = (z - d_{nn})\widehat{P}_{n-1}(z) - \sum_{k=1}^{n-1} d_{k,n}\widehat{P}_{k-1}(z), \qquad n > 1.$$

with $d_{2,1}\widehat{P}_1(z) = (z - d_{11})\widehat{P}_0(z)$, for $n = 1$.

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For D subnormal, we can write

$$d_{n+1,n}\widehat{P}_n(D) = (D - d_{nn}I)\widehat{P}_{n-1}(D) - \sum_{k=1}^{n-1} d_{k,n}\widehat{P}_{k-1}(D), \qquad n > 1$$

Hessenberg Matrix associated to a sum of measures

Theorem (EST 2006, NTCAT06-ICM)

Let μ, $\{\widehat{P}_n\}_{n=1}^{\infty}$, $D = (d_{jk})_{j,k=1}^{\infty}$ *and* $\{D^{(i)}\}$ *be as above.*

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$$d_{k,n} = \sum_{i=1}^{m} p_i \langle D^{(i)} v_{n-1}^{(i)}, v_{k-1}^{(i)} \rangle, \; i = 1, \ldots, m, \; k = 1, \ldots, n \qquad (1)$$

$$w_n^{(i)} = \left[D^{(i)} - d_{nn} I \right] v_{n-1}^{(i)} - \sum_{k=1}^{n-1} d_{k,n} v_{k-1}^{(i)}, \; i = 1, \ldots, m \qquad (2)$$

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When $n = 1$ *we take* $w_1^{(i)} = \left[D^{(i)} - d_{11} I \right] v_0^{(i)}$, $d_{11} = \sum_{i=1}^{m} p_i d_{11}^{(i)}$

$$d_{n+1,n} = \sqrt{\sum_{i=1}^{m} p_i \langle w_n^{(i)}, w_n^{(i)} \rangle}, \qquad (3)$$

$$v_n^{(i)} = \frac{w_n^{(i)}}{d_{n+1,n}}, \qquad v_0^{(i)} = e_0 \qquad i = 1, \ldots, m. \qquad (4)$$

Recurrent algorithm

We have $\{v_0^{(i)}, v_1^{(i)}, \dots, v_{n-1}^{(i)}\}_{i=1}^{m}, D_n = \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ d_{21} & d_{22} & \dots & d_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & d_{nn} \end{pmatrix}$

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$$\boxed{w_n^{(i)}} = \left[D^{(i)} - d_{nn}I\right] v_{n-1}^{(i)} - \sum_{k=1}^{n-1} d_{k,n} v_{k-1}^{(i)} \quad (2)$$

$\downarrow$

$$\boxed{d_{n+1,n}} = \sqrt{\sum_{i=1}^m p_i \langle w_n^{(i)}, w_n^{(i)} \rangle} \quad (3)$$

$\downarrow$

$$\boxed{v_n^{(i)}} = \frac{w_n^{(i)}}{d_{n+1,n}} \quad (4)$$

$\boxed{v_0^{(i)}, v_1^{(i)}, \dots, v_{n-1}^{(i)}}$ $\swarrow$ $\searrow$

$$\boxed{d_{k,n+1}} = \sum_{i=1}^m p_i \langle D^{(i)} v_{n-1}^{(i)}, v_{k-1}^{(i)} \rangle \quad (1)$$

Hessenberg Matrix associated to a sum of measures

The theorem gains in interest if we realize that it can be written in a matricial way.

Corollary

Let $V^{(i)}$ denote the upper triangular matrix with the vectors $v_0^{(i)}$, $v_1^{(i)}$, $v_2^{(i)}$, ...,of ℓ^2, as columns (i.e., $V^{(i)} = (v_0^{(i)}, v_1^{(i)}, v_2^{(i)}, \ldots)$). Then, we have

$$D = \sum_{i=1}^{m} p_i \, [V^{(i)}]^H D^{(i)} V^{(i)}.$$

Hessemberg matrix associated to a self-similar measure

We use the following result of E. Torrano (1987) to apply the above result to self-similar measures.

1. Let D be the Hessenberg matrix associated to a measure μ.
2. Let $\varphi(z) = \alpha z + \beta$ be a similarity, where $\alpha, \beta \in \mathbb{C}$.
3. Let μ_φ be the transformation of this measure by φ.
4. Let D^* be the Hessenberg matrix associated to μ_φ.

