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A Kepstrum approach to Filtering, Smoothing and Prediction.

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Abstract
The kepstrum (or complex cepstrum) method is revisited and applied to the problem of spectral factorization where the spectrum is directly estimated from observations. The solution to this problem in turn leads to a new approach to optimal filtering, smoothing and prediction using the Wiener theory. Unlike previous approaches to adaptive and self-tuning filtering, the technique, when implemented, does not require a priori information on the type or order of the signal generating model. And unlike other approaches - with the exception of spectral subtraction - no state-space or polynomial model is necessary. In this first paper results are restricted to stationary signal and additive white noise.

Notation

τ	sampling interval,
θ	normalized frequency $\omega\tau \quad (0 \le \theta \le 2\pi)$
q^{-1}	backward shift operator: $y_{k-1} = q^{-1} y_k$.
ζ	transform variable z^{-1} corresponding to the backward shift operator.
E[.]	expectation operator
w_k, ξ_k, v_k	white-noise processes
y_k, s_k	signal and message processes.
ε_k	innovations process
$g_{ys}(\zeta)$	cross spectral density transform between signal and message.
$g_{ss}(\zeta)$	spectral density transform of message process.
$\Lambda(\zeta), Z(\zeta)$	spectral factor and normalized spectral factor

1. Introduction

This paper extends the kepstrum approach to spectrum estimation investigated by the writers some years ago [1,2] applying it to the construction of Wiener filters. The kepstrum approach [3,4] evolved from mathematical work on spectral factorization [5,6,7] the word 'kepstrum', due to Robinson, coming from the initial letters of 'Kolmogorov equation power series'.

The kepstrum approach is almost, though not quite, identical with the 'cepstrum' or 'complex cepstrum' approach on which numerous works have appeared with application to echo detection, geophysics, speech processing, vibration signal analysis etc. Bogart et al [8] first coined the name 'cepstrum' to mean the inverse FFT of the logarithm of the modulus of the signal spectrum. This definition was later extended to the 'complex cepstrum' where the logarithm includes also its imaginary part [9,10] , for application to 'homomorphic signal processing', i.e. use of the logarithm of the transfer function to make additive the passage of signals through cascaded operations.

The cepstrum method is essentially a practical approach to signal analysis based on the use of the FFT. From the theoretical point of view the objection may be raised that it is based on a quantity both dependent on sample length and subject to statistical variation. Here the alternative kepstrum approach appears to supply a surer theoretical foundation. The complex cepstrum defined as an inverse FFT array is replaced by the kepstrum defined as the sequence of coefficients in the Kolmogorov series expansion, the relationship between these being that the complex cepstrum is a truncated version of the kepstrum coefficients corresponding to the sample length chosen. The kepstrum is not subject to statistical variation. and so it clarifies the relation between the estimate found empirically from a sample and the true value.

The key underlying technique in this paper is the application of kepstrum analysis to spectral factorization directly from observed signals. We already gave the theory of this method [11,1,2]which was there applied to system identification. Essentially the same method was more recently used by Elliott and Rafaely [12].The present paper will again review the theory of the method and then apply it to Wiener filtering and similar problems, the procedure giving a foundation for a general method of constructing Wiener filters directly from observed data. Although the Wiener filter has been in existence since the 1940s [13] difficulties in computing the filter have led to it rarely being applied in practice in its original form because the required algebraic models of the signal and noise generating processes are unknown. Furthermore, even if this information is known, the spectral factorization leads to complex algebraic expressions from which a procedure must be carried out to enable the filter to be causal. The final answer is complicated and too dependent on poorly known parameters.

The literature has many approaches to adaptive minimum mean-squared error filtering and perhaps the most popular are based on the adaptive noise cancellation [14,15,16] which uses multiple sensors and a least-mean squares (LMS) weight estimation scheme to arrive at an adaptive finite impulse-response (FIR) filter. Self-tuning filters, smoothers and predictors have also been tried which are similar but based instead on an adaptive infinite-impulse response (IIR) methods [17,18] and use extended least-squares (ELS) or maximum recursive likelihood parameter identification on an ARMA model. However both approaches require some form of model order in the adaptive scheme and the computation becomes more intense as the model order increases. In fact many real problems may well need high model orders to cope with such problems as room reverberation. An alternative approach is of course to use Kalman filters [19] which are more suitable for time-varying models but even if such models are known a priori a computationally intense Riccati equation must then be solved and once again the computational complexity increases with model order.

