The Darboux transformation and the complex Toda lattice

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Abstract

It is well known that each solution of the Toda lattice can be represented by a tridiagonal matrix $J(t)$. Under certain restrictions, it is possible to obtain some new solution by using the Darboux transformation of $J(t) - CI$. Our goal is the extension of this fact, which is known for the real lattice, to high order complex Toda lattices as well as to the bi-infinite Toda lattice. In this latter case, we use the factorization LU for block-tridiagonal matrices.

1 The Toda lattice

We study the construction of some solutions $\{\widetilde{\alpha}_n(t)\,,\ \widetilde{\lambda}_n(t)\}\,,\ n \in \mathbb{Z}$, of the Toda complex lattice

$$\left.\begin{array}{rcl} \dot{\alpha}_n(t) &=& \lambda_{n+1}^2(t) - \lambda_n^2(t) \\ \dot{\lambda}_{n+1}(t) &=& \dfrac{\lambda_{n+1}(t)}{2}\left[\alpha_{n+1}(t) - \alpha_n(t)\right] \end{array}\right\}, \quad n \in \mathbb{S}, \tag{1}$$

from another given solution $\{\alpha_n(t)\,,\ \lambda_n(t)\}\,,\ n \in \mathbb{Z}$.

We consider:

1. the semi-infinite problem: $\mathbb{S} = \mathbb{N}$, $\lambda_1 = 0$,
2. the infinite problem: $\mathbb{S} = \mathbb{Z}$,

In [6] the semi-infinite complex problem was analyzed. In the real, infinite case, sufficient conditions for the existence of a new solution were given in [7].

The problem: obtain a similar result to the complex infinite Toda lattice.

2 The generalized Toda lattice

In a more general way, when $\mathbb{S} = \mathbb{N}$ we consider the generalized Toda lattice of order $p \in \mathbb{N}$ (see [1]),

$$\left.\begin{array}{rcl} \dot{J}_{nn}(t) &=& J_{n,n+1}(t)J^p_{n,n+1}(t) - J_{n-1,n}(t)J^p_{n-1,n}(t) \\ \dot{J}_{n,n+1}(t) &=& \frac{1}{2}J_{n,n+1}(t)\left[J^p_{n+1,n+1}(t) - J^p_{n,n}(t)\right] \end{array}\right\} \tag{2}$$

where we denote by $J_{i,j}(t)$ (respectively $J^p_{i,j}(t)$) the entry in the $(i+1)$-row and $(j+1)$-column of matrix $J(t)$ (respectively $(J(t))^p$),

$$J(t) = \begin{pmatrix} \alpha_1(t) & \lambda_2(t) & \\ \lambda_2(t) & \alpha_2(t) & \ddots \\ & \ddots & \ddots \end{pmatrix}, \quad t \in \mathbb{R}\,.$$

The generalized Toda lattice admits a Lax pair representation, i.e. a formulation in terms of the commutator of two operators,

$$\dot{J}(t) = [J(t), K(t)] = J(t)K(t) - K(t)J(t)\,, \text{ where}$$

$$K(t) = \frac{1}{2}\begin{pmatrix} 0 & -J^p_{01}(t) & \cdots & -J^p_{0p}(t) & 0 & \cdots \\ J^p_{01}(t) & 0 & -J^p_{12}(t) & \cdots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & & \\ J^p_{0p}(t) & & & & & \\ 0 & J^p_{1,p+1}(t) & \ddots & & & \\ & & & & & \\ \vdots & 0 & \ddots & & & \end{pmatrix}, \ t \in \mathbb{R}\,.$$

In [2, Th. 1.3], given a solution $J(t)$ of (2), for each $C \in \mathbb{C}$ verifying

$$\det(J_n(t) - C\mathrm{I}_n) \neq 0\,, \quad n \in \mathbb{N}\,, \tag{3}$$

we prove the existence of

$$\widetilde{J}(t) = \begin{pmatrix} \widetilde{\alpha}_1(t) & \widetilde{\lambda}_2(t) & \\ \widetilde{\lambda}_2(t) & \widetilde{\alpha}_2(t) & \ddots \\ & \ddots & \ddots \end{pmatrix}, \quad \Gamma(t) = \begin{pmatrix} 0 & \gamma_2(t) & & \\ \gamma_2(t) & 0 & \gamma_3(t) & \\ & \gamma_3(t) & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

