The complex high order Toda lattices and polynomials generated by three-term recurrence relations

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Introduction

In [1] was studied the construction of a solution $\{\widetilde{\alpha}_n(t)\,,\,\widetilde{\lambda}_n(t)\}\,,\ n\in\mathbb{Z}\,,$ of the Toda real lattice

$$\left.\begin{array}{rcl}\dot{\alpha}_n(t) & = & \lambda_{n+1}^2(t)-\lambda_n^2(t)\\ \dot{\lambda}_{n+1}(t) & = & \dfrac{\lambda_{n+1}(t)}{2}\left[\alpha_{n+1}(t)-\alpha_n(t)\right]\end{array}\right\}, \qquad n\in\mathbb{Z}, \tag{1}$$

from another given solution $\{\alpha_n(t)\,,\,\lambda_n(t)\}\,,\ n\in\mathbb{Z}$.

[1] F. Gesztesy, H. Holden, B. Simon, Z. Zhao, 1993, On the Toda and Kac-van Moerbeke systems, *Trans. Am. Math. Soc.* **339** (2), 849–868.

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from another given solution $\{\alpha_n(t)\,,\,\lambda_n(t)\}\,,\; n \in \mathbb{Z}$.

The connection between both solutions is the Volterra lattice

$$\dot{\gamma}_{n+1}(t) = \gamma_{n+1}(t)\,(\gamma_{n+2}(t) - \gamma_n(t))\,, \quad n \in \mathbb{Z}. \qquad (2)$$

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In both cases, the connection between the solutions of Toda and Volterra lattices is

$$\left.\begin{array}{rclcrcl} \lambda_{n+1}^2(t) & = & \gamma_{2n}^2(t)\gamma_{2n+1}^2(t) & , & \alpha_n(t) & = & \gamma_{2n-1}^2(t)+\gamma_{2n}^2(t) \\ \widetilde{\lambda}_{n+1}^2(t) & = & \gamma_{2n+1}^2(t)\gamma_{2n+2}^2(t) & , & \widetilde{\alpha}_n(t) & = & \gamma_{2n}^2(t)+\gamma_{2n+1}^2(t) \end{array}\right\}$$

(Bäcklund transformation)

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Introduction

Here, we generalize that results for generalized Toda lattices of order $p \in \mathbb{N}$, for all $n = 0, 1, \ldots,$

$$\left.\begin{array}{rcl} \dot{J}_{nn}(t) & = & J_{n,n+1}(t) J^{p}_{n,n+1}(t) - J_{n-1,n}(t) J^{p}_{n-1,n}(t) \\ \dot{J}_{n,n+1}(t) & = & \frac{1}{2} J_{n,n+1}(t) \left[J^{p}_{n+1,n+1}(t) - J^{p}_{n,n}(t) \right] \end{array} \right\}, \qquad (3)$$

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We denote by $J_{i,j}(t)$ (respectively $J^p_{i,j}(t)$) the entry in the $(i+1)$-row and $(j+1)$-column of matrix $J(t)$ (respectively $(J(t))^p$,

$$J(t) = \begin{pmatrix} \alpha_1(t) & \lambda_2(t) & \\ \lambda_2(t) & \alpha_2(t) & \ddots \\ & \ddots & \ddots \end{pmatrix}, \quad t \in \mathbb{R}.$$

In all the following we assume $\lambda_n(t) \neq 0\,,\ n \in \mathbb{N}\,,\ t \in \mathbb{R}$.

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In all the following we assume $\lambda_n(t) \neq 0$, $n \in \mathbb{N}$, $t \in \mathbb{R}$.

Definition

We say that $\{J(t)\}$, $t \in \mathbb{R}$, is a *generalized Toda solution* if (3) is verified.

Introduction

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$$\dot{\Gamma}_{n-1,n}(t) = \frac{1}{2}\Gamma_{n-1,n}(t)\left[(\Gamma^2(t) + C\mathrm{I})^p_{nn} - (\Gamma^2(t) + C\mathrm{I})^p_{n-1,n-1}\right], \quad (4)$$

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being $C \in \mathbb{C}$ and

$$\Gamma(t) = \begin{pmatrix} 0 & \gamma_2(t) & & \\ \gamma_2(t) & 0 & \gamma_3(t) & \\ & \gamma_3(t) & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad t \in \mathbb{R}. \quad (5)$$

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Definition

We say that $\{\Gamma(t)\}$, $t \in \mathbb{R}$, is a solution of the high order Volterra lattice, or a generalized Volterra solution, if we have (4).