2. Kepstrum Analysis

$H(\zeta)$ will represent a discrete transfer function (t.f) written in terms of the inverse of the z-transform variable. If a t.f. $H(\zeta)$ is both stable and minimum phase, i.e. it has no poles or zeros on or inside the unit circle, then it is possible to define the *kepstrum function*

$$K(\zeta) = \ln H(\zeta) \tag{1}$$

as a regular function within the unit circle. This is also called *kepstrum generating function* (k.g.f) because it defines the *kepstrum coefficients* or *kepstrum* which are the coefficients k_n in the expansion

$$K(\zeta) = k_0 + k_1\zeta + k_2\zeta^2 + ... \tag{2}$$

valid within the unit circle. The kepstrum function has the following properties.

Property (i) ***Additivity***

$$\ln\{H_1(\zeta)H_2(\zeta)\} = \ln H_1(\zeta) + \ln H_2(\zeta) \tag{3}$$

Property (ii) ***Inverse***

$$\ln\{1/H(\zeta)\} = -\ln H(\zeta) \tag{4}$$

From these it follows that the kepstrum coefficients are respectively added and negated for the operations of cascading and inversion.

If the kepstrum coefficients are known or can be estimated then the t.f

$$H(\zeta) = \exp K(\zeta) \tag{5}$$

can be computed using a series expansion. Putting $\zeta = e^{-j\theta}$ ($0 \le \theta \le 2\pi$) then

$$\ln H(e^{-j\theta}) = K(e^{-j\theta}) = k_0 + k_1 e^{-j\theta} + k_2 e^{-2j\theta} + ... \tag{6}$$

Then clearly

$$\ln H(e^{-j\theta}) = \ln\left| H(e^{-j\theta})\right| + j\angle H(e^{-j\theta}) \tag{7}$$

where $\angle$ represents phase. From (6) it follows that

$$\ln\left|H(e^{-j\theta})\right| = k_0 + k_1\cos\theta + k_2\cos 2\theta + ... \tag{8}$$

$$\angle H(e^{-j\theta}) = -(k_1\sin\theta + k_2\sin 2\theta + ...) \tag{9}$$

From the Fourier cosine series (8) there follows

$$k_0 = \frac{1}{2\pi}\int_0^{2\pi} \ln\left| H(e^{-j\theta})\right| d\theta$$

$$k_n = \frac{1}{\pi}\int_0^{2\pi} \ln\left| H(e^{-j\theta})\right| \cos n\theta \; d\theta, \quad n = 1,2,3,.. \tag{10}$$

which can also be written over the half-range ($0 \le \theta \le \pi$). From these values of the k, phase can in principal be found using the sine series (9). Thus phase is re-constituted from log amplitude. .For practical use of the above series it is necessary to truncate the number of kepstrum coefficients. Use is made of the FFT and these estimates are known in the literature as 'cepstrum coefficients'.

2.1 Further Properties of the kepstrum

Property (iii) ***Spectral Factorization***

Spectral factorization arises in optimal filtering. The discrete-time spectral density is defined theoretically from the bilateral transform of the autocorrelation function. Denoting this by g(ζ), the spectral density S(θ) is defined as

$$S(\theta) = g(e^{j\theta}) \tag{11}$$

where θ is the normalised frequency. S(θ) is a periodic function of θ and may be defined in any range of 2π but is most commonly defined for $0 \le \theta \le 2\pi$. The problem of spectral factorization is to represent g(ζ) in the form

$$g(\zeta) = \Lambda(\zeta)\Lambda(\zeta^{-1}) \tag{12}$$

giving the corresponding frequency representation of the spectral density

$$S(\theta) = \left|\Lambda(e^{j\theta})\right|^2 \tag{13}$$

where Λ (ζ) is both stable and minimum phase. Here the t.f Λ (ζ^{-1}) is both unstable and non-minimum phase but is normally considered to be an uncausal representation of the spectral factor. Note that (13) implies the positivity of S(θ). The above spectral density may be regarded as resulting from the passage of white noise input w_k having unit variance through a t.f. Λ (ζ) i.e. of a process

$$s_k = \Lambda(q^{-1})\, w_k \tag{14}$$

From (14) follows

$$\ln\left|\Lambda(e^{j\theta})\right| = \frac{1}{2}\ln S(\theta) \tag{15}$$

and now from (10) we may write

$$k_0 = \frac{1}{4\pi}\int_0^{2\pi} \ln S(\theta)\, d\theta$$

$$k_n = \frac{1}{2\pi}\int_0^{2\pi} \ln S(\theta)\cos n\theta\, d\theta \quad n = 1, 2, .. \tag{16}$$