verifying

$$\left.\begin{array}{ll} \lambda^2_{n+1}(t) = \gamma^2_{2n}(t)\gamma^2_{2n+1}(t)\,, & \alpha_n(t) = \gamma^2_{2n-1}(t) + \gamma^2_{2n}(t) + C \\ \widetilde{\lambda}^2_{n+1}(t) = \gamma^2_{2n+1}(t)\gamma^2_{2n+2}(t)\,, & \widetilde{\alpha}_n(t) = \gamma^2_{2n}(t) + \gamma^2_{2n+1}(t) + C \end{array}\right\}$$

such that $\widetilde{J}(t)$ is another solution of (2), and $\Gamma(t)$ is a solution of the Volterra lattice:

$$\dot{\Gamma}_{n-1,n}(t) = \frac{1}{2}\Gamma_{n-1,n}(t)\left[(\Gamma^2(t) + C\mathrm{I})^p_{nn} - (\Gamma^2(t) + C\mathrm{I})^p_{n-1,n-1}\right].$$

3 Relation between the generalized Toda lattice and some polynomials

The matrix $J(t)$ t defines the sequence of polynomials given by

$$\left.\begin{array}{r} P_n(t,z) = (z - \alpha_n(t))P_{n-1}(t,z) - \lambda^2_n(t)P_{n-2}(t,z),\ n \in \mathbb{N}\,, \\ P_{-1}(t,z) \equiv 0,\ P_0(t,z) \equiv 1\,. \end{array}\right\}$$

The main tools in the proof of [2, Th. 1.3]:

a. We have established the dynamic behavior of $P_n(t,z)$,

$$\dot{P}_n(t,z) = -\sum_{j=1}^{p} J^p_{n,n-j}(t)\lambda_{n-j+2}(t)\ldots\lambda_{n+1}(t)P_{n-j}(t,z)\,,$$

b. As was proposed in [6], we use the *kernel polynomials* (cf. [4])

$$Q_n^{(C)}(t,z) = \frac{P_{n+1}(t,z) - \frac{P_{n+1}(t,C)}{P_n(t,C)}P_n(t,z)}{z - C}.$$

where $C \in \mathbb{C}$ verifies (3). The sequence $Q_n^{(C)}(t,C)$ satisfies a three-term recurrence relation whose coefficients define the new generalized solution $\widetilde{J}(t) = \widetilde{J}(t,C)$

4 The new solutions and the Darboux transformation

If we define

$$J^{(1)}(t) := \begin{pmatrix} \alpha_1(t) & \lambda_2(t)^2 & & \\ 1 & \alpha_2(t) & \lambda_3(t)^2 & \\ & 1 & \alpha_3(t) & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

and $C \in \mathbb{C}$ verifies (3), then there exist

$$L(t) = \begin{pmatrix} \gamma_2^2(t) & & \\ 1 & \gamma_4^2(t) & \\ & \ddots & \ddots \end{pmatrix}, \quad U(t) = \begin{pmatrix} 1 & \gamma_3^2(t) & \\ & 1 & \gamma_5^2(t) \\ & & \ddots & \ddots \end{pmatrix}$$

such that $J^{(1)}(t) - C\mathrm{I} = L(t)U(t)$. The new solution is defined by the Darboux transformation of $J^{(1)}(t) - C\mathrm{I}$, this is,

$$\widetilde{J}^{(1)}(t) - C\mathrm{I} = U(t)L(t)\,,$$

being

$$\widetilde{J}^{(1)}(t) := \begin{pmatrix} \widetilde{\alpha}_1(t) & \widetilde{\lambda}_2(t)^2 & & \\ 1 & \widetilde{\alpha}_2(t) & \widetilde{\lambda}_3(t)^2 & \\ & 1 & \widetilde{\alpha}_3(t) & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

5 The infinite Toda lattice

Let us consider (1) with $\mathbb{S} = \mathbb{Z}$ and take the infinite matrix

$$J = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & \alpha_{-1}(t) & \lambda_0(t) & & \\ & \lambda_0(t) & \alpha_0(t) & \lambda_1(t) & \\ & & \lambda_1(t) & \alpha_1(t) & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

The infinite Toda lattice admits also a Lax pair representation. However, in this case it is not possible to use directly the sequences of polynomials associated to J.

