GEOMETRICAL INTERPRETATIONS OF BÄCKLUND TRANSFORMATIONS AND CERTAIN TYPES OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Gauss' *Theorema Egregium* contains a partial differential equation relating the Gaussian curvature K to components of the metric tensor and its derivatives. Well-known partial differential equations such as the Schrödinger equation and the sine-Gordon equation correspond to this PDE for special choices of K and special coördinate systems. The sine-Gordon equation, for example, can be derived via Gauss' equation for $K = -1$ using the Tchebychef net as a coördinate system.

In this thesis we consider a special class of Bäcklund Transformations which correspond to coördinate transformations on surfaces having a specified Gaussian curvature. These transformations lead to Gauss' PDE in different forms and provide a method for solving certain classes of non-linear second order partial differential equations.

In addition, we develop a more systematic way to obtain a coordinate system for a more general class of PDE, such that this PDE corresponds to the Gauss equation.

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Chapter 1

Introduction

1.1 General

The dynamics of interfaces, surfaces, fronts are an important ingredients in numerous nonlinear phenomena arising in classical and quantum physics, and in some cases the dynamics can be modelled by nonlinear partial differential equations (PDEs) that describe the evolution of surfaces in time. As a result of this relationship, the study of the connection between certain types of surfaces and nonlinear PDEs has been one of the classical problems of differential geometry. Curvature, for example, plays an important part in a number of problems of physics and mathematics associated with manifolds.

Often, one has to solve nonlinear PDEs in order to explain the physical phenomena, but solution techniques for nonlinear PDEs are fairly specialized and rare. One of these techniques, a coordinate transformation method, loosely speaking, known as the *Bäcklund Transformation method*, is of interest in this text. It is known [7] that a Bäcklund transformation may be regarded, in geometrical language, as a transformation of a surface S into a new surface $\bar{S}$, where S is a solution of a given PDE, but where the transformed surface $\bar{S}$ may either be a solution of the original PDE or of some other differential equation. Bäcklund transformations, in essence, preserve invariant properties between two differential equations and their solutions, and they relate these equations to one another through a representation of surfaces with the same curvature in some known coordinate systems. They can thus be useful for finding a solution to a given differential equation by relating it to another differential equation with a known solution. In recent times, interest in these transformations have persisted due to their connection with the sine-Gordon equation and its associated soliton theory.

1.2 A Brief Description

The first chapter contains the general introduction and a review of the literature pertaining to the work in this thesis, followed by some definitions and fundamental equations which will be used in the following chapters. In section 1.3 we review some basic definitions which arise in differential geometry. In subsection 1.3.2, the Gauss equation, which plays a central rôle in our discussions, is presented. We then illustrate how some well known PDEs such as the Schrödinger equation, the sine-Gordon equation, the Liouville equation and the Monge-Ampère equation can be generated from the Gauss equation by the appropriate choice of coordinates. In section 1.4 we show how the covariant transformation equations can be used to determine the Bäcklund transformations between two coordinate systems, where each coordinate system represents a specific PDE.

Chapter 2 consists of two major sections. In section 2.1 we look mainly at the solution techniques and Bäcklund transformations developed for various classes of second order quasi-linear partial differential equations [26]. In subsection 2.1.1 we first show how a certain class of second order quasi-linear PDEs of the hyperbolic type can be solved. As an example, a family of solutions for the sine-Gordon equation is derived. The Cauchy problem is then discussed and the sine-Gordon equation is used as an illustration. Further, we establish that the solution obtained for the Cauchy problem of the sine-Gordon equation corresponds to a Beltrami surface. Our approach in deriving solutions through Bäcklund transformations is further illustrated through an example, where a soliton solution of the sine-Gordon equation is used to derive a solution to the Schrödinger equation. Subsections 2.1.2 and 2.1.3 deal with some classes of second order quasi-linear PDEs of the parabolic type and the elliptic type, respectively. Illustrative examples are given wherever appropriate.

In section 2.2, we show how the same technique used in section 2.1 can be implemented to solve a fully non-linear second order PDE, the Monge-Ampère equation, and further discuss the solution to the Cauchy problem for this equation. Finally, we discuss some relationships among the sine-Gordon, the Monge-Ampère and the Schrödinger equations, which Bäcklund transformations elucidate and discuss briefly how a more general class of Monge-Ampère equation can be solved using Bäcklund transformations.