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then we have

$$D^* = \alpha\, U^H D U + \beta I,$$

where $U = \left(\delta_{jk} e^{(k-1)\theta i}\right)_{j,k=1}^{\infty}$, with $\alpha = |\alpha| e^{\theta i}$.

Hessemberg matrix associated to a self-similar measure

Corollary

Let $\Phi = \{\varphi_i(z) = \alpha_i z + \beta_i; p_i\}$ *be an IFSS with probabilities. Let* μ *be the corresponding self-similar measure.*

Hessemberg matrix associated to a self-similar measure

Corollary

Let $\Phi = \{\varphi_i(z) = \alpha_i z + \beta_i; p_i\}$ *be an IFSS with probabilities. Let* μ *be the corresponding self-similar measure. Then, the Hessenberg matrix* D *associated to the self-similar measure* μ *satisfies the following recurrent equation*

$$D = \sum_{i=1}^{m} p_i \, [V^{(i)}]^H \left[\alpha_i [U^{(i)}]^H D U^{(i)} + \beta_i I\right] V^{(i)},$$

where $U = \left(\delta_{jk} e^{(k-1)\theta i}\right)_{j,k=1}^{\infty}$, *with* $\alpha = |\alpha| e^{\theta i}$.

Convergence to Hessemberg matrix

Then we have the following algorithm

$$
\begin{array}{ccccccccc}
\nu & \longrightarrow & \mathcal{T}(\nu) & \longrightarrow & \mathcal{T}^2(\nu) & \cdots & \mathcal{T}^n(\nu) & \longrightarrow & \mu \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
M_\nu & \xrightarrow{meth\ 1} & \mathcal{T}_\Phi(M_\nu) & \xrightarrow{meth\ 1} & \mathcal{T}_\Phi^2(M_\nu) & \cdots & \mathcal{T}_\Phi^n(M_\nu) & \xrightarrow{meth\ 1} & M_\mu \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
D_0 & \xrightarrow{meth\ 2} & D_1 & \xrightarrow{meth\ 2} & D_2 & \cdots & D_n & \xrightarrow{meth\ 2} & D_\mu
\end{array}
$$

Examples

Example I. Let $\mathcal{L}$ be the normalized Lebesgue measure in the interval $[-1,1]$. This is a self-similar measure for the IFSS

$$\Phi = \{\varphi_1(x) = 1/2x - 1/2, \varphi_2(x) = 1/2x + 1/2; p_1 = p_2 = 1/2\}.$$

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Algorithm I.

$$\mathcal{T}_\Phi(M_\nu) = \sum_{i=1}^{2} \frac{1}{2} A_{\varphi_i}^H M_\nu A_{\varphi_i}.$$

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If we iterate the transformation $\mathcal{T}_\Phi$ 30 times starting with the sixth order identity matrix we obtain

$$\begin{pmatrix} 1.0 & 0.0 & 0.33333333 & 0.0 & 0.20000000 & 0.0 \\ 0.0 & 0.33333333 & 0.0 & 0.20000000 & 0.0 & 0.14285714 \\ 0.33333333 & 0.0 & 0.20000000 & 0.0 & 0.14285714 & 0.0 \\ 0.0 & 0.20000000 & 0.0 & 0.14285714 & 0.0 & 0.11111111 \\ 0.20000000 & 0.0 & 0.14285714 & 0.0 & 0.11111111 & 0.0 \\ 0.0 & 0.14285714 & 0.0 & 0.11111111 & 0.0 & 0.09090909 \end{pmatrix}.$$

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This matrix agrees with the 6th order moment matrix $M_{\mathcal{L}}$.

Examples

Then, the 5-section of Jacobi matrix $J_{\mathcal{L},5}$ is

$$\begin{pmatrix} 0.0 & 0.5773502693 & 0.0 & 0.0 & 0.0 \\ 0.5773502691 & 0.0 & 0.5163977795 & 0.0 & -0.7577722133 \cdot 10^{-9} \\ 0.0 & 0.5163977796 & 0.0 & 0.5070925551 & 0.0 \\ 0.3023715782 \cdot 10^{-9} & 0.0 & 0.5070925521 & 0.0 & 0.5039526136 \\ 0.0 & -0.2639315569 \cdot 10^{-8} & 0.0 & 0.5039526419 & 0.0 \end{pmatrix}$$

