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Adding local rotational degrees of freedom to ANC beams

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Abstract

This work shows a simple finite element formulation that enables to impose concentrated moments and rotations to ANC beams which are finite elements that lack rotational degrees of freedom. The idea is based on an specific constraint that expresses in a simple form the relation between the deformation of the beam and the rotation of any of its sections. By controlling this sectional rotation, moments and angles can be easily imposed on any model.

1 Introduction

Finite element beam discretizations based on the Absolute Nodal Coordinate (ANC) Formulation have been recently proposed by Shabana and coworkers [1, 2]. Their greatest advantage over conventional beam models is that these new finite elements avoid the use of rotational degrees of freedom by using an elaborated set of nodal parameters. As a result, their implementation is straightforward, the mass matrix is constant, and they can employ arbitrarily complex material constitutive laws. Moreover, since no special nodal rotation updates are required, ANC beams can be incorporated into existing finite element codes with relative ease. We refer to Romero [3] for a recent comprehensive review.

The absence of rotational degrees of freedom is a great advantage, but it carries some undesirable consequences. From the point of view of nonlinear structural engineering, maybe the most important one is that neither concentrated moments nor rotations can be directly imposed to an ANC beam. These boundary conditions are extremely common in engineering analysis, and unless a simple way to impose them is devised, the use of ANC beams can not be widespread.

In this work we will show how to incorporate rotational degrees of freedom at those points where the boundary conditions are required, and how to link these degrees of freedom to those of the ANC section. From the mathematical standpoint this will amount to imposing that the "rotational part" of the section deformation equals the imposed rotation. How to clearly identify this "rotational part" is the goal of Section 2. From the numerical point of view, we will accomplish this goal by devising a discrete finite element that will transfer the rotational degrees of freedom to the section. To do so, the element must manage to convert the virtual work in the rotational degrees of freedom to virtual work in the ANC node.

2 ANC beams

Beams based on the absolute nodal coordinate formulation (ANC) were originally introduced in Shabana and Yacoub [1, 2] and have been analyzed an improved by Shabana and co-workers (see for example, [4, 5]). The main novelty of the formulation is the deformation parametrization which, for each element e of the beam is of the form:

$$\boldsymbol{\varphi}_e(x,y,z,t) = \boldsymbol{S}_e(x,y,z) \begin{Bmatrix} \boldsymbol{a}_e(t) \\ \boldsymbol{b}_e(t) \\ \boldsymbol{c}_e(t) \end{Bmatrix} . \tag{1}$$

The matrix $\boldsymbol{S}_e$ contains the element interpolation functions and must be of the form:

$$\boldsymbol{S}_e(x,y,z) = \begin{bmatrix} \boldsymbol{J}_N(x,y,z) & \boldsymbol{0}_N & \boldsymbol{0}_N \\ \boldsymbol{0}_N & \boldsymbol{J}_N(x,y,z) & \boldsymbol{0}_N \\ \boldsymbol{0}_N & \boldsymbol{0}_N & \boldsymbol{J}_N(x,y,z) \end{bmatrix} , \tag{2}$$

where $\boldsymbol{0}_N$ is the N-dimensional zero row vector and $\boldsymbol{J}_N$ is an interpolation row vector of the form:

$$\boldsymbol{J}_N(x,y,z) = \langle 1,\ x,\ y,\ z,\ xy,\ xz,\ x^2,\ x^3\ \ldots x^{N-5} \rangle . \tag{3}$$

The constant N determines the degree of polynomial interpolation in the x direction and must be greater or equal to 7. For the rest of the article, as in the original work of Shabana and Yacoub [1, 2], we assume that $N = 8$, effectively setting the order of interpolation in the x direction to 3. In view of equation 1, the number of degrees of freedom per element is $3 \times N$, which for our choice equals 24.

