# ANALYTICAL DESCRIPTION OF CHAOTIC OSCILLATIONS IN A TOROIDAL THERMOSYPHON

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ABSTRACT

Multiple time scales and singular perturbation techniques are used to describe the ordinary and the chaotic oscillations due to natural convection in a fluid loop subject to a known external heat flux. The turbulent flow in the loop is modelled using the hydraulic approximation with a quadratic friction law. No steady solutions exist if the heat is added mainly to the top half and extracted from the bottom half of the loop, and two steady convective solutions may exist if one proceeds otherwise; these convective solutions may loose stability when the heat input is shifted from the side toward the bottom. The instability leads, first, to a periodic convective flow and then, after a period doubling Feigenbaum cascade, to a chaotic motion. An intermittent type transition from limit cycles to chaos is also found in the analysis. The transition to chaos can be described in terms of a non-invertible return map, obtained by singular perturbation techniques for loops with long length, when the system becomes strongly dissipative.

## 1. Introduction and Formulation

Oscillatory motion due to natural convection in closed fluids loops has been described often in literature, after the pioneering work of Keller[3] and Welander[9]. The analyses correspond to different forms of the closed loops and different forms of the external heat input to the loop.

In most of the analyses the flow is modelled using the Boussinesq and the hydraulic engineering approximations to describe the mean velocity and the mean temperature across the pipe. The velocity $v$ is only a function of $t$ if the diameter $D$ of the pipe is constant. The temperature $T$ is a function of $t$ and of the coordinate $l$ along the loop, of a length $L$ large compared with $D$. Wall friction is modelled by a linear or quadratic law in terms of the fluid velocity. The heat exchange between the fluid and the pipe wall is similarly modelled, or the heat input per unit length of pipe and unit time is specified, as we shall do in the present analyses.

The conservation equations take the form*

$$\rho_0 L\frac{dv}{dt} = -\frac{\lambda L}{2D}\rho_0 v|v| + \rho_0 \alpha g \oint T\frac{dz}{dl}dl \ , \tag{1}$$

*The equations describing the flow in a duct of variable section can easily be written in the same form.

*FLUID PHYSICS, LECTURE NOTES OF SUMMER SCHOOLS*
M. G. Velarde and C. I. Christov, eds., World Scientific, 1994.

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial l} = q(l) \;, \tag{2}$$

if the hydraulic approximation, of uniform velocity and temperature across the pipe, is used to describe the flow of a liquid of density weakly dependent on temperature

$$\rho = \rho_0 - \rho_0\alpha(T - T_0) \;, \tag{3}$$

where $T_0$ is the mean temperature along the pipe, not changing with time if the total heat input $\oint q(l)dl = 0\,; \rho_0$ is the corresponding density, and $\alpha$ the thermal expansion coefficient, assumed to be small enough so that $\alpha(T - T_0) \ll 1$ to justify the Boussinesq approximation.

The momentum equation has been integrated along the loop; the last term in eq.(1) results from the buoyancy force, with $z(1)$ equal to mean height of the local cross section of the pipe. The first term in the right hand side of eq.(1) represents the wall friction forces, with $\lambda$ equal to the Darcy-Weissbach friction coefficient that we will consider to be constant; corresponding to large Reynolds number turbulent flows in rough pipes. The heat input to the fluid per unit length of pipe and time has been divided by $\rho_0 c_l \pi D^2/4$, with $c_l$ equal to the specific heat of the fluid, to yield the function q(l) considered to be known in our analysis of Sen et al.[8].

For the solution of eqs.(1) and (2) we shall use a Galerkin type technique, similar to that used by Saltzman[7] in the analysis of convection in a heated horizontal liquid layer. A truncated form of the resulting infinite set of ordinary differential equations was used in the famous analysis of Lorenz[4] that led, as we shall also find for problem (1)-(2), to chaotic oscillations. The method has been used in connection with the toroidal thermosiphon problem by Malkus[5], Yorke and Yorke[10] and Sen et al.[8]

We begin by writing the data $q(l)$ and $dz/dl$, as well as unknown function $T(l,t)$ in the form of a Fourier series expansion, involving the angle variable $\phi = 2\pi l/L$

$$q(l) = \sum_{n=1}^{\infty}(V_n \sin n\phi + W_n \cos n\phi) \;, \tag{4.a}$$

$$\frac{dz}{dl} = \sum_{n=1}^{\infty}(A_n \sin n\phi + B_n \cos n\phi) \;, \tag{4.b}$$

$$T - T_0 = \sum_{n=1}^{\infty}[S_n(t) \sin n\phi + C_n(t) \cos n\phi] \;, \tag{4.c}$$

By the appropriate choice of the origin of $l$ we can write $B_1 = 0$. Notice that for a circular loop $dz/dl = \sin\phi$, so that $A_n = B_n = 0$, except for $n = 1$, when $A_1 = 1$.

When the expansions (4.a)-(4.c) are used with eqs.(1) and (2) we obtain the system

$$\rho_0 L\frac{dv}{dt} = -\frac{\lambda L}{2D}\rho_0 v|v| + \rho_0\alpha g\frac{L}{2}\sum_{n=1}^{\infty}(A_n S_n + C_n B_n) \;, \tag{5}$$

$$\frac{dS_n}{dt} = n\frac{2\pi v}{L}C_n + V_n \ , \tag{6.a}$$
$$\frac{dC_n}{dt} = -n\frac{2\pi v}{L}S_n + W_n \ , \tag{6.b}$$

that determines the unknown functions $v(t)$, $S_n(t)$, $C_n(t)$ in terms of their initial values $v(0)$, $S_n(0)$, $C_n(0)$.