for the coefficients of the kepstrum representation of $\Lambda(\zeta)$

$$\Lambda(\zeta) = k_0 + k_1\zeta + k_2\zeta^2 + \dots \tag{17}$$

These also give the kepstrum coefficients of the bilateral kepstrum expansion of g(.) since from

$$\ln g(\zeta) = \ln\Lambda(\zeta) + \ln\Lambda(\zeta^{-1}) \tag{18}$$

it follows that the log-spectrum will have the bilateral representation

$$\ln g(\zeta) = \dots + k_2\zeta^{-2} + k_1\zeta^{-1} + 2k_0 + k_1\zeta + k_2\zeta^2 + \dots \tag{19}$$

On putting $e^{j\theta}$ for ζ this gives twice the complex form of the Fourier series of $\Lambda(\zeta)$ and again leads to the values (16) of the kepstrum coefficients.

It is convenient also to work with a normalized spectral factor $Z(\zeta)$ which is defined using the autoregressive representation of the signal

$$s_k = b_1 s_{k-1} + b_2 s_{k-2} + \dots \varepsilon_k \tag{20}$$

where ε_k is the innovations sequence which is white noise of zero mean.. Using q^{-1}, the backward shift operator which corresponds in the time-domain to the variable ζ in the frequency domain, this equation may be written

$$s_k = Z(q^{-1})\varepsilon_k \tag{21}$$

where

$$Z(\zeta) = (1 - b_1\zeta - b_2\zeta^2 - \dots)^{-1} \tag{22}$$

Here

$$Z(0) = 1 \tag{23}$$

and comparing (21) and (14) it is seen that

$$\Lambda(\zeta) = Z(\zeta)\,\sigma_\varepsilon \tag{24}$$

where σ_ε is the standard deviation of the innovations sequence.

Then taking logs of (24)

$$\ln\Lambda(\zeta) = \ln Z(\zeta) + \ln\sigma_\varepsilon \tag{25}$$

and, using (8), the zeroth kepstrum coefficient is found as

$$k_0 = \ln\sigma_\varepsilon \tag{26}$$

which from (16) gives rise to the formula of Kolmogorov

$$\sigma_\varepsilon^{\,2} = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \ln S(\theta)\, d\theta\right] \tag{27}$$

From (25),(26) it follows that the kepstrum expansion of $Z(\zeta)$ coincides with that of $\Lambda(\zeta)$ but starting with the second term, i.e.

$$\ln Z(\zeta) = k_1\zeta + k_2\zeta^2 + \dots \tag{28}$$

The amplitude information about the spectrum is thus contained in the constant term, the remaining terms described by $Z(\zeta)$ being independent of amplitude.

Property (iv) ***Impulse response from kepstrum coefficients***
Suppose that the kepstrum function $K(\zeta)$ corresponds to the minimum phase t.f. $H(\zeta)$ having power series

$$H(\zeta) = h_0 + h_1\zeta + h_2\zeta^2 + ... \quad (29)$$

Then $H(\zeta)$ may be represented as

$$H(\zeta) = \exp(k_0 + k_1\zeta + k_2\zeta^2 + ...) = \exp k_0 \; Z(\zeta) \quad (30)$$

where

$$Z(\zeta) = \exp(k_1\zeta + k_2\zeta^2 + k_3\zeta^3 + ...) \quad (31)$$

This is similar to the spectrum factorization case where $\Lambda\ (\zeta)$ was represented as a scaling factor times a normalized function. The function $Z(\zeta)$ may easily be determined by a slight modification of the recursive method as described by Silvia & Robinson [4]. Differentiation of (31) gives

$$dZ(\zeta)/d\zeta = (k_1 + 2k_2\zeta + 3k_3\zeta^2 + ...).Z(\zeta) \quad (32)$$

Now let

$$Z(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + ... \quad (33)$$

where

$$a_0 = Z(0) = 1$$

Then, by differentiating (33) and substitution in (32) and equating coefficients on both sides, there are found for n=1,2,3,…the relations

$$na_n = k_1 a_{n-1} + 2k_2 a_{n-2} + ... + nk_n a_0 = \sum_{r=1}^{n} rk_r a_{n-r} \quad (34)$$

from which the coefficients a_n can be determined recursively.
The inverse of the t.f. $H(\zeta)$ may be determined similarly:

$$H(\zeta)^{-1} = \exp\{-(k_0 + k_1\zeta + k_2\zeta^2 + ...)\} = \exp(-k_0)Z(\zeta)^{-1} \quad (35)$$

$$Z(\zeta)^{-1} = \exp(-k_1\zeta - k_2\zeta^2 - k_3\zeta^3 - ...) \quad (36)$$

The calculation remains the same apart from the different scaling factor and the change in sign of the kepstrum coefficients.