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Algorithm II. $D = \sum_{i=1}^{m} p_i \, [V^{(i)}]^H \left[\alpha_i [U^{(i)}]^H D U^{(i)} + \beta_i I\right] V^{(i)}$

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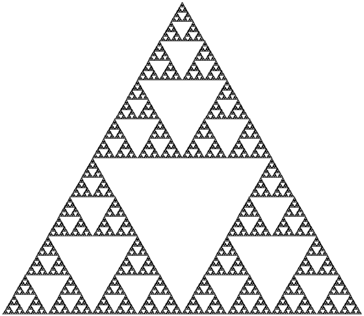
$$D_5^{30} = \begin{pmatrix} 0.0 & 0.5773502692 & 0.0 & -0.2133333332 \cdot 10^{-9} & 0.0 \\ 0.5773502691 & 0.0 & 0.5163977796 & 0.0 & -0.1 \cdot 10^{-9} \\ 0.0 & 0.5163977796 & 0.0 & 0.5070925526 & 0.0 \\ 0.0 & 0.0 & 0.5070925529 & 0.0 & 0.5039526304 \\ 0.0 & 0.0 & 0.0 & 0.5039526307 & 0.0 \end{pmatrix}$$

Examples

Example II. Let T be the Sierpinski triangle with basis on the $[-1, 1]$ interval.

Consider the uniform measure μ on T, i.e., the $\frac{log3}{log2}$-dimensional Hausdorff measure on T.



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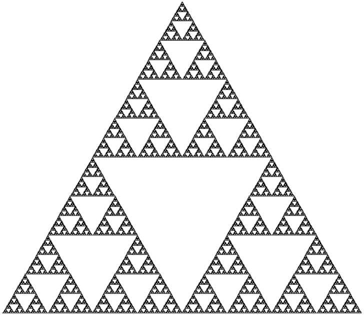

This is a self-similar measure for the IFSS given by

$$\Phi = \left\{ \varphi_1(z) = \frac{1}{2z} - \frac{1}{2}, \varphi_2(z) = \frac{1}{2z} + \frac{1}{2}, \varphi_3(z) = \frac{1}{2z} + \frac{1\sqrt{3}}{2i}; p_i = \frac{1}{3} \right\}$$

Examples

Algorithm I. Applying $\mathcal{T}_\Phi$ 30 times starting with the identity matrix we obtain an approximation of the 4-section of the Hessenberg matrix of the measure μ:

$$\begin{pmatrix} 0+0.5773502693i & 0.3\cdot 10^{-9}+0i & 0-0.4182428890i & -0.2457739408\cdot 10^{-8}+0i \\ 0.6666666673+0.0i & 0+0.5773502691i & 0.1267731382\cdot 10^{-8}+0i & 0-0.3487499915i \\ 0+0i & 0.7888106373+0i & 0+0.5773502706i & 0.1292460659\cdot 10^{-8}+0i \\ -0.406877\cdot 10^{-9}+0i & 0+0.279363\cdot 10^{-9}i & 0.7737179471+0i & 0+0.5773502588i \end{pmatrix}$$

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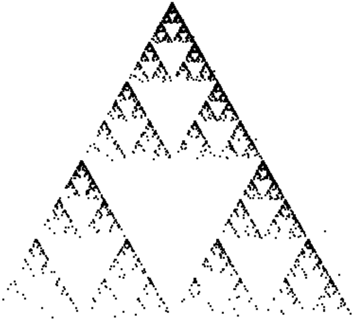
Algorithm II. With only seven iterations, we have

$$\begin{pmatrix} 0+0.572839i & -0.410^{-9}+0i & 0-0.418197i & -0.548635\cdot 10^{-10}-0.635737\cdot 10^{-20}i \\ 0.666692 & 0+0.572839i & -0.110^{-9}-0.380415\cdot 10^{-20}i & 0.106810\cdot 10^{-19}-0.348729i \\ 0 & 0.788866 & -0.108689\cdot 10^{-19}+0.572839i & -0.1610^{-9}-0.521858\cdot 10^{-19}i \\ 0 & 0 & 0.773830-0.258454\cdot 10^{-21}i & -0.797017\cdot 10^{-19}+0.572839i \end{pmatrix}$$

Examples

Example III. Let T be the Sierpinski triangle as above. Consider the invariant for the same IFSS with probabilities $p_1 = \frac{1}{10}, p_2 = \frac{1}{5}, p_3 = \frac{1}{7}$.