Taking $\mathcal{R}_n := \begin{pmatrix} f_n \\ f_{-n+1} \end{pmatrix}$, $n \in \mathbb{N}$, it is possible to change the infinite recurrence relation

$$\lambda_{n+1}(t)f_{n-1}(t,z) + (\alpha_{n+1} - z)f_n(t,z) + \lambda_{n+2}(t)f_{n+1}(t,z) = 0\,, \quad n \in \mathbb{Z}\,,$$

to a semi-infinite recurrence relation,

$$E_n(t)\mathcal{R}_{n-1}(t,z) + (V_n(t) - zI_2)\,\mathcal{R}_n(t,z) + E_{n+1}(t)\mathcal{R}_{n+1}(t,z) = 0\,, \quad n \in \mathbb{N}\,,$$

where E_m, V_m, $m \in \mathbb{N}$, are 2×2-finite matrices. In this way, we can study the infinite case as a semi-infinite vectorial case. The vectors $\mathcal{R}_n$ are not polynomials, but we can prove

$$\mathcal{R}_n = (E_2 \cdots E_n)^{-1}C_n\mathcal{R}_1\,,$$

where the sequence $\{C_n\}$ of 2×2 matrices verifies

$$\left.\begin{array}{r} E_n^2C_{n-1} + (V_n(t) - zI_2)\,C_n + C_{n+1} = 0\ ,\ n \in \mathbb{N} \\ C_0 = O_2\ ,\ C_1 = I_2 \end{array}\right\}$$

i.e.,

$$C_n = \begin{pmatrix} c_{n1}(t,z) & c_{n2}(t,z) \\ c_{n3}(t,z) & c_{n4}(t,z) \end{pmatrix}$$

and for each $i = 1,2,3,4$, c_{ni} is a polynomial in z, $\deg c_{ni} \le n-1$.

Taking $I_{-1} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $W_n := I_{-1}V_n$, $n \in \mathbb{N}$, we can show

$$\left.\begin{array}{rcl} \dot{W}_n &=& E^2_{n+1} - E^2_n \\ \dot{E}_{n+1} &=& \frac{1}{2}E_{n+1}(W_{n+1} - W_n) \end{array}\right\}, \ n = 2,3,\ldots \tag{4}$$

This is, $\{W_n, E_n\}$ is a solution of a semi-infinite matricial Toda lattice, like (1).

6 The infinite Toda lattice and the Darboux transformation

We define

$$J^{(B)} := \begin{pmatrix} V_1 & E_2^2 & & \\ I_2 & V_2 & E_3^2 & \\ & I_2 & V_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

Let $C \in \mathbb{C}$ be such that

$$\det\left(J^{(B)}_{2n}(t) - CI_{2n}\right) \neq 0\,, \quad t \in \mathbb{R}\,,\ n \in \mathbb{N}\,.$$

Then, we know (see [5]) that there exist two blocked matrices

$$L^{(B)} := \begin{pmatrix} A_1 & & & \\ I_2 & A_2 & & \\ & I_2 & A_3 & \\ & & \ddots & \ddots \end{pmatrix}, \quad U^{(B)} := \begin{pmatrix} I_2 & \Gamma_1 & & \\ & I_2 & \Gamma_2 & \\ & & I_2 & \ddots \\ & & & \ddots \end{pmatrix}$$

such that $J^{(B)} - CI = L^{(B)}U^{(B)}$. We define the blocked Darboux transformation of $J^{(B)} - CI$ as

$$\widetilde{J}^{(B)} - CI := U^{(B)}L^{(B)} = \begin{pmatrix} \widetilde{V}_1 - CI_2 & \widetilde{E}_2^2 & & \\ I_2 & \widetilde{V}_2 - CI_2 & \widetilde{E}_3^2 & \\ & I_2 & \widetilde{V}_3 - CI_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

We are researching the two following questions:

1. Can we construct a vectorial solution of hte Toda lattice, like (4), from $\widetilde{J}^{(B)} - CI$?
2. Are the (scalar) entries of $\widetilde{J}^{(B)}$ a new solution of the Toda lattice (1)?

References

[1] A.I. Aptekarev, A. Branquinho, 2003, Padé approximants and complex high order Toda lattices, *J. Comput. Appl. Math.* **155**, 231–237

[2] D. Barrios Rolanía, A. Branquinho, Complex high order Toda lattices, *J. Diff. Eq. Appl.* 15(2) (2009), 197–213

[3] D. Barrios Rolanía, A. Branquinho, A. Foulquié Moreno, Dynamics and interpretation of some integrable systems via multiple orthogonal polynomials, *submited* (2009)

[4] Chihara, T. S., 1978, An Introduction to Orthogonal Polynomials (New York: Gordon and Breach Science Pub.)

[5] E. Isaacson, H. Bishop Keller, Analysis of Numerical Methods (New York: Courant Inst. of Math. Sci., John Wiley & Sons, Inc.)

[6] F. Peherstorfer, On Toda lattices and orthogonal polynomials, *J. Comput. Appl. Math.* 133 (2001) 519-534.

[7] F. Gesztesy, H. Holden, B. Simon, and Z. Zhao. *On the Toda and Kac-van Moerbeke systems*. Trans. Am. Math. Soc., 339(2) (1993) 849-868.