3. Wiener Estimators

The classical least-squares solution for filtering, smoothing and prediction was given independently by Kolmogorov for discrete-time and by Wiener for continuous-time. [6,7,13] In this section we will review results described in detail in a previous paper by Barrett & Moir [20] for the form of the optimal discrete time filter following the Wiener theory. It will be shown that it can be represented using innovations and the solution found using kepstrum coefficients.

We consider the basic problem of a random signal y_k corrupted with additive zero-mean white noise v_k to give a message s_k

$$s_k = y_k + v_k \quad (37)$$

where the signal y_k is modeled as a transfer function driven by zero-mean white noise uncorrelated with $\boldsymbol{v_k}$.

$$y_k = W(q^{-1})\,\xi_k \quad (38)$$

It is assumed that driving and additive noise have variances σ_ξ^2, σ_v^2 respectively. The estimate $\hat{y}_k$ of y_k is found from the s_k process by

$$\hat{y}_k = H(q^{-1})s_k \tag{39}$$

where H(.) has been chosen to minimize mean-squared error

$$E[e_k^2] = E[(y_k - \hat{y}_k)^2] \tag{40}$$

This optimal estimator transfer function is given by

$$H(\zeta) = \left[\frac{g_{ys}(\zeta)}{\Lambda(\zeta^{-1})}\right]_+ \frac{1}{\Lambda(\zeta)} \tag{41}$$

where $g_{ys}(\zeta)$ is the cross-spectral density transform between signal and message $\boldsymbol{s_k}$ which, since noise is uncorrelated with message, takes the form

$$g_{ys}(\zeta) = W(\zeta)W(\zeta^{-1})\sigma_\xi^2 \tag{42}$$

$\Lambda(\zeta)$ is the stable, minimum phase spectral factor found from the spectral density of the message:

$$g_{ss}(\zeta) = \Lambda(\zeta)\Lambda(\zeta^{-1}) = W(\zeta)W(\zeta^{-1})\sigma_\xi^2 + \sigma_v^2 \tag{43}$$

$[]_+$ denotes the realizable part corresponding to the interval $\Im$ of the message (see below) A detailed deduction of this result was given in the 1987 paper cited [20]

.

A further simplification results on using the normalized spectral factor from (24). We can write the term in brackets as a Laurent series

$$\frac{g_{ys}(\zeta)}{Z(\zeta^{-1})\sigma_\varepsilon} = C(\zeta) = \sum_{n=-\infty}^{\infty} c_n \zeta^n \tag{44}$$

when the estimator is

$$H(\zeta) = [C(\zeta)]_+ \frac{1}{Z(\zeta)} \tag{45}$$

where $[C(\zeta)]_+$ is found by truncation of the Laurent series i.e.

$$[C(\zeta)]_+ = \sum_{n \in \Im} c_n \zeta^n \tag{46}$$

Here $\Im$ is a suitably chosen interval depending on three possible cases.

(i) Filtering $\Im = \{0, 1, 2, 3, \ldots\}$

This corresponds to instantaneous estimation $\hat{y}_{k/k}$ i.e. information on s_k is required up to and including time k. The estimator becomes

$$H(\zeta) = 1 - \frac{\sigma_v^2}{\sigma_\varepsilon^2}\frac{1}{Z(\zeta)} \tag{47}$$

(ii) Smoothing $\Im = \{-d, -d+1, -d+2, ..\}, \; d > 0$

This corresponds to a fixed-lag smoothed estimate $\hat{\boldsymbol{y}}_{\boldsymbol{k/k+d}}$ of the signal at time k with information up to and including time k+d. The estimator is

$$H(\zeta) = 1 - \frac{\sigma_v^2}{\sigma_\varepsilon^2}\frac{P_d(\zeta^{-1})}{Z(\zeta)} \tag{48}$$

where the polynomial $P_d(\zeta^{-1})$ is the expansion to d terms of the infinite power series

$$Z(\zeta^{-1})^{-1} = 1 + p_1\zeta^{-1} + p_2\zeta^{-2} + \ldots + p_d\zeta^{-d} + p_{d+1}\zeta^{-d+1} + \ldots \tag{49}$$