For a circular loop the last term in eq.(5) reduces to $\rho_0 \alpha g L S_1/2$, so that (5) and the two eqs.(6.a),(6.b) corresponding to $n = 1$, form a set of three nonlinear equations for the functions $v(t)$, $S_1(t)$, $C_1(t)$; when for laminar flow $\lambda|v|$ is taken as a constant, the resulting equations are very similar to those of the Lorenz system. We shall consider the turbulent case, when for large enough Reynolds numbers $\lambda$ can be taken as constant. The equations will be written in nondimensional form, using $t_0, v_0$ and $S_0 = C_0$, defined by the relations.

$$\frac{\lambda L}{D}\rho_0 v_0^2 = \rho_0 \alpha g L C_0 \ , \qquad \frac{C_0}{t_0} = 2\pi\frac{v_0 C_0}{L} = |W_1| \ ,$$

as units for $t, v, S_1$ and $C_1$.

Then we obtain the system of equations

$$\varepsilon v_t = s - v|v| \ , \tag{7.a}$$
$$s_t = cv + \varepsilon a \ , \tag{7.b}$$
$$c_t = -sv + 1 \ , \tag{7.c}$$

involving only the parameters $\varepsilon$ and $a$ defined by

$$\varepsilon = \frac{4\pi D}{\lambda L} \ , \quad \varepsilon a = \frac{V_1}{|W_1|} \ . \tag{8}$$

The parameter $\varepsilon$ measures the effect of the fluid inertia; $\varepsilon \ll 1$ for long thin pipes.

We consider when writing (7) that $W_1$ is positive, so that the heat is added mainly on the bottom half of the loop and extracted on the upper half. The parameter $\varepsilon a$ measures the vertical asymmetry of the heat input; when $a = 0$ the loop is heated symmetrically with respect to the vertical, while for $a\varepsilon \gg 1$ the heat is added and extracted mainly in the horizontal direction. When heating the loop from above $W_1$ is negative, and then the right hand side of eq.(7.c) should be replaced by $-(sv + 1)$; in this case that we will not discuss in this paper there are no steady solutions to the problem (7). We could account for the effects of forced flow with a pump by adding a constant term $p$ to the right–hand side of (7a).

Eqs.(7) must be solved with initial conditions

$$s = s_1 \ , \quad c = c_1 \ , \quad v = v_1 \quad \text{at} \quad t = 0 \tag{9}$$

that specify the initial state of the system.

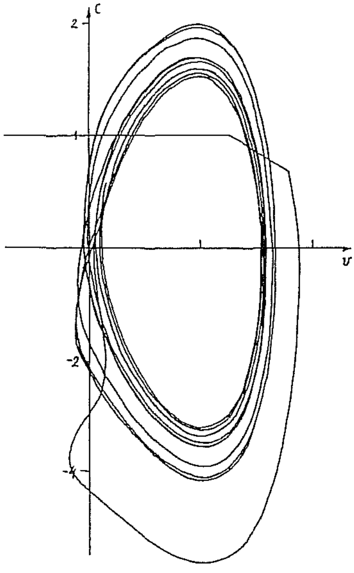



Figure 1: (a)

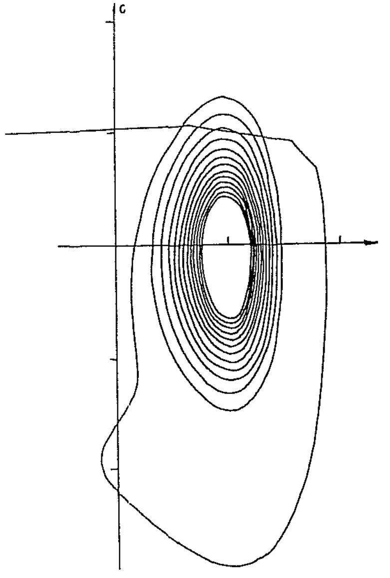



Figure 1: (b)

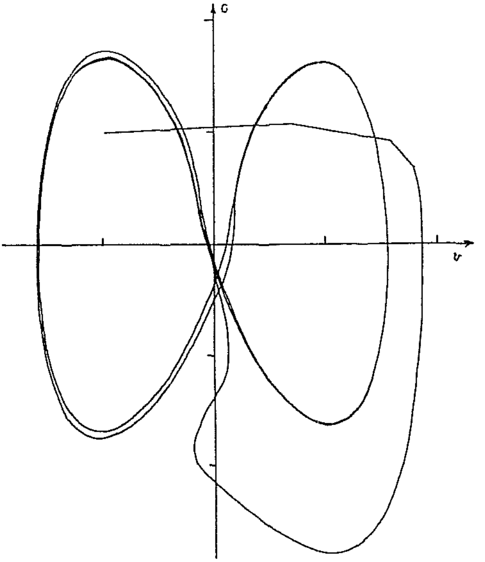



Figure 1: (c)

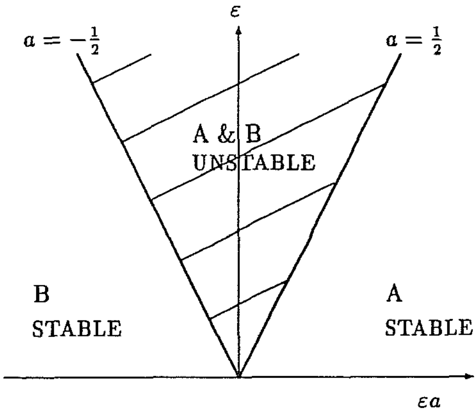



Figure 2:

Typical trajectories are given for $\varepsilon = 0.1$ and various values of a at Fig. 1a–1c.

