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THE EDGE SLIDE GRAPH OF THE *n*-DIMENSIONAL CUBE

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Abstract

The goal of this thesis is to understand the spanning trees of the n-dimensional cube Q_n by understanding their edge slide graph. An edge slide is a move that "slides" an edge of a spanning tree of Q_n across a two-dimensional face, and the edge slide graph is the graph on the spanning trees of Q_n with an edge between two trees if they are connected by an edge slide. Edge slides are a restricted form of an edge move, in which the edges involved in the move are constrained by the structure of Q_n , and the edge slide graph is a subgraph of the tree graph of Q_n given by edge moves.

The signature of a spanning tree of Q_n is the *n*-tuple (a_1, \ldots, a_n) , where a_i is the number of edges in the *i*th direction. The signature of a tree is invariant under edge slides and is therefore constant on connected components. We say that a signature is connected if the trees with that signature lie in a single connected component, and disconnected otherwise. The goal of this research is to determine which signatures are connected.

Signatures can be naturally classified as reducible or irreducible, with the reducible signatures being further divided into $strictly\ reducible$ and quasi-irreducible signatures. We determine necessary and sufficient conditions for (a_1, \ldots, a_n) to be a signature of Q_n , and show that strictly reducible signatures are disconnected. We conjecture that strict reducibility is the only obstruction to connectivity, and present substantial partial progress towards an inductive proof of this conjecture. In particular, we reduce the inductive step to the problem of proving under the inductive hypothesis that every irreducible signature has a "splitting signature" for which the upright trees with that signature and splitting signature all lie in the same component. We establish this step for certain classes of signatures, but at present are unable to complete it for all.

Hall's Theorem plays an important role throughout the work, both in characterising the signatures, and in proving the existence of certain trees used in the arguments.

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Contents

A	Abstract		
A	ckno	wledgement	ii
1		roduction	1
	1.1	Overview of this thesis	2
2	Pre	liminaries	5
	2.1	Spanning trees	5
	2.2	The n-dimensional cube (hypercube)	8
		2.2.1 Counting spanning trees of Q_n	9
		2.2.2 The direction and decoupled degree monomials	10
		2.2.3 Signatures of spanning trees of Q_n	11
		2.2.4 Edge slides	13
		2.2.5 Upright spanning trees	15
		2.2.5.1 Encoding spanning trees of the graph $Q_n \dots \dots \dots$	16
		2.2.5.2 Notation for upright spanning trees of Q_n	18
3	Sign	natures of spanning trees of Q_n	20
	3.1	Characterising signatures of spanning trees of Q_n	20
		3.1.1 Matchings and Hall's Marriage Theorem	20
		3.1.2 Hall's Marriage Theorem and signatures of spanning trees of Q_n	23
		3.1.3 Main result	24
	3.2	Classification of signatures of spanning trees of $Q_n \dots \dots \dots \dots$	27
		3.2.1 The excess of a signature of Q_n	27
	3.3	Summary map	33
4	Loc	al moves on upright spanning trees of Q_n	34
	4.1	Introduction	34
	4.2	Local moves	34
		4.2.1 The V -move	35
		4.2.2 The path move	41
	4.3	The local move graph of Q_n	44
		4.3.1 The local move graph of the irreducible signature (2, 2, 3)	44
	4.4	Summary map	45

CONTENTS iv

5	Spli	tting spanning trees and signatures of Q_n	48
	5.1	Introduction	
	5.2	Splitting the Q_n graph	
	5.3	Upright spanning forests	
		5.3.1 Signatures of rooted spanning forests	. 49
		5.3.2 Rooted spanning forests of \mathcal{F}_n^{i+} and \mathcal{F}_n^{i-}	. 49
		5.3.3 Hall's Theorem and signatures of spanning forests of \mathcal{F}_n^{n+}	. 50
	5.4	The edge slide graph of the irreducible signature $(2, 3, 3, 7)$. 51
	5.5	Amenable splitting signatures	. 56
	5.6	Signature moves	. 57
	5.7	Summary map	. 71
6	Red	lucible signatures of Q_n	72
	6.1	Introduction	. 72
	6.2	Preparatory lemmas	. 73
		6.2.1 Labels of spanning trees of Q_n with a reducible signature	
		6.2.2 Partitioning the n-dimensional cube	
		6.2.3 Edge slides of spanning trees of Q_n with reducible signatures	
		6.2.4 The existence of subtrees with different signatures of spanning trees with	
		reducible signatures	. 78
	6.3	The edge slide graph of a reducible signature of Q_n	
	6.4	Summary map	
7	Irre	ducible signatures of Q_n : irreducible splitting signatures	82
•	7.1	Introduction	
	7.2	Preparatory lemmas	
	1.2	7.2.1 Swapping labels in the upper n face	
		7.2.2 Settled trees	
		7.2.2.1 Swapping labels of a settled tree of Q_n in the upper n face	
	7.3	Rearranging the labels in the upper n face	
	1.0	7.3.1 The $(2, 2, 3)$ splitting signature of an irreducible signature of Q_4	
		7.3.2 Unidirectional splitting signatures	
		7.3.3 Super rich splitting signatures	
	7.4		
	7.4	Summary map	. 122
8	Irre	ducible signatures of Q_n : transforming irreducible splitting signatures	129
	8.1	Introduction	. 129
	8.2	Preparatory lemmas	. 129
		8.2.1 The main technical tool of this chapter	. 129
		8.2.2 Ordering irreducible splitting signatures	133
		8.2.3 Connecting irreducible splitting signatures	
	8.3	Transforming an irreducible splitting signature of a tree	
	8.4	Summary map	
9	Irre	ducible signatures of Q_n : reducible splitting signatures	139
	9.1	Introduction	
	9.2	Preparatory lemmas	