(iii) Prediction $\Im = \{d, d+1, d+2, ...\}, d > 0$

This corresponds to a d-step ahead predictor $\hat{\boldsymbol{y}}_{k/k-d}$ of y at time k with information up to and including time k-d. From the power series expansion for the normalized spectral factor

$$Z(\zeta) = 1 + a_1\zeta + a_2\zeta^2 + ... + a_d\zeta^d + a_{d+1}\zeta^{d+1} + ... \tag{50}$$

follows

$$H(\zeta) = [a_d\zeta^d + a_{d+1}\zeta^{d+1} + a_{d+2}\zeta^{d+2} + ...]\frac{1}{Z(\zeta)} \tag{51}$$

It should also be noted that the above expressions for smoothing and prediction have other forms, e.g. we can define $\hat{y}_{k-d/k}$ and $\hat{y}_{k+d/k}$ if information is taken to be up to and including time k for both cases. This adds an extra delay term for the smoother and a d-step advance for the predictor. The predictor is still causal for this case as can be seen from (51) that there is already a d-step time-delay implicit within the transfer function as it stands.

3.1 Innovations form for estimators

The innovations model must generate the same spectrum as the message via white-noise driving through the transfer function of the spectral factor. Use of (21) results in the following innovations forms.

(i) Filtering and Smoothing (Hagander & Wittenmark [17], Barrett & Moir [20])

$$\hat{y}_{k/k+d} = s_k - \frac{\sigma_v^2}{\sigma_\varepsilon^2}\sum_{n=0}^{d} p_n \varepsilon_{k+n} \tag{52}$$

where for filtering d=0 and for smoothing d>0.

(ii) Prediction (Wittenmark[18], Barrett & Moir[20])

$$\hat{y}_{k/k-d} = \sum_{n=d}^{\infty} a_n \varepsilon_{k-n} \tag{53}$$

For the special case when d=1, the one step ahead predictor becomes

$$H(\zeta) = [a_1\zeta^1 + a_2\zeta^2 + a_3\zeta^3 + ...]\frac{1}{Z(\zeta)} \tag{54}$$

$$= 1 - Z(\zeta)^{-1} \tag{55}$$

The innovations representation is then

$$\hat{y}_{k/k-1} = s_k - \varepsilon_k \tag{56}$$

3.2 The kepstrum method applied to Wiener estimation

In applying the kepstrum method to equations for the Wiener estimators there are several unknown quantities. If the kepstrum coefficients corresponding to the logarithm of the spectral factor $\ln Z(\zeta)$ can be found, then by using (34) the impulse response (or power series expansion) of $Z(\zeta)$ can be found and hence the coefficients $a_i, i = 1, 2, ...$ in (50). Furthermore, by using the inversion property (4), i.e. simply by negating the kepstrum coefficients, the coefficients of the power-series expansion of the inverse of the spectral factor

$$Z(\zeta)^{-1} = 1 + p_1\zeta + p_2\zeta^2 + ... + p_d\zeta^d + p_{d+1}\zeta^{d+1} + ... \tag{57}$$

are easily found. The coefficients of (57) of the causal sequence become identical to the un-causal sequence of (49) by substitution of $\zeta = \zeta^{-1}$.The innovations variance follows from (26)

$$\sigma_\varepsilon^2 = \exp(2k_0) \tag{58}$$

The innovations sequence can be found by applying a whitening filter to the message process. Hence if $Z(\zeta)^{-1}$ can be estimated then

$$\varepsilon_k = Z(\zeta)^{-1} s_k \tag{59}$$

will generate the white innovations sequence. The only unknown is the variance of the white measurement noise which must be assumed to be known a priori or can be estimated when the signal is absent. This is not too unrealistic since a standard way of estimating noise statistics or models is to assume regions of 'noise-alone'. When applied to speech this can be done during the periods between words using word-boundary detection. One such method which works at low SNRs and in realistic environments is given by Agaiby and Moir [21] Similar methods are used for spectral subtraction [22]

4. Frequency-domain adaptive Wiener estimators.

The previous section outlined the method for implementing an innovations based filter, smoother or predictor. The estimation of the kepstrum coefficients of the spectral factor and the corresponding transfer-function, is done by the technique described in Barrett and Moir[2]) using the fast-Fourier transform (FFT) and its inverse . This method uses the periodogram of an observation sequence to derive an estimate of the magnitude and phase of an arbitrary transfer function driven by white-noise.