## 2. Steady Solutions and Stability

Eqs.(7) have two steady solutions

$$v = s = 1\,, \quad c = -\varepsilon a \qquad (A) \tag{10.a}$$

$$v = s = -1\,, \quad c = \varepsilon a \qquad (B) \tag{10.b}$$

corresponding to, say, counter-clockwise and clockwise steady convection. When small perturbation are allowed in the initial conditions away from the steady solution (A) the time dependence of the perturbations is of the form $\exp(\lambda t)$, with $\lambda$ given by

$$\varepsilon\lambda^3 + 2\lambda^2 + (\alpha+1)\varepsilon\lambda + 3 = 0\ . \tag{11}$$

We can use the invariance properties of the system (7) under the transformation

$$v \longrightarrow -v\ , \quad s \longrightarrow -s\ , \quad c \longrightarrow -c\ , \quad \text{and} \quad a \longrightarrow -a\ ,$$

to obtain from eq.(11) the characteristic equation for the stability of the steady solution (B), by replacing $a$ by $-a$.

From eq.(11) we conclude that (A) is stable for $a > 1/2$, and (B) is stable for $a < -1/2$. The stability domain in the parameter plane $(\varepsilon, \varepsilon a)$ is shown in Fig. 2.

Notice that for $\varepsilon \ll 1$, the stability boundaries for (A) and (B) are very close to each other, and thus we may expect that a simplified description of the transient response of the system will be possible in this case of long loops with dominant effects of the friction forces. For $\varepsilon \ll 1$ the roots of eq.(11) are given in first approximation by

$$\lambda = -2/\varepsilon \quad \text{and} \quad \lambda = \pm\sqrt{3/2} - \varepsilon(a - 1/2)/4 \ . \tag{12}$$

Thus the perturbation transients involve three different time scales: $\varepsilon$ associated with the first root, 1 the period and $1/\varepsilon$ the time of growth of the oscillations associated with the remaining roots.

## 3. Non–Linear Transients

In the phase space associated with the autonomous nonlinear system we have a flow with a divergence $-2|v|/\varepsilon$, so that the system is dissipative in general, and strongly dissipative for $\varepsilon \ll 1$.

The method of multiple scales can be used to give a simplified description of the solution of the system (7), with the initial conditions (9), for $\varepsilon \ll 1$.

In a first stage for $t/\varepsilon = t'$ of order unity, when written in terms of $t'$, (7) indicates that in the limit $\varepsilon \to \infty$ the fluid temperature does not change for $t' \approx 1$, so that $c = c_I$, $s = s_I$, while the velocity changes are given by

$$\frac{dv}{dt'} = s_I - v|v| \ , \qquad v(0) = v_I \ . \tag{13}$$

So that in the first short stage $v$ will change from its initial value $v_I$ to the asymptotic value $v_a$ given by

$$v_a|v_a| = s_I \ . \tag{14}$$

Thus in a time of order $\varepsilon$ the point characterizing the state of the system in the phase space moves toward the surface (14); the phase volume shrinks into a nearly two dimensional set around this surface.

The changes in $s$ and $v$ that take place in a second stage, for times $t$ of order unity, will be described using eqs.(7) with as independent variable. In first approximation, for $\varepsilon \ll 1$, and $t \approx 1$, eqs.(7) simplify to

$$s - v|v| = 0 \ , \tag{15.a}$$

$$s_t = cv \ , \tag{15.b}$$

$$c_t = -sv + 1 \ , \tag{15.c}$$

where the effects of inertia have been left out of the momentum equation, and the effects of the asymmetry of heating are out of (15.b). Eqs.(15) lead to the system

$$2v_t = c\frac{v}{|v|} \ , \tag{16.a}$$

$$c_t = 1 - v^2|v| \ , \tag{16.b}$$

to be solved with the initial conditions

$$c(0) = c_I \ , \quad v(0) = v_a \ . \tag{17}$$

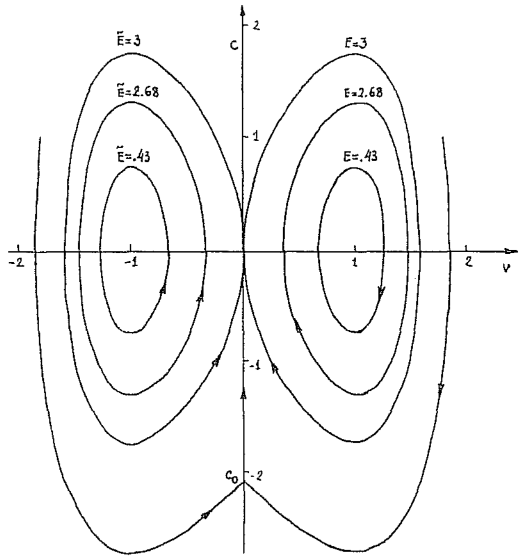



Figure 3:

Eqs.(16) have a first integral

$$\begin{aligned} c^2 + v^4 - 4v + 3 &= E \quad \text{for} \quad v > 0 \,, & (18.\text{a}) \\ c^2 + v^4 + 4v + 3 &= \tilde{E} \quad \text{for} \quad v > 0 \,, & (18.\text{b}) \end{aligned}$$

where $E$ and $\tilde{E}$ are constants, given by the initial conditions (17), unless we wait for times of the order of $1/\varepsilon$, when the effects of the terms neglected in (16) must be taken into account. Eq.(18.a) can be used, together with (16.a) to calculate $v(t)$ by a quadrature. The two forms of (18) reflect the symmetry of eqs.(16) under the transformation $c \to -c$, $v \to -v$, $t \to t$.

If, for example, $s_I < 0$, we must use eq.(18.a) with (16.a), and the solution is periodic if $E < 3$; while for $E > 3$, eq.(18.a) can only be used during a time interval $(0, t_0)$, when $v > 0$. At $t = t_0$, $v(t_0) = 0$. For $E > 3$ and $t > t_0$, we must use the following solution of Eq.(16.a).

$$v = s = 0 \,, \quad c = c_0 + t - t_0 \,, \tag{19}$$

where $c_0 = -(E-3)^{1/2}$. This solution turns out to be stable only for $c < 0$.

The orbits associated with eqs.(16) are given in Fig. 3.

The steady solutions ($c = 0, v = 1$) of (16) are center points of the system, surrounded in the phase plane by periodic orbits corresponding to initial conditions represented by points for which $E < 3$ (or $\tilde{E} < 3$). If the initial conditions are represented by a point outside the domain bounded by the orbits $E = 3$ (or $\tilde{E} = 3$), then the solution of (16) satisfies (18.a) only for $t < t_0$; in the interval $t_0 < t < t_0 - c_0$, we must use eq.(19) if we want to describe well the solution of eqs.(7). During this later period there is no motion if $c < 0$, and the temperature distribution is symmetric with respect to the vertical axis of the loop, with higher temperatures on its upper half; the flow is induced again, as we shall see later, when, due to the external heating, the bottom temperature increases to become higher than that of the top.

The periodic solution of eqs.(16), for $E < 3$, represents well, when $\varepsilon \ll 1$, the solution of eqs.(7), even at times $t \approx 1/\varepsilon$, if $E$ is allowed to change with $\sigma = \varepsilon t$. The evolution of $E$ with $\sigma$ can be obtained by two time scales methods, or in the heuristic form given below.

For initial conditions such that $E < 3, v > 0$ during the second stage of nearly periodic motion, and we can derive from (7) the following equation for $v$.

$$2\frac{d^2v}{dt^2} = 1 - v^3 + \varepsilon\frac{d}{dt}\left(\frac{a}{v} - \frac{v^2}{2} - \frac{1}{v}\frac{d^2v}{dt^2}\right) . \tag{20}$$

If $\varepsilon \ll 1$ we can, neglecting terms of the order of $\varepsilon^2$ , replace in the last term of (20), $d^2v/dt^2$ by its approximate value $(1 - v^3)/2$, and thus obtain, for $t \gg 1/\varepsilon$, the equation

$$2v_{tt} + v^3 - 1 = -\varepsilon\left(a - \frac{1}{2}\right)v^{-2}v_t + O(\varepsilon^2) , \tag{21.a}$$

so that, if we now define $E$ by

$$E = 4v_t + v^4 - 4v + 3 , \tag{22.a}$$

eq.(20) leads to

$$E_t = -4\varepsilon\left(a - \frac{1}{2}\right)v^{-2}v_t^2 + O(\varepsilon^2) . \tag{23.a}$$

Similarly, if the initial conditions are such that $v_a < 0$, and $\tilde{E}$ is defined by

$$\tilde{E} = 4v_t + v^4 + 4v + 3 , \tag{22.b}$$

then the evolution, for $E < 3$, of $\tilde{E}$ is given by

$$\tilde{E}_t = 4\varepsilon\left(a - \frac{1}{2}\right)v^{-2}v_t^2 + O(\varepsilon^2) , \tag{23.b}$$

When we look at the system for times $t = 1$, $E$ does not change significantly and $v$ is a periodic function of $t$, with period $P$ given by

$$P = \int_{V_m}^{V_M} \frac{4dv}{\sqrt{E - 3 + 4v - v^4}} , \tag{24}$$

where $V_m < 1$ and $V_M > 1$ are the two positive roots of

$$E - 3 + 4v - v^4 = 0 \; . \tag{25}$$

The periodic dependence of $v$ on the fast time variable $t$ is given by

$$\int_{V_m}^{v} \frac{2dx}{\sqrt{E-3+4x-x^4}} = \begin{cases} t-\psi & \text{if} \quad 0 < t-\psi < \frac{T}{2} \\ T-t+\psi & \text{if} \quad \frac{T}{2} < t-\psi < T \end{cases} , \tag{26}$$

where in first approximation the phase $\psi$ and amplitude $E$, and therefore $P$, $V_m$ and $V_M$ are functions only of the slow variable $\sigma = \varepsilon t$. To evaluate $E(\sigma)$ we can use (26), together with (23.a) to obtain

$$E_\sigma = -\left(a - \frac{1}{2}\right) G(E) \; , \tag{27}$$

where $G(E) > 0$ is

$$G(E) = \frac{2}{P} \int_{V_m}^{V_M} \frac{4v_t^2}{v^2} dt \; , \tag{28}$$

the average of $4v_t^2 v^{-2}$ on a period. We can also write $G(E)$ in the form

$$G(E) = \frac{4}{P} \int_{V_m}^{V_M} \frac{\sqrt{E-3+4v-v^4}}{v^2} dv \; , \tag{29}$$

and evaluate it in terms of elliptic integrals.

