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HOMOGENIZATION RESULTS FOR CHEMICAL REACTIVE FLOWS THROUGH POROUS MEDIA

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ABSTRACT. This paper deals with the homogenization of a nonlinear problem modelling chemical reactive flows through periodically perforated domains. The chemical reactions take place on the walls of the porous medium. The effective behavior of these reactive flows is described by a new elliptic boundary-value problem containing an extra zero-order term which captures the effect of the chemical reactions.

1. INTRODUCTION

The aim of this paper is to study the homogenization of some chemical reactive flows through periodically perforated domains or porous media. We will focus our attention on a nonlinear problem which describes the motion of a fluid reacting on the boundary of a porous medium.

Let Ω be an open bounded set in $\mathbb{R}^n$ and let us perforate it by holes. As a result, we obtain an open set Ω^ε which will be referred to as being the *perforated domain*; ε represents a small parameter related to the characteristic size of the perforations.

The nonlinear problem studied in this paper concerns the stationary reactive flow of a fluid confined in Ω^ε, of concentration u^ε, reacting on the boundary of the perforations. A simplified version of this problem can be written as follows:

$$
\begin{cases} -D_f \Delta u^\varepsilon = f & \text{in } \Omega^\varepsilon, \\ -D_f \dfrac{\partial u^\varepsilon}{\partial \nu} = a\varepsilon g(u^\varepsilon) & \text{on } S^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}
$$

Here, ν is the exterior unit normal to Ω^ε, $a > 0$, $f \in L^2(\Omega)$ and S^ε is the boundary of our porous medium $\Omega \setminus \overline{\Omega^\varepsilon}$. Moreover, the fluid is assumed to be homogeneous and isotropic, with a constant diffusion coefficient $D_f > 0$.

The semilinear boundary condition on S^ε in problem (1.1) describes the chemical reactions which take place locally at the interface between the reactive fluid and the perforations. From strictly chemical point of view, this situation represents, equivalently, the effective reaction on the walls of the porous medium between the fluid filling Ω^ε and a rigid solid part filling the holes.

The function g in (1.1) is assumed to be given. We shall address here the case of a single-valued maximal monotone graph with $g(0) = 0$, i.e. the case in which g is the

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subdifferential of a convex lower semicontinuous function G. This situation is well illustrated by the following important practical example, left as an open case in [7]:

$$g(v) = |v|^{p-1}v, \quad 0 < p < 1 \quad \text{(Freundlich kinetics)}.$$

The exponent p is called *the order of the reaction.*

The existence and uniqueness of a weak solution of (1.1) can be settled by using the classical theory of semilinear monotone problems (see, for instance, [1] and [9]). As a result, we know that there exists a unique weak solution $u^\varepsilon \in V^\varepsilon \cap H^2(\Omega^\varepsilon)$, where

$$V^\varepsilon = \{v \in H^1(\Omega^\varepsilon) \mid v = 0 \text{ on } \partial\Omega\}.$$

When we associate with Ω^ε the following nonempty convex subset of V^ε:

$$K^\varepsilon = \{v \in V^\varepsilon \mid G(v)|_{S^\varepsilon} \in L^1(S^\varepsilon)\}, \tag{1.2}$$

then u^ε is also known to be characterized as being the unique solution of the following variational problem:

For all $v^\varepsilon \in K^\varepsilon$ find $u^\varepsilon \in K^\varepsilon$ such that

$$D_f \int_{\Omega^\varepsilon} Du^\varepsilon D(v^\varepsilon - u^\varepsilon)\,dx - \int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon)\,dx + a\langle \mu^\varepsilon, G(v^\varepsilon) - G(u^\varepsilon)\rangle \geqslant 0, \tag{1.3}$$

where μ^ε is the linear form on $W_0^{1,1}(\Omega)$ defined by

$$\langle \mu^\varepsilon, \varphi\rangle = \varepsilon \int_{S^\varepsilon} \varphi\, d\sigma, \quad \forall \varphi \in W_0^{1,1}(\Omega).$$

From a geometrical point of view, we shall just consider periodic structures obtained by removing periodically from Ω, with period εY (where Y is a given hyperrectangle in $\mathbb{R}^n$), an elementary hole T which has been appropriately rescaled and which is strictly included in Y, i.e. $\overline{T} \subset Y$.