Solutions generation

Theorem 1

Let $\{J(t)\}$, $t \in \mathbb{R}$, be a generalized Toda solution. Let $C \in \mathbb{C}$ be such that $\det(J_n(t) - C\mathrm{I}_n) \neq 0$ for each $n \in \mathbb{N}$ and $t \in \mathbb{R}$.

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$$\widetilde{J}(t) = \begin{pmatrix} \widetilde{\alpha}_1(t) & \widetilde{\lambda}_2(t) & & \\ \widetilde{\lambda}_2(t) & \widetilde{\alpha}_2(t) & \ddots & \\ & \ddots & \ddots & \end{pmatrix}, \ t \in \mathbb{R},$$ such that the following relations hold

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such that the following relations hold

$$\left.\begin{array}{ll} \lambda^2_{n+1}(t) = \gamma^2_{2n}(t)\gamma^2_{2n+1}(t)\,, & \alpha_n(t) = \gamma^2_{2n-1}(t) + \gamma^2_{2n}(t) + C \\ \widetilde{\lambda}^2_{n+1}(t) = \gamma^2_{2n+1}(t)\gamma^2_{2n+2}(t)\,, & \widetilde{\alpha}_n(t) = \gamma^2_{2n}(t) + \gamma^2_{2n+1}(t) + C \end{array}\right\}$$

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In the above conditions, the sequences $\{\widetilde{\lambda}_{n+1}(t)\}$, $\{\widetilde{\alpha}_n(t)\}$, $\{\gamma_n^2(t)\}$ are unique.

Solutions generation

The main tool in the proof of Theorem 1 is the sequence of polynomials $\{P_n(t, z)\}$, $n \in \mathbb{N}$, associated with the matrix $J(t)$.

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$$\left.\begin{array}{r} P_n(t,z) = (z-\alpha_n(t))P_{n-1}(t,z) - \lambda_n^2(t)P_{n-2}(t,z), \quad n \in \mathbb{N}\,, \\ P_{-1}(t,z) \equiv 0,\; P_0(t,z) \equiv 1\,. \end{array}\right\}$$

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Theorem 2

In the above conditions, $\{J(t)\}$, $t \in \mathbb{R}$, is a generalized Toda solution if and only if for all $t \in \mathbb{R}$

$$\dot{P}_n(t,z) = -\sum_{j=1}^{p} J_{n,n-j}^{p}(t)\lambda_{n-j+2}(t)\ldots\lambda_{n+1}(t)P_{n-j}(t,z)\,, \tag{6}$$

for each $n \in \mathbb{N}$ and each fixed $z \in \mathbb{C}$.

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$$A(t) = \frac{1}{2}\begin{pmatrix} 0 & -J^p_{01}(t) & \cdots & -J^p_{0p}(t) & 0 & \cdots \\ J^p_{01}(t) & 0 & -J^p_{12}(t) & \cdots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & & \\ J^p_{0p}(t) & & & & & \\ 0 & J^p_{1,p+1}(t) & \ddots & & & \\ \vdots & 0 & \ddots & & & \end{pmatrix}, \ t \in \mathbb{R}.$$

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$$\Rightarrow (J(t) - z\mathrm{I})\,\mathcal{P}(t,z) = 0\,, \quad \mathcal{P}(t,z) = (\widehat{p}_0(t,z)\,,\,\widehat{p}_1(t,z),\cdots)^T$$

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Taking derivatives, and taking into account $\dot{J}(t) = [J(t), A(t)]$, we arrive to

$$A(t)\mathcal{P}(t,z) + \dot{\mathcal{P}}(t,z) = \mu\mathcal{P}(t,z) \tag{7}$$

where

$$\mu = \mu\widehat{p}_0(t,z) = -\frac{1}{2}\sum_{j=1}^{p} J_{0j}^{p}(t)\widehat{p}_j(t,z)$$