Eq.(27) must be integrated with the condition

$$E(0) = E_a = c_I^2 + v_a^4 - 4v_a + 3 \; . \tag{30}$$

If $a$ is larger than $1/2$ – the critical value of $a$ for the change of stability of the steady solution $(v = 1, c = 1)$ – $E$ decreases with $\sigma$, from its initial value to 0. If on the other hand $a < 1/2$, $E$ increases with $\sigma$ until it reaches values close to 3, when there are intervals in each cycle with $v \ll 1$ and the two time scales analysis fails.

Notice that when $E$ is close to 3, the period $P$ approaches the value

$$P_0 = \int_0^{\sqrt[3]{4}} \frac{4dv}{\sqrt{4v - v^4}} = 6.131 \; , \tag{31}$$

and the rate of growth of $E$ becomes very large; with the main contribution to the value of $G(E)$ coming from the times of passage of the orbit close to the origin of the $(c, v)$ plane. Thus for small values of $(3 - E)$, eq.(27) takes the form

$$E_\sigma = 4P_0^{-1}(-a + \frac{1}{2}) \left[2\pi(3-E)^{-1/2} - 3.577\right] \; , \tag{32}$$

implying that the value $E = 3$ is reached at a finite value of $\sigma$.

The analysis is similar for initial conditions such that $v_a < 0$. We simply need to replace $E$ by $\tilde{E}$, $a$ by $-a$ and $v$ by $-v$ in eqs.(27) and (29). The solution $v = -1, c = 0$ is unstable for $a > -1/2$ when $E$ grows from its initial value toward 3 and the multiple scale analysis, giving nearly periodic motion, fails

## 4. The Critical Transients

For values of $E$ (or of $\tilde{E}$) close to 3 the effects of inertia and heating asymmetry, neglected in (15), become important during the moments of passage of the orbits close to the origin of the phase space, when most of the change in $E$ takes place. The changes in $E$ during the remaining part of the oscillatory cycle can be neglected in first approximation

In order to describe the transients associated with the passage of the orbits of eqs.(7) close to the origin, $s = v = c = 0$, of the phase space, we shall rewrite eqs.(7) with new variables, measured with the characteristic values of $s, v$ and $c$ in the region where the terms in $\varepsilon$ of eqs.(7) are important; the time is measured with the characteristic time of passage as unit. The new variables are

$$y = v\varepsilon^{-2/3}\ , \quad z = s\varepsilon^{-4/3}\ , \quad x = c\varepsilon^{-1/3}\ , \quad \text{and} \quad \tau = t\varepsilon^{-1/3}\ . \tag{33}$$

We thus obtain from eqs.(7) the system

$$y_\tau = z - y|y|\ , \tag{34.a}$$

$$z_\tau = yx + a\ , \tag{34.b}$$

$$x_\tau = 1\ , \tag{34.c}$$

where a term $-\varepsilon^2 zy$, representing the effect of convection on the bottom temperature changes, has been left out in the r.h.s. of eq.(34.c).

From (34) we can derive the following universal equation

$$y_{xx} + 2|y|y_x - yx - a = 0\ , \tag{35}$$

an Airy equation, perturbed by a non-linear term and a forcing term $a$; the two middle terms in eq.(35) appeared in the outer region, the remaining terms are new. This equation can be used to describe the successive passages of the orbit close to the origin. The initial conditions for eq.(35) are provided by the matching conditions with the incoming outer orbit. The orbit can be characterized, in first approximation, by a given value of $E$, because the changes in $E$ during the orbit are, of the order of $\varepsilon$, small compared with those, of the order of $\varepsilon^{2/3}$, encountered during the passage through the critical inner region close to the origin of the phase space.

Close to the origin, the orbits given by eq.(18), associated with values of $E$ (or $\tilde{E}$) close to 3 take the asymptotic form

$$y - \frac{x^4}{4} = A\ , \tag{36.a}$$

$$-y - \frac{x^4}{4} = \tilde{A}\ , \tag{36.b}$$

where

$$4A\varepsilon^{2/3} = 3 - E \ , \quad 4\tilde{A}\varepsilon^{2/3} = 3 - \tilde{E} \ . \tag{37}$$

Notice that $A$ (or $\tilde{A}$) also characterize the maximum (or minimum) values $V_M$ of $v$. Namely

$$V_M = \sqrt[3]{4} - \frac{A\varepsilon^{2/3}}{3} \ . \tag{38}$$

To insure that the solution of eq.(35) matches (36.a) or (36.b) for large negative values of $x$, we must solve eq.(35) with initial conditions provided by the asymptotic form

$$x \to -\infty : \quad y = \frac{x^2}{4} + A_1 + \frac{1-2a}{x} - 4A_1 \frac{1-2a}{3x^3} + \dots \tag{39.a}$$

or a similar expression

$$x \to -\infty : \quad -y = \frac{x^2}{4} + \tilde{A}_1 + \frac{1+2a}{x} - 4\tilde{A}_1 \frac{1+2a}{3x^3} + \dots \tag{39.b}$$

to match with (36.b), obtained from (39a) by replacing $y$ by $-y$, $A_1$ by $\tilde{A}_1$, and $a$ by $-a$. The expressions (39a) and (39b) match with incoming orbits associated with $A = A_1$ or $A = \tilde{A}_1$.