We shall prove that the solution u^ε, properly extended to the whole of Ω, converges to the unique solution of the following variational inequality:

$$\int_\Omega QDuD(v-u)\,dx \geqslant \int_\Omega f(v-u)\,dx - a\frac{|\partial T|}{|Y \setminus T|}\int_\Omega (G(v) - G(u))\,dx \tag{1.4}$$

for $u \in H_0^1(\Omega)$ and for all $v \in H_0^1(\Omega)$. Here, $Q = ((q_{ij}))$ is the classical homogenized matrix, whose entries are defined as follows:

$$q_{ij} = D_f\Big(\delta_{ij} + \frac{1}{|Y \setminus T|}\int_{Y\setminus T} \frac{\partial \chi_j}{\partial y_i}\,dy\Big) \tag{1.5}$$

in terms of the functions χ_i, $i = 1, \ldots, n$, solutions of the so-called cell problems

$$\begin{cases} -\Delta\chi_i = 0 & \text{in } Y \setminus T, \\ \dfrac{\partial(\chi_i + y_i)}{\partial \nu} = 0 & \text{on } \partial T, \\ \chi_i & Y\text{-periodic.} \end{cases} \tag{1.6}$$

Remark 1.1. For the case of a smooth function g, we refer to [4].

The structure of the paper is as follows: first, let us mention that we shall just focus on the case $n \geq 3$, which will be treated explicitly. The case $n = 2$ is much more simpler and we shall omit to treat it. Section 2 is devoted to the setting of our problem. In Section 3 we formulate the main result of this paper. Section 4 contains some necessary preliminary results. In the last section we give the proof of our main result.

Finally, notice that throughout the paper we denote by C a generic fixed strictly positive constant, whose value can change from line to line.

2. Setting of the problem

Let Ω be a smooth bounded connected open subset of $\mathbb{R}^n$ ($n \geq 3$) and let $Y = [0, l_1[\times \cdots \times [0, l_n[$ be the representative cell in $\mathbb{R}^n$. Denote by T an open subset of Y with smooth boundary ∂T such that $\overline{T} \subset Y$. We shall refer to T as being *the elementary hole.*

Let ε be a real parameter taking values in a sequence of positive numbers converging to zero. For each ε and for any integer vector $k \in \mathbb{Z}^n$, set T_k^ε the translated image of εT by the vector $kl = (k_1 l_1, \ldots, k_n l_n)$

$$T_k^\varepsilon = \varepsilon(kl + T).$$

The set T_k^ε represents the holes in $\mathbb{R}^n$. Also, let us denote by T^ε the set of all the holes contained in Ω, i.e.

$$T^\varepsilon = \bigcup \{ T_k^\varepsilon \mid \overline{T_k^\varepsilon} \subset \Omega,\ k \in \mathbb{Z}^n \}.$$

Set

$$\Omega^\varepsilon = \Omega \setminus \overline{T^\varepsilon}.$$

Hence, Ω^ε is a periodically perforated domain with holes of size of the same order as the period. Remark that the holes do not intersect the boundary $\partial\Omega$.

Let

$$S^\varepsilon = \bigcup \{ \partial T_k^\varepsilon \mid \overline{T_k^\varepsilon} \subset \Omega,\ k \in \mathbb{Z}^n \}.$$

So

$$\partial\Omega^\varepsilon = \partial\Omega \cup S^\varepsilon.$$

We shall also use the following notations:

$$\begin{aligned} &|\omega| = \text{ the Lebesgue measure of any measurable subset } \omega \text{ of } \mathbb{R}^n, \\ &\chi_\omega = \text{ the characteristic function of the set } \omega, \\ &Y^* = Y \setminus \overline{T} \quad \text{and} \quad \rho = \frac{|Y^*|}{|Y|}. \end{aligned} \tag{2.1}$$

Moreover, for an arbitrary function $\psi \in L^2(\Omega^\varepsilon)$, we shall denote by $\widetilde{\psi}$ its extension by zero inside the holes.