The generic method for estimating kepstrum coefficients is shown in *Fig.* 1 below.

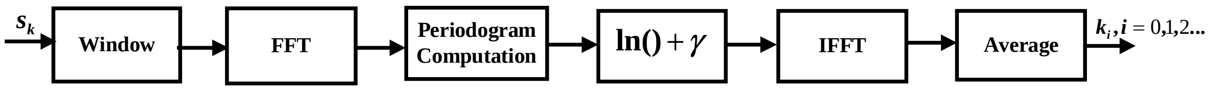

Fig. 1 **Estimation of kepstrum coefficients**

In the above diagram, after windowing the data, the N point FFT (or DFT) is defined using the commonly used notation $W_N = e^{-2\pi j/N}$ as

$$X_k = \sum_{i=0}^{N-1} x_i W_N^{ik},\ k = 0,1,2...N-1 \tag{60}$$

and its inverse. as

$$x_i = \frac{1}{N}\sum_{k=0}^{N-1} X_k W_N^{-ik},\ i = 0,1,2...N-1 \tag{61}$$

The periodogram is the estimate of spectral density at each frequency-bin and is found from the FFT

$$\hat{S}(e^{j\theta})\Big|_{\theta=2\pi k/N} = \frac{1}{N}\left|X_k\right|^2,\ k = 0,1,2,..,N-1 \tag{62}$$

After taking the natural log of periodogram it is found that there exists a bias equal in magnitude to minus Euler's constant γ =0.577215.., (Wahba[23], Barrett & Moir[2], Ephrain & Rahim[24]) Therefore in fig. 1 shows Euler's constant added to un-bias the periodogram. Finally the kepstrum coefficients can, if desired, be averaged for stationary signals this contributing a form of convergence to the algorithm. Of course this will not be possible if the time-series is non-stationary. It should be noted that since these are estimates, strictly speaking these should be termed 'cepstrum coefficients' but kepstrum is maintained throughout the paper to avoid confusion.

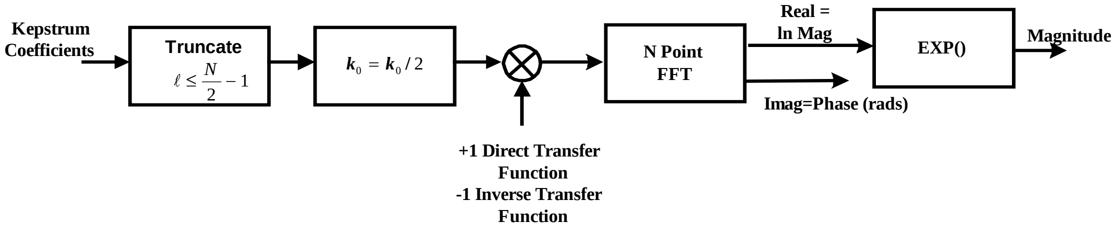

***Fig. 2*.** **Estimation of minimum-phase spectral factor frequency response.**

Fig. 2 shows a method whereby the frequency response of the spectral factor (or its inverse) can be estimated accurately with a few additional steps. The zeroth kepstrum coefficient must be halved and the whole kepstrum series truncated at a suitable value $\ell \leq \frac{N}{2}-1$. Points above ℓ are set to zero *maintaining an N point sequence*. Truncating the series gives a smoothing of the estimated frequency response but too much truncation can remove peaks in a spectrum. By introducing a sign change by multiplying the kepstrum coefficients by minus one the inverse of the spectral factor can be estimated. An FIR kepstrum polynomial can now be defined

$$K_\ell(\zeta) = k_0 + k_1\zeta + k_2\zeta^2 + ... + k_\ell\zeta^\ell \qquad (63)$$

whose coefficients can be negated for the purposes of estimating the inverse spectral factor or left unchanged and the direct spectral factor estimated. The final FFT in *Fig. 2* and the steps after it are not used here but could be employed if frequency-domain convolution is used. Consider the following two possible cases.

(i) Sign change in kepstrum coefficients.

