New results on lower bounds for the number of ($\leq k$)-facets (extended abstract)

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Abstract

In this paper we present three different results dealing with the number of ($\leq k$)-facets of a set of points:

(i) We give structural properties of sets in the plane that achieve the optimal lower bound $3\binom{k+2}{2}$ of ($\leq k$)-edges for a fixed $k \leq \lfloor n/3 \rfloor - 1$;

(ii) We show that the new lower bound $3\binom{k+2}{2} + 3\binom{k-\lfloor \frac{n}{3} \rfloor+2}{2}$ for the number of ($\leq k$)-edges of a planar point set is optimal in the range $\lfloor n/3 \rfloor \leq k \leq \lfloor 5n/12 \rfloor - 1$;

(iii) We show that for $k < n/4$ the number of $(\leq k)$-facets of a set of n points in $\mathbb{R}^3$ in general position is at least $4\binom{k+3}{3}$, and that this bound is tight in that range.

Keywords: $(\leq k)$-edges, $(\leq k)$-facets

1 Introduction

In this paper we deal with the problem of giving lower bounds to the number of $(\leq k)$-facets of a set of points S: An oriented simplex with vertices at points of S is said to be a *j-facet* of S if it has exactly j points in the positive side of its affine hull. Similarly, the simplex is said to be an *$(\leq k)$-facet* if it has at most k points in the positive side of its affine hull.

The number of j-facets of S is denoted by $e_j(S)$ and $E_k(S) = \sum_{j=0}^{k} e_j(S)$ is the number of $(\leq k)$-facets (the set S can be omitted if it is clear from the context). Giving bounds on these quantities, and on the number of the companion concept of k-set, is one of the central problems in Discrete and Computational Geometry, and has a long history that we will not try to summarize here. Chapter 8.3 is a complete and up to date survey of results and open problems in the area.

Regarding lower bounds for $E_k(S)$, which is the main topic of this paper, the problem was first studied by Edelsbrunner et al. in $\mathbb{R}^2$ due to its connections with the complexity of higher order Voronoi diagrams. In that paper it was stated that $E_k(S) \geq 3\binom{k+2}{2}$ and an example was given showing that the bound is tight if $k \leq \lfloor n/3 \rfloor - 1$. Unfortunately, the proof of the bound was not correct and a correct proof, based on circular sequences, was independently found by Abrego and Fernández-Merchant and Lovász et al. where the problem was revisited due to its strong connection with the rectilinear crossing number of the complete graph or, equivalently, with the number of convex quadrilaterals in a set of points.

This lower bound was slightly improved for $k \geq \lfloor \frac{n}{3} \rfloor$ by Balogh and Salazar , again using circular sequences. Recently, and based on the observation that it suffices to proof the bound for sets with triangular convex hull, we have shown that, in $\mathbb{R}^2$,

$$E_k(S) \geq 3\binom{k+2}{2} + \sum_{j=\lfloor \frac{n}{3} \rfloor}^{k} (3j - n + 3). \tag{1}$$

In this paper we deal with three different problems related to lower bounds for $E_k(S)$ in $\mathbb{R}^2$ and $\mathbb{R}^3$. Due to the lack of space, proofs will be omitted from this extended abstract.

2 Optimal $(\leq k)$-set vectors

For $S \subset \mathbb{R}^2$ and for fixed $k \leq \lfloor \frac{n}{3} \rfloor - 1$, we say that $E_k(S)$ is *optimal* if $E_k(S) = 3\binom{k+2}{2}$.

Theorem 2.1 *If $E_k(S)$ is optimal, then S has a triangular convex hull.*

Corollary 2.2 *If $E_k(S)$ is optimal, then the outermost $\lceil \frac{k}{2} \rceil$ layers of S are triangles.*

Theorem 2.3 *If $E_k(S)$ is optimal, then $E_j(S)$, $0 \leq j \leq k$, is optimal.*

3 Tightness of the lower bound for $(\leq k)$-edges in $\mathbb{R}^2$

In this section we show a point configuration which proves that the lower bound for $E_k(S)$ is tight for $0 \leq k \leq \lfloor \frac{5n}{12} \rfloor - 1$. This solves an open conjecture Consider the configuration in Figure 1, which is composed of three rotationally symmetric chains, each one associated to a convex hull vertex, fulfilling the following properties (where left and right are considered with respect to the corresponding convex hull vertex):

- The first part of the chain is slightly convex to the right and contains $\frac{3n}{12}$ points, with a hole between the first $\frac{2n}{12}$ points (which we call subchain A) and the remaining $\frac{n}{12}$ points (called subchain B).
- Each chain is completed with a subchain C, composed of another $\frac{n}{12}$ points slightly convex to the left.
- All the lines spanned by two points in $A \cup B$ leave to the right the next chain in counterclockwise order, and to the left both the points in C and those in the remaining chain.

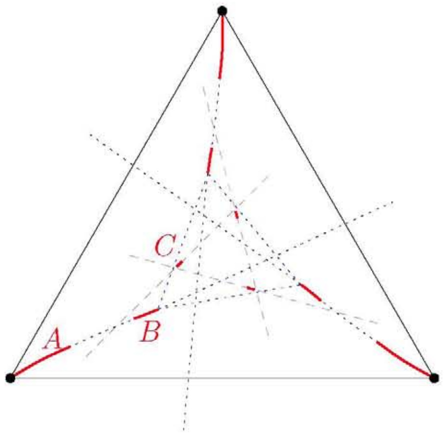

Fig. 1. Configuration for which the bound for $E_k(S)$ is tight.

- All the lines spanned by two points in C separate subchains A and B. Furthermore, they leave to the right both other subchains of type C.
- The triangle defined by the innermost points of chains of type B contains all the chains of type C.

Theorem 3.1 *For the point configuration S defined above and $\lfloor \frac{n}{3} \rfloor \leq k \leq \lfloor \frac{5n}{12} \rfloor - 1$,*

$$E_k(S) = 3\binom{k+2}{2} + 3\binom{k - \frac{n}{3} + 2}{2}.$$

4 A lower bound for $(\leq k)$-facets in $\mathbb{R}^3$

Throughout this section $S \subset \mathbb{R}^3$ will be a set of n points in general position. Given $p, q, r \in S$, recall that the triangle pqr is a *j-facet* of S if the plane containing it partitions S into two subsets with cardinality j and $n - 3 - j$ (we consider unoriented j-facets and, therefore, $j \leq \frac{n-3}{2}$). In this section we give a lower bound for the number of $(\leq k)$-facets of S. To the best of our knowledge, this is the first lower bound for this quantity.

Motivated by the standard definition of convex position, we say that a set of points is in *simplicial position* if its convex hull is a simplex. As in [2], we first try to show that the number of $(\leq k)$-facets is minimized for sets in simplicial position using continuous motion. The events when the number of j-facets changes are called *mutations* and have been previously considered by Andrzejak

Unlike in the planar case, when a vertex of the convex hull is moved to infinity, there are mutations for which the number of ($\leq k$)-facets increases. Using the concept of *centerpoint* what we can prove is the following:

Theorem 4.1 *Let S be a set of n points with $h > 4$ extreme points. There exists a set S_1 of n points in simplicial position and such that $E_k(S_1) \leq E_k(S)$ for every $0 \leq k < \frac{n}{4}$.*

Using this result we can apply induction and show that:

Theorem 4.2 *Let S be a set of n points in $\mathbb{R}^3$ in general position. If $k < \frac{n}{4}$, the number of ($\leq k$)-facets of S is at least $4\binom{k+3}{3}$. Furthermore, this bound is tight.*

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