CONTENTS v

	9.2.1	The existence of a vertex with label n that does not lie above the reducing	î D
	9.2.2	set	
		natures	. 140
9.		nsforming a tree with a reducible splitting signature into a tree with an	
	irred	lucible splitting signature	. 157
9.	4 Sum	mary map	. 159
10 T	wo infi	nite families of irreducible signatures of Q_n .	161
10	0.1 The	irreducible signature $\mathcal{I}_n^{(-1)}$. 161
	10.1.	.1 The edge slide graph of signature $\mathcal{I}_n^{(-1)}$ of Q_n	. 162
	10.1.	.2 Counting the number of upright spanning trees of Q_n with signature $\mathcal{I}_n^{(-)}$	$^{-1)} 164$
10	0.2 The	irreducible signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$. 166
	10.2.	.1 The edge slide graph of signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$ of Q_n	. 167
	10.2.	.2 Counting the number of upright spanning trees of Q_n with signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$	$^{-1)}169$
11 D	iscussio	on	174
Refe	rences		177
Index			

List of Figures

2.1	An example of a spanning tree	6
2.2	An edge move	6
2.3	An example of the tree graph of a connected graph	7
2.4	The construction of 4-cube	9
2.5	Upward and downward edges	11
2.6	A signature of a spanning tree of 2-cube	
2.7	An edge slide	13
2.8	A downward edge slide	
2.9	The edge slide graph of 2-cube	15
2.10	An upright spanning tree of 3-cube	16
2.11	Encoding a spanning tree of 3-cube	17
2.12	Notation for an upright spanning tree of 4-cube	19
3.1	A bipartite graph describing the common interests between women and men	21
3.2	A bipartite graph describing the work experience of the candidates	21
3.3	Perfect matching graph for Problem 2	23
3.4	A bipartite graph of vertices of 3-cube and a signature of 3-cube	24
3.5	An upright spanning tree of Q_3 with signature $(1, 3, 3) \dots \dots \dots \dots$	25
3.6	A bipartite graph between vertices of n -cube and a signature of n -cube	26
4.1	The process of sliding an edge in direction i	
4.2	The V -move	
4.3	The process of the V-move in the case $i \neq j$	
4.4	The process of the V-move in the case $i = j$	
4.5	The path move	
4.6	The process of the path move	42
4.7	Diagram for the proof of Lemma 4.2.6	43
4.8	Upright spanning trees of Q_3 with signature $(2, 2, 3)$	46
4.9	The local move graph of signature $(2, 2, 3) \dots \dots \dots \dots \dots \dots$	47
5.1	A rooted upright spanning forest of 3-cube	49
5.2	An upright spanning forest of 3-cube with signature $(1, 2, 3) \dots \dots \dots$	50
5.3	Sliding a root of an upright spanning forest of 2-cube resulting from splitting an	
	upright spanning tree of 3-cube with the irreducible signature $(2, 2, 3) \dots \dots$	52
5.4	Diagram 1 for the proof of Proposition 5.4.2	53
5.5	Diagram 2 for the proof of Proposition 5.4.2	
5.6	Diagram 3 for the proof of Proposition 5.4.2	54