The solution of (35) with the initial conditions (39a) must be obtained numerically; it shows the following asymptotic behaviour, for large $x$,

$$x \to \infty : \quad y = \frac{x^2}{4} + A_2 + \frac{1-2a}{x} - 4A_2 \frac{1-2a}{3x^3} + \dots \tag{40.a}$$

or

$$x \to \infty : \quad -y = \frac{x^2}{4} + \tilde{A}_2 + \frac{1+2a}{x} - 4\tilde{A}_2 \frac{1+2a}{3x^3} + \dots \tag{40.b}$$

For each value of $a$ there is a range of values of $A_1$ that leads to the asymptotic form (40.) with

$$A_2 = R(A_1, a) \ . \tag{41.a}$$

For all the other values of $A_1$, the asymptotic form is of the form (40A) with

$$\tilde{A}_2 = \tilde{R}(A_1, a) \ . \tag{41.b}$$

Similarly, if we solve eq.(35) with initial conditions given by (39b), we obtain for large $x$ the asymptotic forms (40a) or (40b) with

$$A_2 = L(\tilde{A}_1, a) \ , \tag{42.a}$$

$$\tilde{A}_2 = \tilde{l}(\tilde{A}_1, a) \ . \tag{42.b}$$

The functions $R$, $\tilde{R}$, $L$ and $\tilde{L}$ of $A_1$ or $\tilde{A}_1$ have been obtained numerically, by solving eq.(35) with the initial conditions (39a) and are shown in Figs. 4–6 for representative positive values of $a$. We can use the symmetry properties of eqs.(7) under the

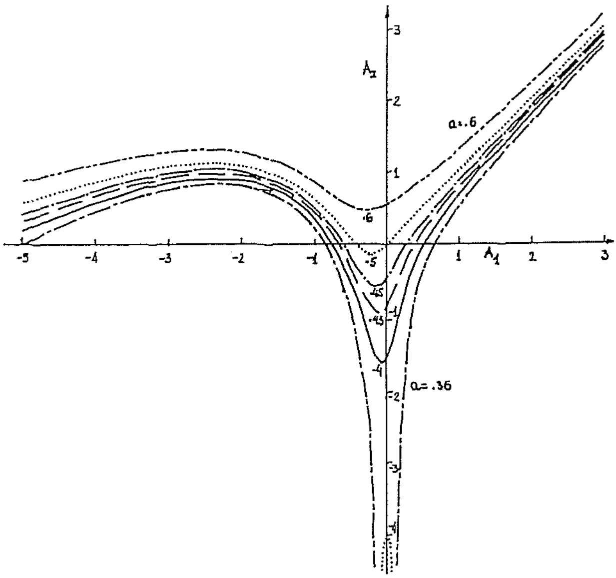



Figure 4:

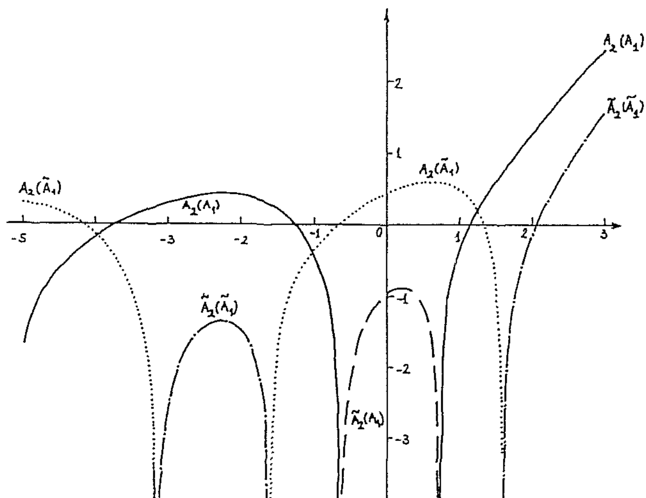



Figure 5: $a = 0.2$

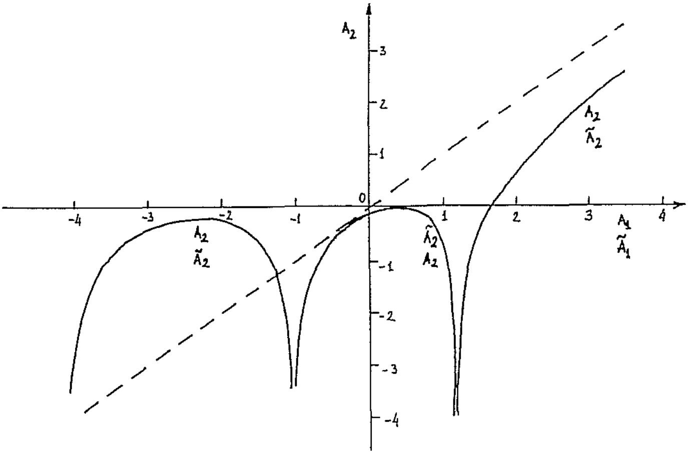


Figure 6: $a = 0$.

transformation ($v, s, c, a$ to $-v, -s, c, -a$) to obtain the functions $R$, $\tilde{R}$, $L$ and $\tilde{L}$ for negative values of $a$.