As already mentioned, we are interested in studying the behavior of the solution, in such a perforated domain, of the following problem:

$$\begin{cases} -D_f \Delta u^\varepsilon = f & \text{in } \Omega^\varepsilon, \\ -D_f \dfrac{\partial u^\varepsilon}{\partial \nu} = a\varepsilon g(u^\varepsilon) & \text{on } S^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

We shall treat the case in which the function g appearing in (2.2) is a single-valued maximal monotone graph in $\mathbb{R}\times\mathbb{R}$, satisfying the condition $g(0)=0$. Moreover, if we denote by $D(g)$ the domain of g, i.e. $D(g)=\{\xi\in\mathbb{R}\mid g(\xi)\neq\emptyset\}$, then we suppose that $D(g)=\mathbb{R}$. We also assume that g is continuous and there exist $C\geqslant 0$ and an exponent q, with $0\leqslant q<n/(n-2)$, such that

$$|g(v)|\leqslant C(1+|v|^q). \tag{2.3}$$

This situation is well illustrated by the following important practical example:

$$g(v)=|v|^{p-1}v,\quad 0<p<1\quad \text{(Freundlich kinetics)}.$$

We know that in this case there exists a lower semicontinuous convex function G from $\mathbb{R}$ to $]-\infty,+\infty]$, G proper, i.e. $G\not\equiv+\infty$ such that g is the subdifferential of G, $g=\partial G$ (G is an indefinite "integral" of g). Let $G(v)=\int_0^v g(s)\,ds$.

Let us introduce the functional space $V^\varepsilon=\{v\in H^1(\Omega^\varepsilon)\mid v=0 \text{ on } \partial\Omega\}$, with $\|v\|_{V^\varepsilon}=\|\nabla v\|_{L^2(\Omega^\varepsilon)}$. Define the convex set

$$K^\varepsilon=\{v\in V^\varepsilon\mid G(v)|_{S^\varepsilon}\in L^1(S^\varepsilon)\}. \tag{2.4}$$

For a given function $f\in L^2(\Omega)$ the weak solution of the problem (2.2) is also the unique solution of the following variational inequality:

For all $v^\varepsilon\in K^\varepsilon$ find $u^\varepsilon\in K^\varepsilon$ such that

$$D_f\int_{\Omega^\varepsilon}Du^\varepsilon D(v^\varepsilon-u^\varepsilon)\,dx-\int_{\Omega^\varepsilon}f(v^\varepsilon-u^\varepsilon)\,dx+a\langle\mu^\varepsilon,G(v^\varepsilon)-G(u^\varepsilon)\rangle\geqslant 0. \tag{2.5}$$

Notice that there exists a unique weak solution $u^\varepsilon\in V^\varepsilon\cap H^2(\Omega^\varepsilon)$ of the above variational inequality (see [1]).

3. The main result

First, let us notice that it is well-known that the solution u^ε of the variational inequality (2.5) is also the unique solution of the minimization problem:

$$u^\varepsilon\in K^\varepsilon,\quad J^\varepsilon(u^\varepsilon)=\inf_{v\in K^\varepsilon}J^\varepsilon(v),$$

where

$$J^\varepsilon(v)=\frac{1}{2}D_f\int_{\Omega^\varepsilon}|Dv|^2\,dx+a\langle\mu^\varepsilon,G(v)\rangle-\int_{\Omega^\varepsilon}fv\,dx.$$

Introduce the following functional defined on $H_0^1(\Omega)$:

$$J^0(v)=\frac{1}{2}\int_\Omega QDvDv\,dx+a\frac{|\partial T|}{|Y^*|}\int_\Omega G(v)\,dx-\int_\Omega fv\,dx.$$

The main result of this paper is the following:

Theorem 3.1. *One can construct an extension $P^\varepsilon u^\varepsilon$ of the solution u^ε of the variational inequality (2.5) such that*

$$P^\varepsilon u^\varepsilon\rightharpoonup u\quad \textit{weakly in } H_0^1(\Omega),$$

where u is the unique solution of the minimization problem:

$$\textit{Find } u\in H_0^1(\Omega) \textit{ such that } J^0(u)=\inf_{v\in H_0^1(\Omega)}J^0(v). \tag{3.1}$$

Moreover, $G(u) \in L^1(\Omega)$*. Here,* $Q = (q_{ij})$ *is the classical homogenized matrix, whose entries are defined as follows:*

$$q_{ij} = D_f\Big(\delta_{ij} + \frac{1}{|Y^*|}\int_{Y^*} \frac{\partial \chi_j}{\partial y_i}\,\mathrm{d}y\Big) \tag{3.2}$$

in terms of the functions χ_i, $i = 1,\dots,n$, *solutions of the so-called cell problems*

$$\begin{cases} -\Delta\chi_i = 0 & \text{in } Y^*,\\ \dfrac{\partial(\chi_i + y_i)}{\partial \nu} = 0 & \text{on } \partial T,\\ \chi_i & Y\text{-periodic.}\end{cases} \tag{3.3}$$

4. Preliminary results

4.1. **An extension result.** The solution u^ε of problem (2.2) being defined only on Ω^ε, we need to extend it to the whole of Ω to be able to state the convergence result. In order to do that, let us recall the following well-known extension result (see [3]):

Lemma 4.1.1. *There exists* $P^\varepsilon \in \mathscr{L}(L^2(\Omega^\varepsilon); L^2(\Omega)) \cap \mathscr{L}(V^\varepsilon; H_0^1(\Omega))$ *a linear continuous extension operator and a positive constant* C*, independent of* ε*, such that*

$$\big\|P^\varepsilon v\big\|_{L^2(\Omega)} \leqslant C\,\|v\|_{L^2(\Omega^\varepsilon)}$$

and

$$\big\|\nabla P^\varepsilon v\big\|_{L^2(\Omega)} \leqslant C\,\|\nabla v\|_{L^2(\Omega^\varepsilon)},$$

for any $v \in V^\varepsilon$.

An immediate consequence of the previous lemma is the following Poincaré's inequality in V^ε:

Lemma 4.1.2. *There exists a positive constant* C*, independent of* ε*, such that*

$$\|v\|_{L^2(\Omega^\varepsilon)} \leqslant C\,\|\nabla v\|_{L^2(\Omega^\varepsilon)},$$

for any $v \in V^\varepsilon$.

4.2. **A convergence result.** In order to get the effective behavior of our solution u^ε we have to pass to the limit in (2.5). Let us introduce therefore, for any $h \in L^{s'}(\partial T)$, $1 \leqslant s' \leqslant \infty$, the linear form μ_h^ε on $W_0^{1,s}(\Omega)$ defined by

$$\langle \mu_h^\varepsilon, \varphi\rangle = \varepsilon \int_{S^\varepsilon} h\Big(\frac{x}{\varepsilon}\Big)\varphi\,\mathrm{d}\sigma, \quad \forall \varphi \in W_0^{1,s}(\Omega),$$

with $1/s + 1/s' = 1$. It is proved in [2] that

$$\mu_h^\varepsilon \to \mu_h \quad \text{strongly in } (W_0^{1,s}(\Omega))', \tag{4.1}$$

where $\langle \mu_h, \varphi\rangle = \mu_h \int_\Omega \varphi\,\mathrm{d}x$, with $\mu_h = (1/|Y|)\int_{\partial T} h(y)\,\mathrm{d}\sigma$. In the particular case when $h \in L^\infty(\partial T)$ or even h is constant, we have

$$\mu_h^\varepsilon \to \mu_h \quad \text{strongly in } W^{-1,\infty}(\Omega).$$

In what follows, we denote by μ^ε the above introduced measure in the particular case in which $h = 1$. Notice that in this case μ_h becomes $\mu_1 = |\partial T|/|Y|$.