Examples

Algorithm I. Applying $\mathcal{T}_\Phi$ 7 times starting with the identity matrix we obtain an approximation of the 4-section of the Hessenberg matrix of the measure μ:

$$\begin{pmatrix} 0.0992 + 1.2029i & -0.2046 - 0.1459i & -0.1799 \cdot 10^{-4} - 0.3176i & -0.0123 + 0.0555i \\ 0.5538 + 0.1359 \cdot 10^{-9} i & 0.1439 + 0.8415i & 0.0208 - 0.0718i & -0.0396 - 0.3027i \\ 0.5688 \cdot 10^{-9} + 1.7342 \cdot 10^{-21} i & 0.6848 + 0.5367 \cdot 10^{-9} i & 0.0390 + 0.7027i & 0.0117 - 0.0461i \\ 0.5398 \cdot 10^{-8} + 0.8097 \cdot 10^{-9} i & 0.7127 \cdot 10^{-8} - 0.2649 \cdot 10^{-9} i & 0.7116 - 0.2392 \cdot 10^{-9} i & 0.07365 + 0.6745i \end{pmatrix}$$

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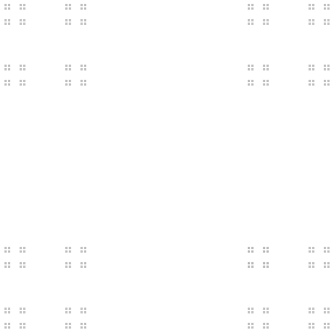
Algorithm II. With seven iterations, we have

$$\begin{pmatrix} 0.099218 + 1.202963i & -0.204629 - 0.145941i & -0.0000179 - 0.317680i & -0.012314 + 0.055542i \\ 0.5538131313 & 0.143933 + 0.841541i & 0.020889 - 0.0718614i & -0.039695 - 0.302772i \\ 0 & 0.684812 + 2.05958 \cdot 10^{-12}i & 0.0390029 + 0.702786i & 0.011747 - 0.046155i \\ 0 & 0 & 0.711680 + 1.54964 \cdot 10^{-12}i & 0.0736565 + 0.674541i \end{pmatrix}$$

Examples

Example IV. Let C be the plane Cantor set.



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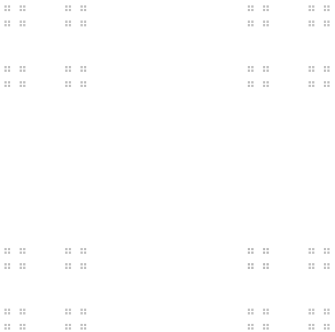

Consider the uniform measure μ on this set.
This measure is self-similar for de following IFSS

$$\Phi = \left\{ \varphi_1(z) = \frac{1}{4}z + \frac{1+i}{2}z, \right.$$
$$\varphi_2(z) = \frac{1}{4}z + \frac{1-i}{2}z,$$
$$\varphi_3(z) = \frac{1}{4}z + \frac{-1+i}{2}z,$$
$$\left. \varphi_4(z) = \frac{1}{4}z + \frac{-1-i}{2}z; p_i = \frac{1}{4} \right\}$$

Examples

Algorithm I. Applying $\mathcal{T}_\Phi$ 10 times starting with the identity matrix we obtain an approximation of the 5-section of the Hessenberg matrix of the measure μ:

$$\begin{pmatrix} 0 & 0 & 0 & -0.5534617900 & 0 \\ 0.7302967432 & 0 & 0 & 0 & -0.1728136409 \\ 0 & 0.7720611578 & 0 & 0 & 0 \\ 0 & 0 & 0.8042685429 & 0 & 0 \\ 0 & 0 & 0 & 0.6168489579 & 0 \end{pmatrix}$$

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Algorithm II. With only seven iterations, we have

$$\begin{pmatrix} 0+0i & 0+0i & 0+0i & -0.5534617832+0i & 0+0i \\ 0.7302967446 & 0+0i & 0+0i & 0+0i & -0.1728136428+0i \\ 0 & 0.7720611608 & 0+0i & 0+0i & -4.0\cdot 10^{-11}+0i \\ 0 & 0 & 0.8042685477 & 0+0i & 0+0i \\ 0 & 0 & 0 & 0.6168489731 & 0+0i \end{pmatrix}$$