Proof of Theorem 2

From (7) we obtain

$$\dot{\widehat{p}}_n(t,z) = -\frac{1}{2}\sum_{j=1}^{p} J^p_{n,n-j}(t)\widehat{p}_{n-j}(t,z)$$
$$+\frac{1}{2}\sum_{j=1}^{p} J^p_{n,n+j}(t)\widehat{p}_{n+j}(t,z) + \mu\widehat{p}_n(t,z)\,, \quad n = 0, 1 \ldots .$$

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$$\Rightarrow \sum_{j=1}^{p} J^p_{n,n-j}(t)\widehat{p}_{n-j}(t,z) + J^p_{nn}(t)\widehat{p}_n(t,z) + \sum_{j=1}^{p} J^p_{n,n+j}(t)\widehat{p}_{n+j}(t,z) = z^p\widehat{p}_n(t,z)\,,\ n = 0,1\ldots, \quad (8)$$

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$$+ \sum_{j=1}^{p} J^p_{n,n+j}(t)\widehat{p}_{n+j}(t,z) = z^p \widehat{p}_n(t,z)\,,\ n = 0, 1 \ldots, \quad (8)$$

Taking $n = 0$ we obtain $\mu = -\frac{1}{2}\left(z^p - J^p_{00}(t)\right)$ and

$$\dot{\widehat{p}}_n(t,z) = -\sum_{j=1}^{p} J^p_{n,n-j}(t)\widehat{p}_{n-j}(t,z) + \frac{\left(J^p_{00}(t) - J^p_{nn}(t)\right)\widehat{p}_n(t,z)}{2}\,. \quad (9)$$

Proof of Theorem 2

Since $\{J(t)\}$ is a generalized Toda solution,

$$\dot{\lambda}_i(t) = \dot{J}_{i-2,i-1}(t) = \frac{1}{2}\lambda_i(t)\left(J^p_{i-1,i-1}(t) - J^p_{i-2,i-2}(t)\right), \quad i \in \mathbb{N}.$$

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$$\Longrightarrow \frac{d}{dt}\left(\lambda_2(t)\cdots\lambda_{n+1}(t)\right) = \frac{1}{2}\lambda_2(t)\cdots\lambda_{n+1}(t)\left(J^p_{nn}(t) - J^p_{00}(t)\right) .$$

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Taking derivatives in $P_n(t,z) = \left(\lambda_2(t)\cdots\lambda_{n+1}(t)\right)\widehat{p}_n(t,z)$, from (9) we arrive to

$$\dot{P}_n(t,z) = -\sum_{j=1}^{p} J^p_{n,n-j}(t)\lambda_{n-j+2}(t)\ldots\lambda_{n+1}(t)P_{n-j}(t,z) \qquad (10)$$

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$$-(z-\alpha_{n+1}(t))\sum_{j=0}^{p-1} J^{p}_{n,n-j-1}(t)\lambda_{n-j+1}(t)\ldots\lambda_{n+1}(t)P_{n-j-1}(t,z)$$

$$\begin{aligned} - \quad & 2\lambda_{n+1}(t)\dot{\lambda}_{n+1}(t)P_{n-1}(t,z) \qquad (11) \\ & +\lambda^2_{n+1}(t)\sum_{j=1}^{p} J^{p}_{n-1,n-j-1}(t)\lambda_{n-j+1}(t)\ldots\lambda_{n}(t)P_{n-j-1}(t,z)\,. \end{aligned}$$

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which conduces to

$$\dot{\alpha}_{n+1}(t) = \dot{J}_{nn}(t) = J_{n,n+1}(t)J^p_{n,n+1}(t) - J_{n-1,n}(t)J^p_{n-1,n}(t)$$

(first part of (3)).

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(first part of (3)).