The topics in Chapter 3 pertain to a systematic way of obtaining a coordinate system corresponding to a more general class of PDEs which can be interpreted as the Gauss equation. This complements the material in Chapter 2, where we established some useful solution techniques via Bäcklund transformations for some classes of PDEs. It is noted that in generalising the technique to include a non-constant Gaussian curvature function, we extend significantly to class of PDEs for which this solution method is available.

Section 3.1 provides a brief introduction to the remainder of Chapter 3. Section 3.2 deals with the preliminaries required for the sections to follow. We also provide with a brief review of the literature pertaining to the material in Chapter 3 in this section.

In section 3.3 a complete characterisation is given for the class of differential equations of type

$$u_t = F\left(K(x,t), u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}\right).$$

Illustrative examples such as the generalised Burgers equation and the generalised KdV equation are provided to show how we can, in principle, determine the coordinate systems for these types of equations.

Section 3.4 consists the complete characterisation for the class of differential equations of type

$$u_{xt} = F\left(K(x,t), u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}\right).$$

Once again, we provide illustrative examples to show how we can determine the coordinate systems for these types of equations. The generalised sine-Gordon equation and the generalised sinh-Gordon equation are used as examples.

In Chapter 4, we conclude the thesis by summing up particular results and proposing certain matters which need further investigation.

1.3 Some Geometrical Aspects

In this section, we review some basic definitions which arise in differential geometry [7, 34, 35]. Let S be a surface in E^3, Euclidean 3-space, and let Γ be a curve on S. If (u, v) denote curvilinear coordinates on S, then the curve Γ can be described by an implicit relationship of the form

$$\phi(u, v) = 0.$$

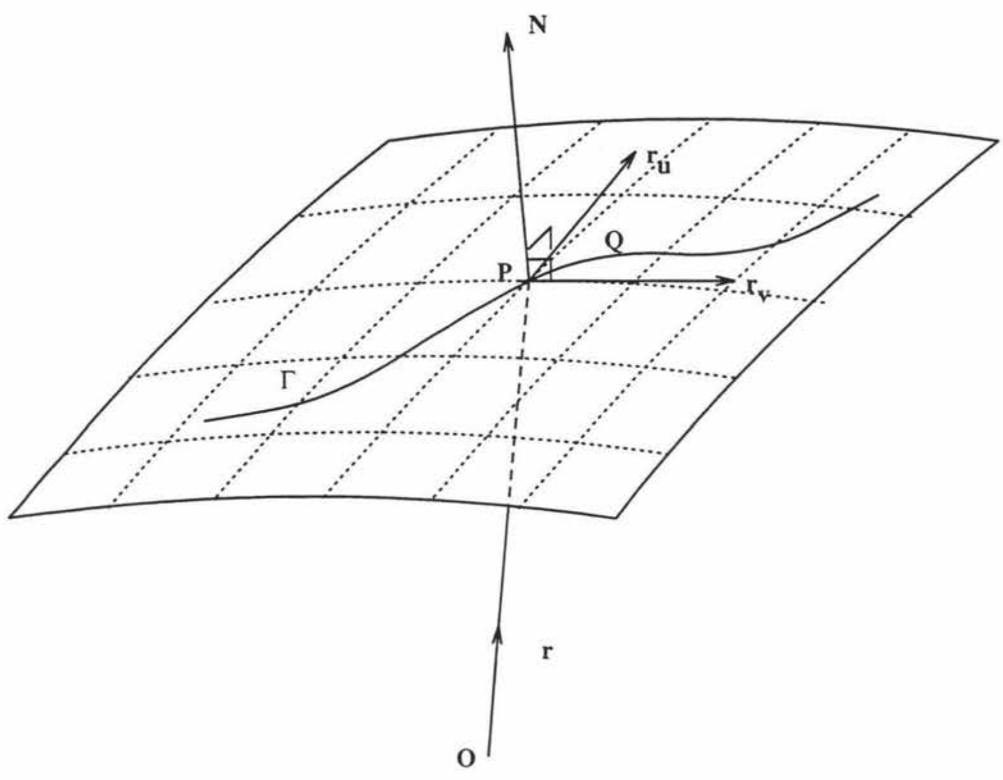

Figure 1.1: The surface S and the curve Γ

The curve Γ defined above can also be given in parametric form:

$$u = u(t), \qquad v = v(t). \tag{1.1}$$

Let $\mathbf{r}$ be the position vector of a point P on the curve. Then the vector $d\mathbf{r}/dt = \dot{\mathbf{r}}$, given by

$$\dot{\mathbf{r}} = \mathbf{r}_u \, \dot{u} + \mathbf{r}_v \, \dot{v}, \tag{1.2}$$

is tangent to the curve and therefore to the surface (cf. Fig.1.1). Here the subscripts u and v denote partial differentiation with respect to u and v respectively. Equation (1.2) can also be written (in a form independent of the choice of parameter) as,

$$d\,\mathbf{r} = \mathbf{r}_u \, du + \mathbf{r}_v \, dv. \tag{1.3}$$

If Q is in a neighbourhood of P on the curve, then the distance ds, between P and Q on the curve can be expressed as

$$I = ds^2 = d\mathbf{r}.d\mathbf{r} = E\,du^2 + 2F\,du\,dv + G dv^2, \tag{1.4}$$

where

$$E = \mathbf{r}_u.\mathbf{r}_u \,,\; F = \mathbf{r}_u.\mathbf{r}_v \,,\; G = \mathbf{r}_v.\mathbf{r}_v. \tag{1.5}$$

The quadratic form in equation (1.4) is called the *first fundamental form* for the surface S.

The functions E, F and G depend on u and v and are called the *components of the metric tensor* or the *components of the first fundamental form.*

The quantity

$$|\mathbf{r}_u \wedge \mathbf{r}_v| = H = \sqrt{EG - F^2}\,, \tag{1.6}$$

corresponds to the *differential area element.* The angle θ between the coordinate curves is

$$\cos\,\theta = \frac{\mathbf{r}_u.\mathbf{r}_v}{|\mathbf{r}_u|\,|\mathbf{r}_v|} = \frac{F}{\sqrt{EG}}. \tag{1.7}$$

If $\mathbf{t}$ is the unit tangent vector at P to the curve Γ on the surface S and $\mathbf{N}$ is the unit surface normal, then the curvature vector of Γ at P, $\mathbf{k}$, can be decomposed as

$$d\,\mathbf{t}/ds = \mathbf{k} = \mathbf{k}_n + \mathbf{k}_g\,,$$

where $\mathbf{k}_n$ is parallel to $\mathbf{N}$ and orthogonal to $\mathbf{k}_g$ (see Fig. 1.2).

The vector $\mathbf{k}_g$ is called the *tangential curvature vector* or *geodesic curvature vector* and