If we notice that, after leaving the inner region of the phase space, $A$ (or $\tilde{A}$) keep their values, when terms of order $\varepsilon^{1/3}$ are neglected, the orbit that brings the phase point back to the origin, we understand that the relations (41) and (42) provide us with one–dimensional return maps that describe the dynamics of the system. Notice also that, as a consequence of the relation (38), these return maps can also be used to relate the successive maximum values $v_M$ of $v$.

The strongly dissipative character of eqs.(7) for $\varepsilon \to 0$, enabled us to obtain the one–dimensional return maps by asymptotic techniques. Fowler and McGuinness[2] used also these techniques to obtain a one-dimensional return map from the Lorenz equations for large values of the Rayleigh and Prandtl numbers, when the Lorenz system becomes strongly dissipative.

The strongly dissipative character of the problem makes the return maps non invertible, a typical feature of the return maps with chaotic behaviour.

## 5. The Analysis of the Return Maps

The typical form of the return maps for $a > 1/2$ is shown in Fig. 4. The successive values of a will finally grow monotonically, independently of the initial value of $A$; thus the system will end up in the steady solution ($v = s, c = -a\varepsilon$), after a slowly-damped oscillation that can be described by eqs.(26) and (27).

For values of $a < 1/2$ the maps take a form shown in Figs. 4–6. These maps have a fixed point for a negative value of $A$ that tends to zero when $a \to 1/2$. The fixed point represents a periodic oscillation of the mean turbulent flow in the fluid loop,

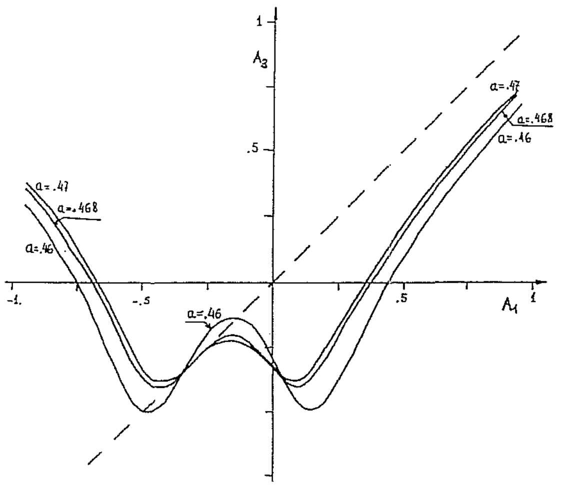



Figure 7:

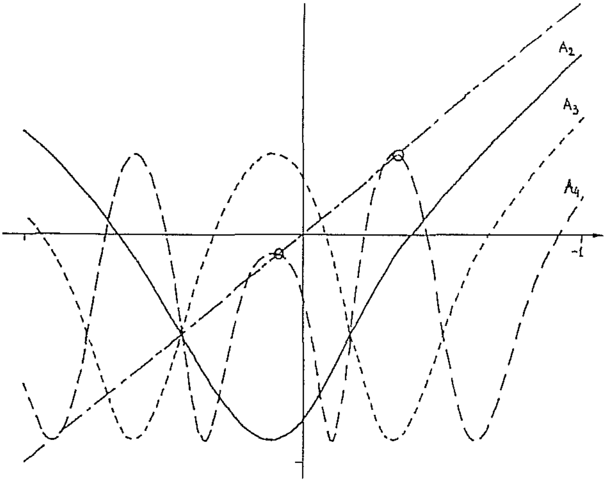
with small negative velocities during a small time interval in each cycle. Thus $a = 1/2$ corresponds to a bifurcation point from stable steady convection, with velocity $v = 1$, to periodic oscillations with maximum velocity $4^{1/3}$ and a minimum velocity zero; the bifurcated orbit is close to the separatrix orbit $E = 3$ of Fig. 3.

The form of the return maps for the transition value $a = 1/2$ is shown in Fig. 3. In this case $a = 1/2$, all the positive values of $A$ are fixed points associated with periodic oscillations. There is a range of values of A that can be reached with initial conditions associated with negative values of $A$.

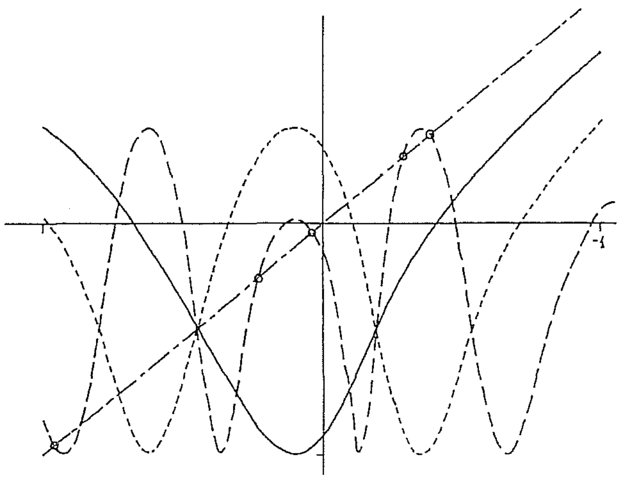
When $a$ falls below 0.469 there is a loss of stability of the bifurcated periodic motion by period doubling to a new periodic motion. Thus the second return maps shown in Fig. 7 indicate the existence for $a = 0.46$ of two stable points of the second return map associated with a periodic orbit (of $2T_0$ in the limit $\varepsilon \to 0$).