By comparing the coefficients of z^{n-1} in (11) and computing,

$$\dot{\lambda}_{n+1}(t) = \dot{J}_{n-1,n}(t) = \frac{1}{2}J_{n-1,n}(t)\left[J^p_{n,n}(t) - J^p_{n-1,n-1}(t)\right]$$

(second part of (3). □

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- $Q_n^{(C)}(t,z) = \dfrac{P_{n+1}(t,z) - \dfrac{P_{n+1}(t,C)}{P_n(t,C)} P_n(t,z)}{z - C}, \quad n = 0, 1, \ldots,$

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Lemma 1 (Chihara)

$Q_n^{(C)}(t,z) = (z - \widetilde{\alpha}_n(t)) Q_{n-1}^{(C)}(t,z) - \widetilde{\lambda}_n^2(t) Q_{n-2}^{(C)}(t,z), \quad n \in \mathbb{N}\,,$
$(Q_{-1}^{(C)} \equiv 0\,,\; Q_0^{(C)} \equiv 1)$, being

Bäcklund transformation

- $\{J(t)\}$ generalized Toda solution,
- $\{P_n(t,z)\}$ given by the recurrence relation,
- $C \in \mathbb{C} \ : \ P_n(t,C) \neq 0\,,\ n \in \mathbb{N}\,,\ t \in \mathbb{R}\,,$
- $Q_n^{(C)}(t,z) = \dfrac{P_{n+1}(t,z) - \dfrac{P_{n+1}(t,C)}{P_n(t,C)} P_n(t,z)}{z - C}, \quad n = 0, 1, \ldots,$

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$$\left.\begin{aligned} \widetilde{\alpha}_n(t) &= \frac{P_{n+1}(t,C)}{P_n(t,C)} + \alpha_{n+1}(t) - \frac{P_n(t,C)}{P_{n-1}(t,C)} \\ \widetilde{\lambda}_n^2(t) &= \lambda_n^2(t) \frac{P_{n-2}(t,C) P_n(t,C)}{P_{n-1}^2(t,C)} \end{aligned}\right\}$$

Bäcklund transformation

$$\{P_n(t,z)\} \longrightarrow \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right.$$

Bäcklund transformation

$$\{P_n(t,z)\} \longrightarrow \begin{cases} \gamma_1(t) := 0, \\ \gamma_{2n}^2(t) := -\dfrac{P_n(t,C)}{P_{n-1}(t,C)}, \\ \gamma_{2n+1}^2(t) := -\lambda_{n+1}^2(t)\dfrac{P_{n-1}(t,C)}{P_n(t,C)}. \end{cases}$$

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From Lemma 1, for each $n \in \mathbb{N}$ we have

$$\lambda_{n+1}^2(t) = \gamma_{2n}^2(t)\gamma_{2n+1}^2(t)\,, \quad \alpha_n(t) = \gamma_{2n-1}^2(t) + \gamma_{2n}^2(t) + C\,,$$

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$\Longrightarrow$ The coefficients of the recurrences relation associated with $\{P_n(t,z)\}$ and $\{Q_n^{(C)}(t,z)\}$ are related by a *Bäcklund transformation*.

Proof of Theorem 1: uniqueness

1) Uniqueness of sequences $\{\widetilde{\alpha}_n(t)\}$, $\{\widetilde{\lambda}_n(t)\}$, $\{\gamma_n^2(t)\}$:

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$$J^{(1)}(t) := \begin{pmatrix} \alpha_1(t) & \lambda_2(t)^2 & & \\ 1 & \alpha_2(t) & \lambda_3(t)^2 & \\ & 1 & \alpha_3(t) & \ddots \\ & & \ddots & \ddots \end{pmatrix} \Rightarrow$$

$$\det\left(C\mathrm{I}_n - J_n^{(1)}(t)\right) = \det\left(C\mathrm{I}_n - J_n(t)\right) \neq 0\,,\ t \in \mathbb{R}\,,\ n \in \mathbb{N}$$

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$\Rightarrow \exists L(t)$ lower triangular matrix and $\exists U(t)$ upper triangular matrix such that $J^{(1)}(t) - C\mathrm{I} = L(t)U(t)$

Proof of Theorem 1: uniqueness

If the diagonal entries of $U(t)$ are $u_{ii} = 1$, then the matrices $L(t)\,,\ U(t)$ are uniquely determined. Therefore,

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$$\lambda_{n+1}^2(t) = \gamma_{2n}^2(t)\gamma_{2n+1}^2(t)\,, \quad \alpha_n(t) - C = \gamma_{2n-1}^2(t) + \gamma_{2n}^2(t) \qquad (12)$$

$$\Rightarrow L(t) = \begin{pmatrix} \gamma_2^2(t) & & \\ 1 & \gamma_4^2(t) & \\ & \ddots & \ddots \end{pmatrix}, \quad U(t) = \begin{pmatrix} 1 & \gamma_3^2(t) & \\ & 1 & \gamma_5^2(t) \\ & & \ddots & \ddots \end{pmatrix}$$