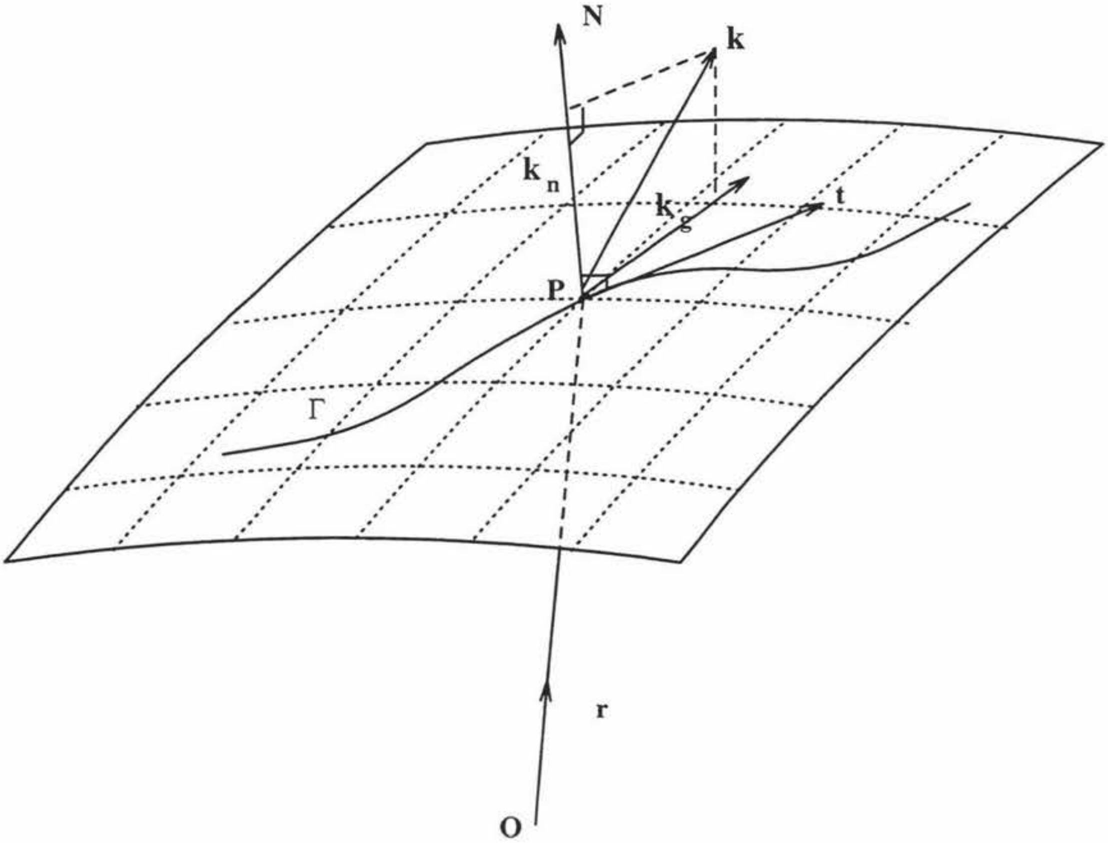

Figure 1.2: The *normal* and *tangential* curvature vectors

the vector $\mathbf{k}_n$ is called the *normal curvature vector*. The latter can be expressed by

$$\mathbf{k}_n = \kappa_n \mathbf{N},$$

where κ_n is known as the *normal curvature*. The normal curvature is given by

$$\kappa_n = \frac{e\,du^2 + 2f\,du\,dv + g\,dv^2}{E\,du^2 + 2F\,du\,dv + G dv^2} \tag{1.8}$$

where, in terms of vector triple products,

$$e = \frac{(\mathbf{r}_{uu}, \mathbf{r}_u, \mathbf{r}_v)}{H}, \quad f = \frac{(\mathbf{r}_{uv}, \mathbf{r}_u, \mathbf{r}_v)}{H}, \quad g = \frac{(\mathbf{r}_{vv}, \mathbf{r}_u, \mathbf{r}_v)}{H}. \tag{1.9}$$

The numerator of equation (1.8), written as

$$II = -\,d\mathbf{r}.d\mathbf{N} = e\,du^2 + 2f\,du\,dv + g\,dv^2 \tag{1.10}$$

is defined as the *second fundamental form.* The functions e, f and g are known as the *components of the second fundamental form.*

1.3.1 Gaussian and Mean Curvatures

The normal curvature given in equation (1.8), when considered in the direction $\lambda = du/dv$ is

$$\kappa_n = \frac{e + 2f\,\lambda + g\,\lambda^2}{E + 2F\,\lambda + G\,\lambda^2} = \kappa_n(\lambda)\,. \tag{1.11}$$

Extrema for κ_n w.r.t λ are characterized by

$$d\kappa_n/d\lambda = 0\,,$$

and this condition implies

$$\kappa_n = \frac{II}{I} = \frac{f + g\lambda}{F + G\lambda} = \frac{e + f\lambda}{E + F\lambda}\,.$$

The above equation indicates that

$$(Fg - Gf)\,\lambda^2 + (Eg - Ge)\,\lambda + (Ef - Fe) = 0,$$

which determines two directions dv/du, in which κ_n obtains an extreme value, unless II vanishes or unless II and I are proportional. One value must be maximum, the other a minimum. These directions are called the *directions of principal curvature* or *curvature directions* and the corresponding values for κ_n denoted by κ_1 and κ_2 are defined as the *principal curvatures.*

The quantities

$$\mathcal{H} = \frac{1}{2}\,(\kappa_1 + \kappa_2) = \frac{Eg - 2fF + eG}{2\,(EG - F^2)} \tag{1.12}$$

and

$$K = \kappa_1 \kappa_2 = \frac{eg - f^2}{EG - F^2} \tag{1.13}$$

are invariants, and are called respectively the *mean curvature* and the *Gaussian curvature* of the surface.