Let F be a continuously differentiable function, monotonously non-decreasing and such that $F(v) = 0$ if and only if $v = 0$. We suppose that there exist a positive constant C and an exponent q, with $0 \leqslant q < n/(n-2)$, such that

$$\left|\frac{\partial F}{\partial v}\right| \leqslant C(1+|v|^q). \tag{4.2}$$

Let us prove now that for any $\varphi \in \mathcal{D}(\Omega)$ and for any $v^\varepsilon \rightharpoonup v$ weakly in $H_0^1(\Omega)$, we get

$$\varphi F(v^\varepsilon) \rightharpoonup \varphi F(v) \quad \text{weakly in } W_0^{1,\overline{q}}(\Omega), \tag{4.3}$$

where

$$\overline{q} = \frac{2n}{q(n-2)+n}.$$

In order to prove (4.3), let us first note that

$$\sup \|\nabla F(v^\varepsilon)\|_{L^{\overline{q}}(\Omega)} < \infty. \tag{4.4}$$

Indeed, from the growth condition (4.2) imposed to F, we get

$$\begin{aligned}\int_\Omega \left|\frac{\partial F}{\partial x_i}(v^\varepsilon)\right|^{\overline{q}} \mathrm{d}x &\leqslant C\int_\Omega \left(1+|v^\varepsilon|^{q\overline{q}}\right)\left|\frac{\partial v^\varepsilon}{\partial x_i}\right|^{\overline{q}} \mathrm{d}x \\ &\leqslant C\left(1+\left(\int_\Omega |v^\varepsilon|^{q\overline{q}\gamma}\,\mathrm{d}x\right)^{1/\gamma}\right)\left(\int_\Omega |\nabla v^\varepsilon|^{\overline{q}\delta}\,\mathrm{d}x\right)^{1/\delta},\end{aligned}$$

where we have taken γ and δ such that $\overline{q}\delta = 2$, $1/\gamma + 1/\delta = 1$ and $q\overline{q}\gamma = 2n/(n-2)$. Notice that from here we get $\overline{q} = 2n/(q(n-2)+n)$. We also have $\overline{q} > 1$ since $0 \leqslant q < n/(n-2)$. Now, as

$$\sup \|v^\varepsilon\|_{L^{\frac{2n}{n-2}}(\Omega)} < \infty,$$

we immediately get (4.4). Hence, to obtain (4.3), it only remains to prove that

$$F(v^\varepsilon) \to F(v) \quad \text{strongly in } L^{\overline{q}}(\Omega). \tag{4.5}$$

But this is just a consequence of the following well-known result (see [9]):

Theorem 4.2.1. *Let $H : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, i.e.*

(a) *For every v the function $H(\cdot, v)$ is measurable with respect to $x \in \Omega$.*

(b) *For every (a.e.) $x \in \Omega$, the function $H(x,\cdot)$ is continuous with respect to v.*

Moreover, if we assume that there exists a positive constant C such that

$$|H(x,v)| \leqslant C(1+|v|^{r/t}),$$

with $r \geqslant 1$ and $t < \infty$, then the map $v \in L^r(\Omega) \mapsto H(x, v(x)) \in L^t(\Omega)$ is continuous in the strong topologies.

Indeed, since

$$|F(v)| \leqslant C(1+|v|^{q+1}),$$

then by applying the above theorem for $H(x,v) = F(v)$, $t = \overline{q}$ and $r = (2n/(n-2)) - r'$, with $r' > 0$ such that $q+1 < r/t$ and using the compact injection $H^1(\Omega) \hookrightarrow L^r(\Omega)$ we easily get (4.5).

5. Proof of the main result

This section is devoted to the proof of Theorem 3.1.