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$$\Rightarrow \{\gamma_n^2(t)\} \text{ is the unique sequence verifying (12)}$$

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$$\Rightarrow \{\gamma_n^2(t)\} \text{ is the unique sequence verifying (12)}$$

$\Rightarrow \{\widetilde{\alpha}_n(t)\}$, $\{\widetilde{\lambda}_n(t)\}$ are the unique sequences satisfying

$$\widetilde{\lambda}_{n+1}^2(t) = \gamma_{2n+1}^2(t)\gamma_{2n+2}^2(t)\,, \quad \widetilde{\alpha}_n(t) = \gamma_{2n}^2(t) + \gamma_{2n+1}^2(t) + C$$

Proof of Theorem 1: generalized Volterra solution

2) $\underline{\{\Gamma(t)\}\text{ is a generalized Volterra solution}}$:

Proof of Theorem 1: generalized Volterra solution

2) $\{\Gamma(t)\}$ is a generalized Volterra solution:

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$$\Gamma^2(t) + C\mathrm{I} = \begin{pmatrix} \alpha_1(t) & 0 & \lambda_2(t) & 0 & & \\ 0 & \widetilde{\alpha}_1(t) & 0 & \widetilde{\lambda}_2(t) & \ddots & \\ \lambda_2(t) & 0 & \alpha_2(t) & 0 & \lambda_3(t) & \ddots \\ 0 & \widetilde{\lambda}_2(t) & 0 & \widetilde{\alpha}_3(t) & \ddots & \ddots \\ & \ddots & \ddots & \ddots & & \end{pmatrix}$$

2) $\{\Gamma(t)\}$ is a generalized Volterra solution:

$$\left.\begin{array}{ll} \lambda_{n+1}^2(t) = \gamma_{2n}^2(t)\gamma_{2n+1}^2(t)\,, & \alpha_n(t) = \gamma_{2n-1}^2(t) + \gamma_{2n}^2(t) + C \\ \widetilde{\lambda}_{n+1}^2(t) = \gamma_{2n+1}^2(t)\gamma_{2n+2}^2(t)\,, & \widetilde{\alpha}_n(t) = \gamma_{2n}^2(t) + \gamma_{2n+1}^2(t) + C \end{array}\right\} \Rightarrow$$

$$\Gamma^2(t) + C\mathrm{I} = \begin{pmatrix} \alpha_1(t) & 0 & \lambda_2(t) & 0 & & \\ 0 & \widetilde{\alpha}_1(t) & 0 & \widetilde{\lambda}_2(t) & \ddots & \\ \lambda_2(t) & 0 & \alpha_2(t) & 0 & \lambda_3(t) & \ddots \\ 0 & \widetilde{\lambda}_2(t) & 0 & \widetilde{\alpha}_3(t) & \ddots & \ddots \\ & \ddots & \ddots & \ddots & & \end{pmatrix}$$

$\Rightarrow$ the matrix $\Gamma^2(t) + C\mathrm{I}$ "interlace" $J(t)$ and $\widetilde{J}(t)$

Proof of Theorem 1: generalized Volterra solution

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Lemma 2

For each $m \in \mathbb{N}$ and $j, k = 0, 1, \ldots,$

$$\left(\Gamma^2(t) + CI\right)^m_{jk} = \begin{cases} 0 & , \quad j+k \text{ odd} \\ J(t)^m_{\frac{j}{2},\frac{k}{2}} & , \quad j,k \text{ even} \\ \widetilde{J}(t)^m_{\frac{j-1}{2},\frac{k-1}{2}} & , \quad j,k \text{ odd} \end{cases}$$

(Proof: By induction on m)

Proof of Theorem 1: generalized Volterra solution

To obtain $\dot{\Gamma}(t)$ we use

$$\frac{\dot{P}_n(t,C)}{P_n(t,C)} = -\sum_{j=1}^{p} J^{p}_{n,n-j}(t)\,\frac{\widehat{p}_{n-j}(t,C)}{\widehat{p}_n(t,C)}\,, \qquad n \in \mathbb{N}\,.$$