1.3.2 The Gauss Equation and some well-known PDEs

A key result in classical differential geometry is Gauss' *Theorema Egregium* [34], which asserts that the Gaussian curvature depends purely on the components of the first fundamental form. Specifically, we have the *Gauss Equation*:

$$K(u,v) = \frac{1}{2H}\left(\left(\frac{F}{HE}E_v - \frac{1}{H}G_u\right)_u + \left(\frac{2}{H}F_u - \frac{1}{H}E_v - \frac{F}{HE}E_u\right)_v\right). \tag{1.14}$$

This equation will play a central rôle in our discussion. Many nonlinear and, some linear PDEs of interest, correspond to the Gauss equation on a surface of prescribed curvature parametrized in an appropriate coordinate system. In certain coordinate systems the Gauss equation takes a particularly simple form. Well known partial differential equations such as the Schrödinger equation, the sine-Gordon equation, the Liouville equation and the Monge-Ampère equation are the classical examples[4, 18]. We illustrate below how these PDEs can be generated from the Gauss equation by the appropriate choice of coordinates.

1.3.2.1 The Schrödinger Equation

Our first example is the Schrödinger equation,

$$\psi_{uu} + K(u,v)\,\psi = 0,$$

which, as will be seen, corresponds to the Gauss equation for surfaces of Gaussian curvature $K(u,v)$ in geodesic polar coordinates.

In a neighbourhood of every point on a smooth surface, a geodesic polar coordinate system exists[34]; hence, we can always construct such a local coordinate system for the surface with Gaussian curvature $K(u,v)$. For a geodesic polar coordinate system $E = 1$ and $F = 0$, equation (1.4) reduces to

$$ds^2 = du^2 + Gdv^2,$$

and equation (1.14) becomes,

$$K(u,v) = -G^{-1/2}(G^{1/2})_{uu}.$$

Using $H = \sqrt{G}$ we have,

$$H_{uu} + K(u,v)H = 0. \tag{1.15}$$

The solution to Schrödinger's equation (1.15) thus corresponds to the differential area element for a surface of curvature $K(u,v)$ in the geodesic coordinates.

1.3.2.2 The sine-Gordon Equation

When $E = G = 1$, the coordinate system forms a *Tchebychef Net* [6, 34], which exists for sufficiently smooth surfaces[34], and equation (1.4) becomes,

$$ds^2 = du^2 + 2Fdudv + dv^2.$$

If θ is the angle through which the coordinate vector $\mathbf{r}_u$ must be turned to bring it into coincidence with $\mathbf{r}_v$ then we have,

$$F = \cos\theta$$

(from equation (1.7)). Now equation (1.14) takes the form

$$K = \frac{1}{\sqrt{1-F^2}}\left(\frac{1}{H}F_u\right)_v$$

i.e.

$$\theta_{uv} = -K(u,v)\sin\theta. \tag{1.16}$$

This is a second order hyperbolic PDE for the function θ, with u = constant and

$v =$ constant as the characteristics. For $K(u,v) = -1$, we get the familiar *sine-Gordon Equation*,

$$\theta_{uv} = \sin\theta\,. \tag{1.17}$$

1.3.2.3 The Liouville Equation

Let $E = G = 0$ so that, the coordinate curves are the minimal lines. We note that this makes the surface representation complex. Equation (1.14) becomes,

$$(\ln F)_{uv} + KF = 0$$

i.e.

$$\Phi_{uv} + K\,e^{\Phi} = 0, \tag{1.18}$$

where

$$F = e^{\Phi}.$$

For $K =$ constant, equation (1.18) corresponds to the *Liouville Equation.*

1.3.2.4 The Monge-Ampère Equation

Consider a surface described by

$$\mathbf{r} = (u, v, Z(u,v))\ .$$

Then the components E, F and G of the first fundamental form, for graphical coordinates will be given by

$$E = 1 + Z_u^2\,, \quad F = Z_u\,Z_v\,, \quad G = 1 + Z_v^2\,,$$

and thus equation (1.4) becomes

$$ds^2 = (1 + Z_u^2)du^2 + 2Z_uZ_vdudv + (1 + Z_v^2)dv^2.$$

The Gauss equation (1.14) reduces to

$$K(u,v) = \frac{Z_{uu}\,Z_{vv} - Z_{uv}^2}{(1 + Z_u^2 + Z_v^2)^2}, \tag{1.19}$$

which also can be written as

$$Z_{uu}\,Z_{vv} - Z_{uv}^2 - K(u,v)\left(1 + Z_u^2 + Z_v^2\right)^2 = 0$$

which is an equation of the *Monge-Ampère* type.

Certain partial equations can thus be interpreted as statements of Gauss' Theorem on a surface of curvature K in an appropriate coordinate system. This observation motivates a strategy for solving these equations based on Bäcklund transformations which correspond to curvilinear coordinate transformations on the surface defined intrinsically by K.