Proof. Let u^ε be the solution of the variational inequality (2.5) and let $P^\varepsilon u^\varepsilon$ be the extension of u^ε given by Lemma 4.1.1. It is not difficult to see that $P^\varepsilon u^\varepsilon$ is bounded in $H_0^1(\Omega)$. So by extracting a subsequence, one has

$$P^\varepsilon u^\varepsilon \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega). \tag{5.1}$$

Let $\varphi \in \mathscr{D}(\Omega)$. Classical regularity results imply $\chi_i \in L^\infty$. The boundedness of χ_i and φ ensure that there exists $M \geqslant 0$ such that

$$\left\|\frac{\partial \varphi}{\partial x_i}\right\|_{L^\infty}\|\chi_i\|_{L^\infty} < M.$$

Let

$$v^\varepsilon = \varphi + \sum_i \varepsilon \frac{\partial \varphi}{\partial x_i}(x)\chi_i\left(\frac{x}{\varepsilon}\right). \tag{5.2}$$

Then $v^\varepsilon \in K^\varepsilon$ and we can take it as a test function in (2.5). Moreover, $v^\varepsilon \to \varphi$ strongly in $L^2(\Omega)$.

Let us compute Dv^ε:

$$Dv^\varepsilon = D\varphi + \sum_i \frac{\partial \varphi}{\partial x_i}(x) D\chi_i\left(\frac{x}{\varepsilon}\right) + \varepsilon \sum_i D\frac{\partial \varphi}{\partial x_i}(x)\chi_i\left(\frac{x}{\varepsilon}\right).$$

So

$$Dv^\varepsilon = \sum_i \frac{\partial \varphi}{\partial x_i}(x)\left(\mathbf{e}_i + D\chi_i\left(\frac{x}{\varepsilon}\right)\right) + \varepsilon \sum_i D\frac{\partial \varphi}{\partial x_i}(x)\chi_i\left(\frac{x}{\varepsilon}\right),$$

where $\mathbf{e}_i$, $1 \leqslant i \leqslant n$, are the elements of the canonical basis in $\mathbb{R}^n$.

Using v^ε as a test function in (2.5) we can write

$$D_f \int_{\Omega^\varepsilon} Du^\varepsilon Dv^\varepsilon \,\mathrm{d}x \geqslant \int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon)\,\mathrm{d}x + D_f \int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon \,\mathrm{d}x - a\langle \mu^\varepsilon, G(v^\varepsilon) - G(u^\varepsilon)\rangle.$$

In fact, we have

$$\begin{aligned} &D_f \int_\Omega DP^\varepsilon u^\varepsilon (\widetilde{Dv^\varepsilon})\,\mathrm{d}x \\ &\geqslant \int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon)\,\mathrm{d}x + D_f \int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon \,\mathrm{d}x - a\langle \mu^\varepsilon, G(v^\varepsilon) - G(u^\varepsilon)\rangle. \end{aligned} \tag{5.3}$$

Denote

$$\rho Q \mathbf{e}_j = \frac{1}{|Y^*|} D_f \int_{Y^*} (D\chi_j + \mathbf{e}_j)\,\mathrm{d}y, \tag{5.4}$$

where $\rho = |Y^*|/|Y|$. Neglecting the term $\varepsilon \sum_i D\frac{\partial \varphi}{\partial x_i}(x)\chi_i(\frac{x}{\varepsilon})$ which actually tends strongly to zero, we can immediately pass to the limit in the left-hand side of (5.3). Hence

$$D_f \int_\Omega DP^\varepsilon u^\varepsilon \widetilde{Dv^\varepsilon}\,\mathrm{d}x \to \int_\Omega \rho Q Du D\varphi \,\mathrm{d}x. \tag{5.5}$$

It is not difficult to pass to the limit in the first term of the right-hand side of (5.3). Indeed, since $v^\varepsilon \to \varphi$ strongly in $L^2(\Omega)$, we get

$$\int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon)\,\mathrm{d}x = \int_\Omega f\chi_{\Omega^\varepsilon}(v^\varepsilon - P^\varepsilon u^\varepsilon)\,\mathrm{d}x \to \int_\Omega f\rho(\varphi - u)\,\mathrm{d}x. \tag{5.6}$$