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The above ratio can be written in terms of diagonal and subdiagonal entries of $(J(t))^j$, $j=1,\ldots,p-1$:

Lemma 3

For each $m,n\in\mathbb{N}$ we have

$$-\sum_{j=1}^{m} J^m_{n,n-j}(t)\frac{\widehat{p}_{n-j}(t,C)}{\widehat{p}_n(t,C)} = \gamma^2_{2n+1}(t)B^{(m)}_{nn}(t)+\lambda_{n+1}(t)B^{(m)}_{n,n-1}(t)\,,$$

where $B^{(m)}(t) := C^{m-1}\mathrm{I}+C^{m-2}J(t)+\cdots+(J(t))^{m-1}$.

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Lemma 4

For each $m, n \in \mathbb{N}$ we have

$$-\sum_{j=1}^{m}\left(\Gamma^2(t)+CI\right)^m_{2n,2n-2j}\frac{\widehat{p}_{n-j}(t,C)}{\widehat{p}_n(t,C)}$$

$$+\sum_{j=1}^{m}\left(\Gamma^2(t)+CI\right)^m_{2n-2,2n-2j-2}\frac{\widehat{p}_{n-j-1}(t,C)}{\widehat{p}_{n-1}(t,C)} \qquad (13)$$

$$=\left(\Gamma^2(t)+CI\right)^m_{2n-1,2n-1}-\left(\Gamma^2(t)+CI\right)^m_{2n-2,2n-2}$$

Proof of Lemma 4: We want to prove

$$S(n,m,t) - S(n-1,m,t) = \widetilde{J}^{m}_{n-1,n-1}(t) - J^{m}_{n-1,n-1}(t) \qquad (14)$$

where

$$S(n,m,t) = \gamma^2_{2n+1} B^{(m)}_{nn}(t) + \lambda_{n+1}(t) B^{(m)}_{n,n-1}(t)\,. \qquad (15)$$

Proof of Theorem 1: generalized Volterra solution

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Proof of Theorem 1: generalized Volterra solution

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- Assume that (14) holds for $m \in \mathbb{N}$. From

$$S(n,m+1,t) = CS(n,m,t) + \lambda_{n+1}(t) J^{m}_{n,n-1}(t) + \gamma^2_{2n+1} J^{m}_{nn}(t)$$

we arrive to (14) in $m+1$

□

Proof of Theorem 1: generalized Volterra solution

We know:

$$\left.\begin{array}{r} \text{Lemmas 2 and 4} \\[1em] \dot{\gamma}_{2n}(t) = \frac{1}{2}\gamma_{2n}(t)\left(\frac{\dot{P}_n(t,C)}{P_n(t,C)} - \frac{\dot{P}_{n-1}(t,C)}{P_{n-1}(t,C)}\right) \\[1em] \frac{\dot{P}_n(t,C)}{P_n(t,C)} = -\sum_{j=1}^{p} J^{p}_{n,n-j}(t)\frac{\widehat{p}_{n-j}(t,C)}{\widehat{p}_n(t,C)} \end{array}\right\} \Longrightarrow$$

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$\Longrightarrow$ for each odd n we have $\dot{\Gamma}_{n-1,n}(t) = \dot{\gamma}_{n+1}(t)$

$$= \frac{1}{2}\Gamma_{n-1,n}(t)\left[(\Gamma^2(t) + CI)^p_{nn} - (\Gamma^2(t) + CI)^p_{n-1,n-1}\right] \tag{16}$$

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In the following we analyze the case when n is an even number.

Proof of Theorem 1: generalized Volterra solution

$$\left.\begin{array}{r} \{J(t)\} \text{ generalized Toda solution} \\ \lambda^2_{n+1}(t) = \gamma^2_{2n}(t)\gamma^2_{2n+1}(t) \end{array}\right\} \Rightarrow$$