1.4 Gauss Equation and Bäcklund Transformations

Given a PDE, the idea here is to first find a coordinate system such that the PDE corresponds to the Gauss equation for a surface of known Gaussian curvature. Then we seek another PDE that can be solved, and determine a coordinate system such that this PDE corresponds to the Gauss equation for the same Gaussian curvature. Using the covariant transformation equations for the two determined coordinate systems yields a system of non-linear PDEs. Solutions to this system define the Bäcklund transformations between the two coordinate systems, thus enabling us to obtain solutions to the given PDE by transforming the known solution of the other PDE.

In order to further describe this method, let us consider two partial differential equations $\mathcal{D}(\phi) = 0$ and $\mathcal{E}(\chi) = 0$ which are of the same order. Assume that the PDE $\mathcal{D}(\phi) = 0$ is the given equation to be solved and the other is a PDE with a known solution.

Further, we assume that these two PDEs can be identified as the Gauss equation with the same K, and that the corresponding components of their first fundamental forms are E, F, G and $\hat{E}, \hat{F}, \hat{G}$ respectively. Let the respective coordinates be (u,v) and (x,y)

(see Fig. 1.3).

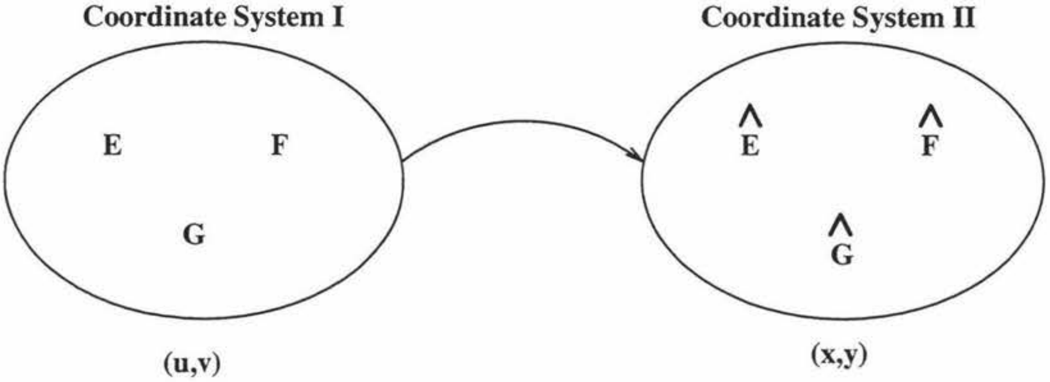

Figure 1.3: Coordinate transformation from coordinate system I to the coordinate system II.

From the tensor formula,

$$\hat{g}_{ij} = g_{lm} \frac{\partial X^l}{\partial \hat{X}_i} \cdot \frac{\partial X^m}{\partial \hat{X}_j}, \tag{1.20}$$

for coordinate transformations, where $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$, $\hat{g}_{11} = \hat{E}$, $\hat{g}_{12} = \hat{g}_{21} = \hat{F}$, $\hat{g}_{22} = \hat{G}$ and then by using the specific values for $\hat{g}_{ij}$'s and g_{lm}'s we obtain the system

$$E\,u_x^2 + 2F\,u_x\,v_x + G\,v_x^2 = \hat{E} \tag{1.21}$$

$$E\,u_x\,u_y + F\,(u_x\,v_y + v_x\,u_y) + G\,v_x\,v_y = \hat{F} \tag{1.22}$$

and

$$E\,u_y^2\,, + 2F\,u_y\,v_y + G\,v_y^2 = \hat{G}. \tag{1.23}$$

We need to solve this system of non-linear PDEs to determine the required Bäcklund transformations.

When applying the method described above in solving a PDE, we are aware of the fact that we may have difficulties, first in identifying the given PDE as the Gauss equation; i.e., to determine the corresponding coordinate system, and then in solving the system of PDEs which determines the Bäcklund transformations. The latter could be relatively harder than the original problem. Further, it should be noted that, imposing different initial conditions on this system of PDEs yields different Bäcklund transformations. This shows that all the solutions to the given PDE cannot be obtained by using one set of Bäcklund transformations, and thus we only end up with certain classes of solutions. This certainly is a weakness in our method, especially when we are looking for all possible solutions.