LIST OF FIGURES vii

5.7	Diagram 4 for the proof of Proposition 5.4.2	55
6.1	The induced subgraphs of 3-cube	
6.2	Spanning trees of the induced subgraphs of 3-cube	
6.3	Edge slides on a spanning tree of 3-cube with a reducible signature	77
7.1	Diagram for the proof of Lemma 7.2.1	85
7.2	Diagram for Case 1 for the proof of Lemma 7.2.2	86
7.3	Diagram for Case 2 for the proof of Lemma 7.2.2	86
7.4	Diagram for the proof of Lemma 7.2.7	88
7.5	Diagram for Corollary 7.2.10	89
7.6	Diagram for the proof of Corollary 7.2.10	
7.7	A settled tree of 4-cube	93
7.8	Diagram for Case 1 for the proof of Lemma 7.2.14	95
7.9	Diagram for Case 2(a) for the proof of Lemma 7.2.14	96
7.10	Diagram 1 for Case 2(b) for the proof of Lemma 7.2.14	97
7.11	Diagram 2 for Case 2(b) for the proof of Lemma 7.2.14	98
7.12	Diagram for Case (b(II)) for the proof of Lemma 7.2.15 \dots	99
7.13	Diagram for the proof of Lemma 7.2.15	100
7.14	Diagram 1 for Example 7.2.16	106
	Diagram 2 for Example 7.2.16	
7.16	Diagram 3 for Example 7.2.16	108
7.17	Diagram for Case 1(a) for the proof of Lemma 7.2.19	109
	Diagram for Case $1(b(I(i)))$ for the proof of Lemma 7.2.19	
7.19	Diagram for Case $1(b(I(ii)))$ for the proof of Lemma 7.2.19	110
7.20	Diagram 1 for Case $1(b(II))$ for the proof of Lemma 7.2.19	111
	Diagram 2 for Case 1(b(II)) for the proof of Lemma 7.2.19	
7.22	Diagram 3 for Case $1(b(II))$ for the proof of Lemma 7.2.19	113
7.23	Diagram 1 for Case 2 for the proof of Lemma 7.2.19	113
	Diagram 2 for Case 2 for the proof of Lemma 7.2.19	
	Diagram 3 for Case 2 for the proof of Lemma 7.2.19	
7.26	Diagram 4 for Case 2 for the proof of Lemma 7.2.19	115
	Diagram 5 for Case 2 for the proof of Lemma 7.2.19	
7.28	Diagram 1 for the proof of Theorem 7.3.1 for signature $(2, 3, 4, 6) \dots \dots$	123
7.29	Diagram 2 for the proof of Theorem 7.3.1 for signature $(2, 3, 5, 5) \dots \dots$	124
	Diagram 3 for the proof of Theorem 7.3.1 for signature $(2, 4, 4, 5) \dots \dots$	
	Diagram 4 for the proof of Theorem 7.3.1 for signature (3, 3, 3, 6)	
	Diagram 5 for the proof of Theorem 7.3.1 for signature $(3, 3, 4, 5) \dots \dots \dots$	
7.33	Diagram 6 for the proof of Theorem 7.3.1 for signature $(3, 4, 4, 4) \dots \dots$	128
8.1	Diagram for the proof of Lemma 8.2.1	130
8.2	Diagram 1 for Case $(a(I(i)))$ for the proof of Lemma 8.2.1	137
8.3	Diagram 2 for Case $(a(I(i)))$ for the proof of Lemma 8.2.1	137
8.4	Diagram for Case (b) of the proof of Lemma 8.2.1	138
9.1	Flowchart (part 1) for the proof of Lemma 9.2.2	141
9.2	Flowchart (part 2) for the proof of Lemma 9.2.2	142
9.3	Diagram for Case (a) for the proof of Lemma 9.2.2	144

LIST OF FIGURES viii

9.4	Diagram 1 for Case (b) for the proof of Lemma 9.2.2
9.5	Diagram 2 for Case (b) for the proof of Lemma 9.2.2
9.6	Diagram 1 for Case $(b(II(i)))$ for the proof of Lemma 9.2.2
9.7	Diagram 2 for Case $(b(II(i)))$ for the proof of Lemma 9.2.2
9.8	Diagram 1 for Case (b(II(ii))) for the proof of Lemma 9.2.2
9.9	Diagram 2 for Case (b(II(ii))) for the proof of Lemma 9.2.2
	Flowchart (part 1) for the proof of Lemma 9.2.3
	Flowchart (part 2) for the proof of Lemma 9.2.3
	Diagram 1 for Case (a) for the proof of Lemma 9.2.3
	Diagram 2 for Case (a) for the proof of Lemma 9.2.3
	Diagram 1 for Case (b) for the proof of Lemma 9.2.3
	Diagram 2 for Case (b) for the proof of Lemma 9.2.3
	Diagram 1 for Case (b(I)) for the proof of Lemma $9.2.3$
	Diagram 2 for Case (b(I)) for the proof of Lemma 9.2.3 $\dots \dots \dots$
	Diagram 1 for Case (b(II)) for the proof of Lemma 9.2.3 $\dots \dots \dots$
	Diagram 2 for Case (b(II)) for the proof of Lemma 9.2.3 $\dots \dots \dots$
	Diagram 1 for the proof of Theorem 9.1.2
9.21	Diagram 2 for the proof of Theorem 9.1.2
10.1	Diagram 1 for the proof of Theorem 10.1.1
	Diagram 2 for the proof of Theorem 10.1.1
	Diagram 3 for the proof of Theorem 10.1.1
	Diagram 4 for the proof of Theorem 10.1.1
	Diagram 1 for the proof of Theorem 10.2.1
	Diagram 2 for the proof of Theorem 10.2.1
	Diagram 3 for the proof of Theorem 10.2.1