Proof of Theorem 1: generalized Volterra solution

$$\left.\begin{array}{r}\{J(t)\}\text{ generalized Toda solution}\\ \lambda_{n+1}^2(t)=\gamma_{2n}^2(t)\gamma_{2n+1}^2(t)\end{array}\right\}\Rightarrow$$

$$2\lambda_{n+1}(t)\dot{\lambda}_{n+1}(t)\overset{(16)}{=}2\gamma_{2n}^2(t)\gamma_{2n+1}(t)\dot{\gamma}_{2n+1}(t)$$

$$+\gamma_{2n}^2(t)\gamma_{2n+1}^2(t)\left[\left(\Gamma^2(t)+CI\right)^p_{2n-1,2n-1}-\left(\Gamma^2(t)+CI\right)^p_{2n-2,2n-2}\right].$$

Proof of Theorem 1: generalized Volterra solution

$$\left.\begin{array}{r}\{J(t)\}\text{ generalized Toda solution}\\[2ex] \lambda_{n+1}^2(t)=\gamma_{2n}^2(t)\gamma_{2n+1}^2(t)\end{array}\right\}\Rightarrow$$

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$$+\gamma_{2n}^2(t)\gamma_{2n+1}^2(t)\left[\left(\Gamma^2(t)+CI\right)^p_{2n-1,2n-1}-\left(\Gamma^2(t)+CI\right)^p_{2n-2,2n-2}\right].$$

Dividing by $\lambda_{n+1}^2(t)=\gamma_{2n}^2(t)\gamma_{2n+1}^2(t)$ we arrive to

$$2\frac{\dot{\gamma}_{2n+1}(t)}{\gamma_{2n+1}(t)}=\left(\Gamma^2(t)+CI\right)^p_{2n,2n}-\left(\Gamma^2(t)+CI\right)^p_{2n-1,2n-1},$$

which is (16) when n is an even number.

Proof of Theorem 1: generalized Toda solution

For each $n = 0, 1, \ldots,$ we want to prove

$$\left.\begin{array}{rcl} \dot{\widetilde{J}}_{nn}(t) & = & \widetilde{J}_{n,n+1}(t)\,\widetilde{J}^{p}_{n,n+1}(t) - \widetilde{J}_{n-1,n}(t)\,\widetilde{J}^{p}_{n-1,n}(t) \\ \dot{\widetilde{J}}_{n,n+1}(t) & = & \frac{1}{2}\widetilde{J}_{n,n+1}(t)\left[\widetilde{J}^{p}_{n+1,n+1}(t) - \widetilde{J}^{p}_{n,n}(t)\right] \end{array}\right\}$$

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We have $\widetilde{\alpha}_n(t) = \dfrac{P_{n+1}(t,C)}{P_n(t,C)} + \alpha_{n+1}(t) - \dfrac{P_n(t,C)}{P_{n-1}(t,C)}$

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For each $n = 0, 1, \dots,$ we want to prove

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We have $\widetilde{\alpha}_n(t) = \dfrac{P_{n+1}(t,C)}{P_n(t,C)} + \alpha_{n+1}(t) - \dfrac{P_n(t,C)}{P_{n-1}(t,C)}$

$$\Rightarrow \widetilde{\alpha}_n(t) = \alpha_{n+1}(t) - \gamma^2_{2n+2}(t) + \gamma^2_{2n}(t)\,.$$

Proof of Theorem 1: generalized Toda solution

For each $n = 0, 1, \ldots,$ we want to prove

$$\left.\begin{array}{rcl} \dot{\widetilde{J}}_{nn}(t) & = & \widetilde{J}_{n,n+1}(t)\,\widetilde{J}^{p}_{n,n+1}(t) - \widetilde{J}_{n-1,n}(t)\,\widetilde{J}^{p}_{n-1,n}(t) \\ \dot{\widetilde{J}}_{n,n+1}(t) & = & \frac{1}{2}\,\widetilde{J}_{n,n+1}(t)\left[\widetilde{J}^{p}_{n+1,n+1}(t) - \widetilde{J}^{p}_{n,n}(t)\right] \end{array}\right\}$$

We have $\widetilde{\alpha}_n(t) = \dfrac{P_{n+1}(t,C)}{P_n(t,C)} + \alpha_{n+1}(t) - \dfrac{P_n(t,C)}{P_{n-1}(t,C)}$

$$\Rightarrow \widetilde{\alpha}_n(t) = \alpha_{n+1}(t) - \gamma^2_{2n+2}(t) + \gamma^2_{2n}(t)\,.$$

Computing and using the fact that $\{J(t)\}$ and $\{\Gamma(t)\}$ are, respectively, generalized Toda and Volterra solution, we arrive to

$$\dot{\widetilde{J}}_{n-1,n-1}(t) = \dot{\widetilde{\alpha}}_n(t) = \widetilde{J}_{n-1,n}(t)\,\widetilde{J}^{p}_{n-1,n}(t) - \widetilde{J}_{n-2,n-1}(t)\,\widetilde{J}^{p}_{n-2,n-1}(t)\,,$$

which is the first part of Toda equation.