The vector $\langle \boldsymbol{a}_e(t),\ \boldsymbol{b}_e(t),\ \boldsymbol{c}_e(t) \rangle^T$ contains the coefficients of the element interpolation functions, and can be associated with nodal degrees of freedom in the following way. Let $\boldsymbol{H}_e^\alpha$, for $\alpha = 1, 2$, be the set of 12 variables associated with the local node α belonging to element e:

$$\boldsymbol{H}_e^\alpha(t) = \left\langle \boldsymbol{\varphi}(\boldsymbol{X}_e^\alpha, t),\ \frac{\partial \boldsymbol{\varphi}(\boldsymbol{X}_e^\alpha, t)}{\partial x},\ \frac{\partial \boldsymbol{\varphi}(\boldsymbol{X}_e^\alpha, t)}{\partial y},\ \frac{\partial \boldsymbol{\varphi}(\boldsymbol{X}_e^\alpha, t)}{\partial z} \right\rangle^T . \tag{4}$$

The first vector in $\boldsymbol{H}_e^\alpha$ is simply the nodal position of the local α-th node at time t, and the remaining vectors the three tangent vectors to the local coordinate curves x, y, and z at the deformed configuration.

It can be verified that for the interpolation chosen above, the deformation $\boldsymbol{\varphi}_e$ can also be written as:

$$\boldsymbol{\varphi}_e(x,y,z,t) = \boldsymbol{D}_e(x,y,z) \begin{Bmatrix} \boldsymbol{H}_e^1(t) \\ \boldsymbol{H}_e^2(t) \end{Bmatrix} , \tag{5}$$

for a new interpolation matrix $\boldsymbol{D}_e$ related to $\boldsymbol{S}_e$. Although completely equivalent to expression (1), this new interpolation has the advantage that the set of variables $\langle \boldsymbol{H}_e^1,\ \boldsymbol{H}_e^2 \rangle^T$ has a more clear geometrical meaning, which has been further explored in Sopanen [5]

Let us define $\boldsymbol{G} = [\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3]$, the three tangent vectors of the deformed beam. This tensor is invertible and, by the polar decomposition theorem, can be expressed as the product $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$, where $\boldsymbol{R}$ is a rotation tensor, and $\boldsymbol{U}$ is a symmetric, positive definite tensor. Since the goal of this work is to control the rotational part of the beam deformation, this is equivalent to imposing the value of $\boldsymbol{G}$ to be equal to a given rotation, leaving $\boldsymbol{U}$ unmodified. In other words, the geometrical interpretation of the beam degrees of freedom and the polar decomposition theorem shows that by imposing $\boldsymbol{R}$ to be as desired, the whole section of the beam will rotated as wanted.

To formulate such constraint mathematically, let $\boldsymbol{\theta}$ be a rotation vector and let $\exp[\boldsymbol{\theta}]$ be its associated rotation tensor. Since we would like to control the rotation part of the motion $\boldsymbol{R}$ by imposing the rotation vector $\boldsymbol{\theta}$ only, it would suffice to impose the constraint $\boldsymbol{R} = \exp[\boldsymbol{\theta}]$. The problem is, however, that there is no close form expression from the rotational part $\boldsymbol{R}$ of a given vector $\boldsymbol{G}$, which has to be obtained by means of an algorithm. The following result shows an equivalent statement which can be used explicitly to impose the desired constraint:

Theorem 2.1 *Let $\boldsymbol{F}$ be a second order tensor with positive determinant. Then there exists a unique rotation vector $\boldsymbol{\theta} \in \mathbb{R}^3$ such that*

$$\operatorname{skew}\Big[\exp[-\hat{\boldsymbol{\theta}}]\boldsymbol{F} \Big] = \boldsymbol{0} . \tag{6}$$

Furthermore, $\exp[\boldsymbol{\theta}]$ is equal to $\boldsymbol{R}$, the rotation that appears in the polar decomposition of $\boldsymbol{F}$.

