IMAGE REGISTRATION UNDER CONFORMAL DIFFEOMORPHISMS

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Doctor of Philosophy
in
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Declaration

It is hereby declare that this dissertation is my own work. It is being submitted for the degree of Doctor of Philosophy in Mathematics at the Massey University, Palmerston North. It has not been submitted before for any degree or examination at this or any other institution.

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Muhammad Yousuf Tufail
(Candidate's Signature)

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Glory and Praise to Allah, the most gracious and merciful. *Indeed, the worst of living creatures in the sight of Allah are the deaf and dumb who do not use reason.*

- Al-Quran (8: 22)

Thousands and hundreds thousands salam to Prophet Muhammad Sall Allāhu ʿalayhi wa-sallam. *No two things have been combined better than knowledge and patience.*

- Sahih Bukhari (Book 11: Hadees 644)

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Abstract

Image registration is the process of finding an alignment between two or more images so that their appearance matches. It has been widely studied and applied to several fields, including medical imaging and biology (where it is related to morphometrics). In biology, one motivation for image registration comes from the work of Sir D'Arcy Thompson. In his book *On Growth and Form* he presented several examples where a grid superimposed onto a two-dimensional image of one species was smoothly deformed to suggest a transformation to an image of another species. His examples include relationships between species of fish and comparison of human skulls with higher apes.

One of Thompson's points was that these deformations should be as 'simple' as possible. In several of his examples, he uses what he calls an *isogonal* transformation, which would now be called conformal, i.e., angle-preserving. His claims of conformally-related change between species were investigated further by Petukhov, who used Thompson's grid method as well as computing the cross-ratio (which is an invariant of the Möbius group, a finite-dimensional subgroup of the group of conformal diffeomorphisms) to check whether sets of points in the images could be related by a Möbius transformation. His results suggest that there are examples of growth and evolution where a Möbius transformation cannot be ruled out. In this thesis, we investigate whether or not this is true by using image registration, rather than a point-based invariant: we develop algorithms to construct conformal transformations between images, and use them to register images by minimising the sum-of-squares distance between the pixel intensities. In this way we can see how close to conformal the image relationships are.

We develop and present two algorithms for constructing the conformal transformation, one based on constrained optimisation of a set of control points, and one based on gradient flow. For the first method we consider a set of different penalty terms that aim to enforce conformality, based either on discretisations of the Cauchy-Riemann equations, or geometric principles, while in the second the conformal transformation is represented as a discrete Taylor series. The algorithms are tested on a variety of datasets, including synthetic data (i.e., the target is generated from the source using a known conformal transformation; the easiest possible case), and real images, including some that are not actually conformally related. The two methods are compared on a set of images that include Thompson's fish example, and a small dataset demonstrating the growth of a human skull. The conformal growth model does appear to be validated for the skulls, but interestingly, not for Thompson's fish.