Proof of Theorem 1: generalized Toda solution

$\widetilde{\lambda}_{n+1}^2(t) = \gamma_{2n+1}^2(t)\gamma_{2n+2}^2(t)$. Taking derivatives and dividing by $\widetilde{\lambda}_{n+1}^2(t)$,

Proof of Theorem 1: generalized Toda solution

$\widetilde{\lambda}^2_{n+1}(t) = \gamma^2_{2n+1}(t)\gamma^2_{2n+2}(t)$. Taking derivatives and dividing by $\widetilde{\lambda}^2_{n+1}(t)$,

$$\frac{\dot{\widetilde{\lambda}}_{n+1}(t)}{\widetilde{\lambda}_{n+1}(t)} = \frac{\dot{\gamma}_{2n+1}(t)}{\gamma_{2n+1}(t)} + \frac{\dot{\gamma}_{2n+2}(t)}{\gamma_{2n+2}(t)} .$$

Proof of Theorem 1: generalized Toda solution

$\widetilde{\lambda}_{n+1}^2(t) = \gamma_{2n+1}^2(t)\gamma_{2n+2}^2(t)$. Taking derivatives and dividing by $\widetilde{\lambda}_{n+1}^2(t)$,

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Taking into account that $\{\Gamma(t)\}$ is a generalized Volterra solution,

$$\begin{aligned}\frac{\dot{\widetilde{\lambda}}_{n+1}(t)}{\widetilde{\lambda}_{n+1}(t)} &= \frac{1}{2}\left[\left(\Gamma^2(t)+CI\right)^p_{2n,2n} - \left(\Gamma^2(t)+CI\right)^p_{2n-1,2n-1}\right] \\ &+ \frac{1}{2}\left[\left(\Gamma^2(t)+CI\right)^p_{2n+1,2n+1} - \left(\Gamma^2(t)+CI\right)^p_{2n,2n}\right] \\ &= \frac{1}{2}\left[\left(\Gamma^2(t)+CI\right)^p_{2n+1,2n+1} - \left(\Gamma^2(t)+CI\right)^p_{2n-1,2n-1}\right],\end{aligned}$$

which is the second part of Toda equation. □

Proof of Theorem 1: generalized Toda solution

$\widetilde{\lambda}^2_{n+1}(t) = \gamma^2_{2n+1}(t)\gamma^2_{2n+2}(t)$. Taking derivatives and dividing by $\widetilde{\lambda}^2_{n+1}(t)$,

$$\frac{\dot{\widetilde{\lambda}}_{n+1}(t)}{\widetilde{\lambda}_{n+1}(t)} = \frac{\dot{\gamma}_{2n+1}(t)}{\gamma_{2n+1}(t)} + \frac{\dot{\gamma}_{2n+2}(t)}{\gamma_{2n+2}(t)}.$$

Taking into account that $\{\Gamma(t)\}$ is a generalized Volterra solution,

$$\begin{aligned}\frac{\dot{\widetilde{\lambda}}_{n+1}(t)}{\widetilde{\lambda}_{n+1}(t)} &= \frac{1}{2}\left[\left(\Gamma^2(t)+CI\right)^p_{2n,2n} - \left(\Gamma^2(t)+CI\right)^p_{2n-1,2n-1}\right] \\ &\quad + \frac{1}{2}\left[\left(\Gamma^2(t)+CI\right)^p_{2n+1,2n+1} - \left(\Gamma^2(t)+CI\right)^p_{2n,2n}\right] \\ &= \frac{1}{2}\left[\left(\Gamma^2(t)+CI\right)^p_{2n+1,2n+1} - \left(\Gamma^2(t)+CI\right)^p_{2n-1,2n-1}\right],\end{aligned}$$

which is the second part of Toda equation. □

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