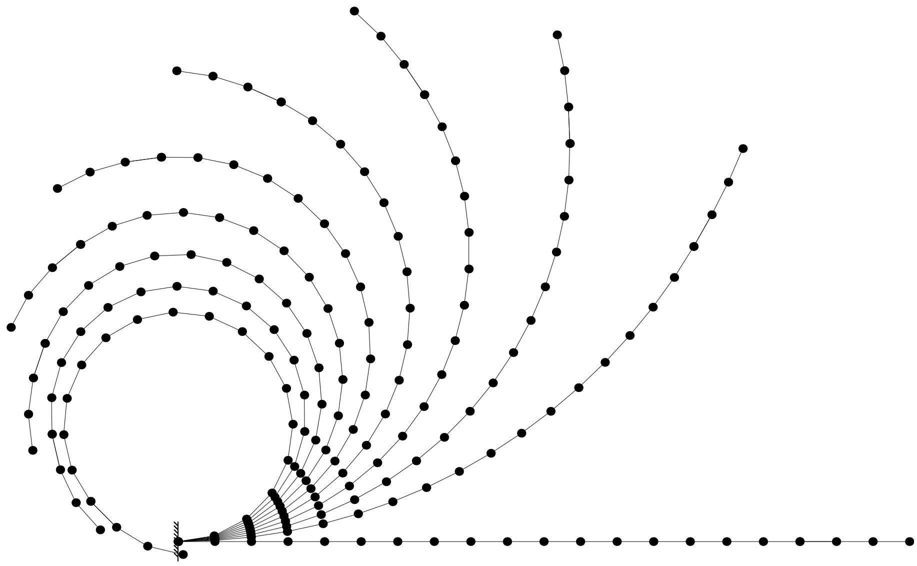

Figure 1: Deformation sequence of a cantilever beam under a concentrated moment at its tip.

In the previous equation we have employed the $\operatorname{skew}[]$ operator defined on second order tensors by $\operatorname{skew}[\boldsymbol{A}] = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T)$. In other words, if we impose the constraint

$$\boldsymbol{g}(\boldsymbol{\theta}, \boldsymbol{G}) = \boldsymbol{0} \quad \text{with} \quad \boldsymbol{g}(\boldsymbol{\theta}, \boldsymbol{G}) = \operatorname{skew}\Big[\exp[-\hat{\boldsymbol{\theta}}(t)]\boldsymbol{G}(t)(\boldsymbol{G}^o)^{-1}\Big] \,, \tag{7}$$

then the rotation part of the section deformation $\boldsymbol{R}$ will be identical to the rotation $\exp[\boldsymbol{\theta}]$.

3 The finite element formulation

To implement the proposed constraint we assume that the beam equations result from the minimization of a potential energy $\Pi(\boldsymbol{\varphi})$. Then, using the classical penalty method, the constraint can be approximately imposed by minimizing the augmented potential energy

$$\Pi^\kappa(\boldsymbol{\varphi}, \boldsymbol{\theta}) = \Pi(\boldsymbol{\varphi}) + \Pi^g(\boldsymbol{\theta}, \boldsymbol{G}) \,, \qquad \Pi^g(\boldsymbol{\theta}, \boldsymbol{G}) = \frac{\kappa}{2}\, |\boldsymbol{g}(\boldsymbol{\theta}, \boldsymbol{G})|^2 \,, \tag{8}$$

where κ is a large positive number.

To obtain the equations of the internal forces resulting from this connecting element, the first variation of (8) should be calculated. The tangent stiffness is simply obtained by calculating the second variation of the same equation.

4 Numerical example

Imposing a simple torque on a ANC beam is a complex task as can be seen, for example, in Sopanen [5]. With the previous formulation, the task becomes trivial. To illustrate this, we use a the classical example of a cantilever beam of length $L = 1$, and bending stiffness $EI = 175000$. When a concentrated moment of modulus $M = 2\pi EI/L$ is applied at the beam's tip, it should bend 2π radians. In order to perform such simulation we simply add a new fictitious node at the beam's tip, a node that only has rotational degrees of freedom. After imposing the concentrated moment the beam bends as the figure 1. The errors at the tip are not due to the imposed moment, but rather related to the beam accuracy.

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