For the third term of the right-hand side of (5.3), by assuming the growth condition (2.4) for the single-valued maximal monotone graph g and by using the preliminary results from Section 4.2 (more precisely (4.3) written for G and for $v^\varepsilon = P^\varepsilon u^\varepsilon$), we get

$$G(P^\varepsilon u^\varepsilon) \rightharpoonup G(u) \quad \text{weakly in } W_0^{1,\overline{q}}(\Omega)$$

which in combination with the convergence (4.1) written for $h = 1$, leads to

$$\langle \mu^\varepsilon, G(P^\varepsilon u^\varepsilon)\rangle \to \frac{|\partial T|}{|Y|}\int_\Omega G(u)\,\mathrm{d}x.$$

In a similar manner, we obtain

$$\langle \mu^\varepsilon, G(v^\varepsilon)\rangle \to \frac{|\partial T|}{|Y|}\int_\Omega G(\varphi)\,\mathrm{d}x$$

and hence we get

$$a\langle \mu^\varepsilon, G(v^\varepsilon) - G(P^\varepsilon u^\varepsilon)\rangle \to a\frac{|\partial T|}{|Y|}\int_\Omega (G(\varphi) - G(u))\,\mathrm{d}x. \tag{5.7}$$

It remains to pass to the limit in the second term of the right-hand side of (5.3) only. We start by writting down the subdifferential inequality

$$D_f\int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon\,\mathrm{d}x \geqslant D_f\int_{\Omega^\varepsilon} Dw^\varepsilon Dw^\varepsilon\,\mathrm{d}x + 2D_f\int_{\Omega^\varepsilon} Dw^\varepsilon(Du^\varepsilon - Dw^\varepsilon)\,\mathrm{d}x, \tag{5.8}$$

for any $w^\varepsilon \in H_0^1(\Omega)$. We follow the same procedure as above and by choosing

$$w^\varepsilon = \overline{\varphi} + \sum_i \varepsilon\frac{\partial\overline{\varphi}}{\partial x_i}(x)\chi_i\Big(\frac{x}{\varepsilon}\Big),$$

where $\overline{\varphi}$ enjoys similar properties as the corresponding φ, we may pass to the limit in the right-hand side of inequality (5.8) and we get

$$\liminf_{\varepsilon\to 0} D_f\int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon\,\mathrm{d}x \geqslant \int_\Omega \rho Q D\overline{\varphi} D\overline{\varphi}\,\mathrm{d}x + 2\int_\Omega \rho Q D\overline{\varphi}(Du - D\overline{\varphi})\,\mathrm{d}x,$$

for any $\overline{\varphi} \in \mathscr{D}(\Omega)$. But since $u \in H_0^1(\Omega)$, by density, we conclude

$$\liminf_{\varepsilon\to 0} D_f\int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon\,\mathrm{d}x \geqslant \int_\Omega \rho Q Du Du\,\mathrm{d}x. \tag{5.9}$$

We put together (5.5)–(5.7) and (5.9) and obtain

$$\int_\Omega \rho Q Du D\varphi\,\mathrm{d}x \geqslant \int_\Omega f\rho(\varphi - u)\,\mathrm{d}x + \int_\Omega \rho Q Du Du\,\mathrm{d}x - a\frac{|\partial T|}{|Y|}\int_\Omega (G(\varphi) - G(u))\,\mathrm{d}x,$$

for any $\varphi \in \mathscr{D}(\Omega)$ and hence by density for any $v \in H_0^1(\Omega)$.

So, finally, we get

$$\int_\Omega Q Du D(v - u)\,\mathrm{d}x \geqslant \int_\Omega f(v - u)\,\mathrm{d}x - a\frac{|\partial T|}{|Y^*|}\int_\Omega (G(\varphi) - G(u))\,\mathrm{d}x,$$

which gives exactly the limit problem (3.1). This ends the proof of Theorem 3.1. □

Remark 5.1. We can treat in a similar manner the case of a multi-valued maximal monotone graph, which includes various semilinear classical boundary-value problems such as Dirichlet or Neumann problems, Robin boundary conditions, Signorini's unilateral conditions, climatization problems (see, for instance, [1, 2, 6]). For the non-stationary case, see also [5] and [8].

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