Chebyshev expansion for the component functions of the almost-Mathieu operator

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The component functions $\{\Psi_n(\epsilon)\}$ $(n \in \mathcal{Z}^+)$ from difference Schrödinger operators, can be formulated in a second order linear difference equation. Then the Harper equation, associated to almost-Mathieu operator, is a prototypical example. Its spectral behavior is amazing. Here, due the cosine coefficient in Harper equation, the component functions are expanded in a Chebyshev series of first kind, $T_n(cos2\pi\theta)$. It permits us a particular method for the θ variable separation. Thus, component functions can be expressed as an inner product, $\Psi_n(\epsilon,\lambda,\theta) = \vec{T}_{[\frac{n(n-1)}{2}]}(cos2\pi\theta) \cdot \vec{A}_{[\frac{n(n-1)}{2}]}(\epsilon,\lambda)$. A matrix block transference method is applied for the calculation of the vector $\vec{A}_{[\frac{n(n-1)}{2}]}(\epsilon,\lambda)$. When θ is integer, $\Psi_n(\epsilon)$ is the sum of component from $\vec{A}_{[\frac{n(n-1)}{2}]}$. The complete family of Chebyshev Polynomials can be generated, with fit initial conditions. The continuous spectrum is one band with Lebesgue measure equal to 4. When θ is not integer the inner product Ψ_n can be seen as a perturbation of vector $\vec{T}_{[\frac{n(n-1)}{2}]}$ on the sum of components from the vector $\vec{A}_{[\frac{n(n-1)}{2}]}$. When $\theta = \frac{p}{q}$, with p and q coprime, periodic perturbation appears, the connected band from the integer case degenerates in q sub-bands. When θ is irrational, ergodic perturbation produces that one band spectrum from integer case degenerates to a Cantor set. Lebesgue measure is $L_\sigma = 4(1-|\lambda|)$, $0 < |\lambda| \leq 1$. In this situation, the series solution becomes critical.

1 Chebyshev expansion of the component functions.

The Almost-Mathieu operator appears in some approximated quantum models of energy spectra, This operator can be formulated via a second order linear difference equation, known as Harper equation:

$$\Psi_{n+1}(\epsilon) = (\epsilon - 2\lambda\cos(n2\pi\theta + \nu))\Psi_n(\epsilon) - \Psi_{n-1}(\epsilon). \tag{1}$$

The family of component functions $\{\Psi_n(\epsilon,\lambda,\theta,\nu)\}$, depends of ϵ, the energy, as primary parameter, and the other parameters related with the particular characteristics of the system in study. When θ is irrational, ergodic case, many work has been generated, focussed on spectrum analysis.

The form of (1) suggests a solution in series of Chebyshev polynomials of first kind. Also, when $n \to \infty$, this series type converges in $\mathcal{L}_2$, Product properties of Chebyshev polynomials of first kind $T_n(\omega)$ are used. Here, without loss of generality, $\nu = 0$. Indeed, when θ is irrational the spectrum does not depend on ν. In other situations, variations on ν only produces shifts in all spectrum bands. This series must agree with Eq. (1). Thus, for n finite, the series is truncated.

$$\Psi_n(\epsilon) = \sum_{k=0}^{[\frac{n(n-1)}{2}]} a_k^{(n)}(\epsilon,\lambda)T_k(\omega). \tag{2}$$

With $[x]$ the integer component of x, and $\omega = 2\pi\theta$. Eq. (2) is introduced in (1) and coefficients from $T_n(\omega)$ with equal n are matched. A recurrent expression for the coefficients $a_k^{(n)}(\epsilon,\lambda)$ are obtained. The compact form is:

$$\begin{aligned} a_k^{(n+1)} =& -\lambda a_{k-n}^{(n)}\sigma(n-1) + \epsilon a_k^{(n)}(1-\sigma(\frac{(n)(n-1)}{2})) - \lambda a_{n-k}^{(n)}(1-\sigma(n)) \\ & - \lambda a_{k+n}^{(n)}(1-\delta_{k,0} - \sigma(\frac{(n)(n-3)}{2})) - a_k^{(n-1)}(1-\sigma(\frac{(n-1)(n-2)}{2})). \end{aligned} \tag{3}$$

With $\sigma(k)$ the Heaviside step function and $\delta_{k,0}$ the Kronecker delta function, $0 \leq k \leq \frac{(n+1)(n)}{2}$.

1.1 Variable separation and inner product.

The parameters are separated. The coefficients of the series depend from ϵ and λ, the Chebyshev Polynomials $T_n(\omega)$ depend from θ. Eq. (2) can be seen as an inner product, $\Psi_n(\epsilon,\lambda,\theta) = \vec{T}^T_{[\frac{n(n-1)}{2}]}(\omega) \cdot \vec{A}_{[\frac{n(n-1)}{2}]}(\epsilon,\lambda)$. The vector $\vec{T}_{[\frac{n(n-1)}{2}]}$, with components $t_i = \cos(2i\pi\theta)$, $i = 0,1,...[\frac{n(n-1)}{2}]$, the vector $\vec{A}_{[\frac{n(n-1)}{2}]}$ is generated via the recursion from (3).