With decreasing values of $a$ we encounter period doubling Feigenbaum[1] cascade that leads to chaotic motion for $a < 0.43$, with windows of odd periodic orbits. Thus the first, second and third return maps, plotted in Fig. 8, show the existence of stable and unstable three period orbits for $a = 0.43$ and $a = 0.425$, respectively.

For $a > 0.37$ we do not find values $\tilde{A}_2$ of the type (41b) associated with a range of values of $A_1$. That is, when we enter the critical region of the phase plane $c, v$ from the right we leave the region toward the right. Thus for all values of $a > 0.37$ the periodic or chaotic oscillations correspond to orbits circling the point $v = 1, -c = 0$. However for $a < 0.37$ the return maps as those associated with $a = 0.36$ and $a = 0$, plotted in Figs. 3 and 6, show that for some interval of values of $A_1$, we obtain values $A_2$ of the type (41b), leading to orbits that circle the other steady solution $v = -1, c = 0$.




(a) $a = 0.43$



(b) $a = 0.425$

Figure 8:

Thus in the interval (0.37-0.44) of $a$ we have chaotic oscillations of the type sketched in Fig. 1b, while in the interval (0.37) the chaotic oscillations are of the form sketched in Fig. 1c. In the first case the maximum and minimum values of $v$ are $4^{1/3}$ and 0 in the limit $\varepsilon \to 0$, while these values are $4^{1/3}$ and $-4^{1/3}$ for $|a| < 0.37$.

The change $\delta_{outer}E$ of $E$ in the outer part of the trajectory, of order $\varepsilon$, is represented by the second term in the right hand side of eq.(32). It can be written as $\delta_{outer}E_2 = 4(a-0.5)3.5777\varepsilon$, and it leads to changes in $A$ in the outer part of each orbit given by $\delta_{outer}A = 3.577(-a+0.5)\varepsilon^{1/3}$, equivalent to a shift of the one dimensional return maps upwards by an amount $\delta_{outer}A$.

In the case $a = 0$, or for small values of $a$, the upwards shift of the return maps may be enough, even for moderately small values of $\varepsilon$, to lead to an unstable and a stable limit cycle. For values of $\varepsilon$ smaller than the critical these cycles disappear, and we find chaotic intermittences of the type I, described by Pomeau and Manneville[6].

## 6. Effects of Longitudinal Thermal Diffusion

If we want to account for the effects of longitudinal thermal diffusion we must add a term $\alpha_T \partial^2 T/\partial l^2$ to the right hand side of eq.(2). Here $\alpha_T$ is the longitudinal effective thermal diffusivity that, following Taylor can be approximated in the turbulent flow case by $10.1 v^* D/2$, with the friction velocity $v^* = v(\lambda/8)^{1/2}$.

Then longitudinal thermal diffusion effects introduce the terms

$$-\alpha_T(2\pi n/L)^2\{S_n, C_n\}$$

in the right hand sides of eqs.(6), and the terms $-\varepsilon b|v|\{s,c\}$ – in the right hand sides of eqs.(7.b), (7.c). Here $\epsilon b = 2\pi\alpha_T/v_0 L = 8.8\varepsilon\lambda^{3/2}$, so that in practical turbulent flow cases $b \ll 1$, and thermal diffusion can be neglected. These effects would be important if $b$ were of order one, as we shall see now.

The presence of these terms modifies slightly, for $\varepsilon \ll 1$ and $b \approx 1$, the old steady solutions and, more strongly, the stability properties of the old solution.

For example, these effects introduce a term $-16bvv_t^2$ in the right hand side of eq.(23.a), so that eq.(27) is replaced by

$$E_\sigma = -\left(a - \frac{1}{2}\right) G(E) - 4bH(E) \ , \tag{27'}$$

where

$$H(E) = \frac{2}{P}\int_0^{P/2} vv_t^2 dt = \frac{4}{P}\int_{V_m}^{V_M} v\sqrt{E - 3 + 4v - v^4}\,dv \ .$$

For small values of $E$, $P = 4\pi/\sqrt{6}$, $G = H = E/2$, so that eq.(27') takes the form

$$E_\sigma = -\left[\left(a - \frac{1}{2}\right) + 4b\right]\frac{E}{2} \ , \tag{27''}$$

and the solution

$$v = s = 1 \ , \quad c = -\varepsilon(a - b) \ , \tag{A'}$$

becomes unstable only for $a < a_c = 1/2 - 4b$. At $a = a_c$ we encounter a Hopf reverse bifurcation leading to an unstable limit cycle with an amplitude $E_c$ given by

$$\left(a - \frac{1}{2}\right) G(E_c) = 4bH(E_c) \ ,$$

such that $E_c \to 3$ for $a = 1/2$.

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