1.2 Transference Block Matrix for the vector $\vec{A}_{[\frac{n(n-1)}{2}]}$.

Matrix transference method can be used to find a suitable linear recursion map This recursion permits us the achievement of $\vec{A}_{[\frac{n(n-1)}{2}]}$. The simplest recursion is the linear first order recursion one. With this purpose, it is necessary to work with a double vector $\vec{\hat{A}}$, which contains both vectors, $\vec{A}_{[\frac{n(n-1)}{2}]}$ and $\vec{A}_{[\frac{(n-1)(n-2)}{2}]}$. $\vec{\hat{A}}_n^T = (\vec{A}^T_{[\frac{n(n-1)}{2}]}, \vec{A}^T_{[\frac{(n-1)(n-2)}{2}]})$.

The vector recursion, for $n \geq 2$, with initial vector $\vec{\hat{A}}_1 = \begin{pmatrix} \Psi_1 \\ \Psi_0 \end{pmatrix}$, is $\vec{\hat{A}}_n = \mathbf{M}_{n,n-1}\vec{\hat{A}}_{n-1}$, with the block matrix:

$$\mathbf{M}_{n,n-1} = \left(\begin{array}{c|c|c} \epsilon\mathbf{I}_{n-1} - \lambda\mathbf{L}_{n-1} & -\lambda\mathbf{I}_{[\frac{(n-1)(n-4)}{2}]+1} & -\mathbf{I}_{[\frac{(n-2)(n-3)}{2}]+1} \\ \hline -\lambda(\mathbf{I}_1 + \mathbf{I}_{n-1}) & \epsilon\mathbf{I}_{[\frac{(n-1)(n-4)}{2}]+1} & \mathbf{0} \\ \hline \mathbf{0} & -\lambda(\mathbf{I}_{[\frac{(n-1)(n-4)}{2}]+1} & \mathbf{0} \\ \hline \mathbf{I}_{[\frac{(n-1)(n-2)}{2}]+1} & \mathbf{0} & \mathbf{0} \end{array}\right). \tag{4}$$

Rows and columns are labelled from 0 to $n-1$. The $\mathbf{I}_n$ matrix is the identity matrix of order n. $\mathbf{L}_n$ is the matrix with component $l_{i,j} = \delta_{n,i+j}$, Kronecker delta, with $0 \leq i,j \leq n-1$. The $\mathbf{L}_n$ matrix mixes coefficients and it complicates the recursion. For example, the block matrix $\mathbf{M}_{3,2}$ is:

$$\left(\begin{array}{cc|c} \epsilon & 0 & -1 \\ 0 & \epsilon-\lambda & 0 \\ \hline -2\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

When θ is integer, $\vec{T}_{[\frac{n(n-1)}{2}]} = \vec{1}_{[\frac{n(n-1)}{2}]}$, with $\frac{n(n-1)}{2}+1$ vector components $t_i = 1$. The Ψ_n function is equal to the sum of component from the vector $\vec{A}_{[\frac{n(n-1)}{2}]}$. The complete family of Chebyshev Polynomials can be generated, with fit initial conditions. For example, if $\Psi_0 = 1$, and $\Psi_1 = 2(\frac{\epsilon}{2} - \lambda cos(\nu))$, then, $\Psi_n = U_n(\frac{\epsilon}{2} - \lambda cos(\nu))$, Chebyshev Polynomials of second kind, in ϵ variable. Observe that, in this problem, this is the unique family of monic polynomials, [3], that converges in $\mathcal{L}_2$-morm.

2 Continuous spectrum and the vector $\vec{T}_{[\frac{n(n-1)}{2}]}$.

When θ is integer, Eq. (1) is trivial. The continuous spectrum appears in the band $[-2+2\lambda cos(\nu), 2+2\lambda cos(\nu)]$, located into the compact $[-4,4]$, with Lebesgue measure equal to 4.

When θ is not integer, the inner product Ψ_n can be seen as a perturbation of vector $\vec{T}_{[\frac{n(n-1)}{2}]}$ on the sum of components from vector $\vec{A}_{[\frac{n(n-1)}{2}]}$. For $\theta = \frac{p}{q}$, with p and q coprime, $\vec{T}_{[\frac{n(n-1)}{2}]}$ has q-periodic components, and one periodic perturbation appears. Now, the connected spectrum band, from the θ integer case, degenerates in q sub-bands.

If θ is irrational, the component from the $\vec{T}_{[\frac{n(n-1)}{2}]}$ are quasi-periodic. The perturbation on the sum becomes ergodic. This produces that the continuous spectrum, of integer θ case, degenerates to a Cantor set, with Lebesgue measure $L_\sigma = 4(1-|\lambda|)$, $0 < |\lambda| \leq 1$, The solutions from Eq.(2) in these Cantor sets are critical. A rigorous argument for this situation is an open line.

References

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