Assuming the *inverse* spectral factor $Z(\zeta)^{-1}$ is to be estimated from the kepstrum coefficients of the natural logarithm of $\ln(1/Z(\zeta))$. From (34) the coefficients of the power series expansion of $Z(\zeta)^{-1}$ are found from

$$np_n = -\sum_{r=1}^{n} rp_{n-r}k_r, n = 1,2...\ell \qquad (64)$$

and we have

$$Z(\zeta)^{-1} \cong 1 + p_1\zeta + p_2\zeta^2 + ... + p_\ell\zeta^\ell \qquad (65)$$

which has been approximated to ℓ coefficients.

(ii) No sign change in the kepstrum coefficients

Here the *direct* spectral factor $Z(\zeta)$ is to be estimated from the kepstrum coefficients corresponding to $\ln Z(\zeta)$. From (34) the coefficients of the power series expansion of $Z(\zeta)$ are estimated from

$$na_n = \sum_{r=1}^{n} ra_{n-r}k_r, n = 1,2...\ell \qquad (66)$$

and we have

$$Z(\zeta) \cong 1 + a_1\zeta + a_2\zeta^2 + ... + a_\ell\zeta^\ell \qquad (67)$$

which has been approximated to ℓ coefficients. In (i) and (ii) above, the two sets of polynomial coefficients a_i, p_i are required in the prediction problem and only p_i in the smoothing problem. The innovations variance can be found in (ii) from $\exp(k_0) = \sigma_\varepsilon$ giving $\sigma_\varepsilon^2 = \exp(2k_0)$.

4.1 Innovations sequence estimation

This approach to obtaining the innovations sequence is to recognize that the kepstrum generating function has finite order and has the form of a polynomial $K_\ell(\zeta)$ which would via *Fig. 2* have given rise to a smoothed spectral (or inverse spectral) factor frequency response. We require the inverse spectral factor and this in turn generates a polynomial given by (49) which is the whitening filter. The estimated innovations then becomes

$$\hat{\varepsilon}_k = \sum_{n=0}^{\ell} p_n s_{k-n} \qquad (68)$$

where $p_0 = 1$. In simulations it has been found that, for most realistic problems, 40 terms or less are sufficient for good accuracy. For problems where the number of terms is greater than around 60, frequency-domain FFT convolution will be faster, with, for example, the overlap-save or the overlap-add method Mitra 2001[25] making full use of *Fig. 2.* The direct convolution method is used here in the following two algorithms.

Algorithm 4.1 Filtering (d=0) and Smoothing (d>0)

For the message s_k, assume σ_v^2 is either known or can be estimated a priori . Then for each batch of data $s_k, k = 0,1...N-1$ where N is the FFT length, do the following:

Step 1. Estimate the kepstrum coefficients from *Fig.* 1.

Step 2. The kepstrum coefficients are preserved up to some point $\ell \leq \frac{N}{2} - 1$ and the rest set to zero (It is assumed that ℓ >>d.) Set $k_o = k_o / 2$.

Step 3. Estimate the polynomial coefficients $p_n, n = 1,2...\ell$ from (64) and calculate $\frac{1}{\sigma_\varepsilon^2} = \exp(-2k_0)$.

Step 4. Estimate the innovations $\hat{\varepsilon}_k$ from (68)

Step 5. The filtered (d=0) or d steps smoothed estimate of the signal is found from (52)

$$\hat{y}_{k/k+d} = s_k - \frac{\sigma_v^2}{\sigma_\varepsilon^2} \sum_{n=0}^{d} p_n \hat{\varepsilon}_{k+n} \tag{69}$$

Algorithm 4.2 Prediction (d>0)

Assume a message s_k, σ_v^2 is not required. Then for each batch of data $s_k, k = 0,1,.., N-1$ where N is the FFT length, do the following:

Steps 1 to 4 are identical to Algorithm 4.1.

Step 5. Estimate the polynomial coefficients $a_n, n = 1,2...\ell$ from (66).

Step 6. Estimate the d>0 steps ahead predicted estimate from (53)

$$\hat{y}_{k/k-d} = \sum_{n=d}^{\ell} a_n \hat{\varepsilon}_{k-n} \tag{70}$$

where the series is truncated at ℓ terms.

5. Illustrative Examples

A simulated example is shown first the signal consisting of unit variance white-noise passing though a low-pass Butterworth filter of order fifteen and cut-off frequency 410Hz. The sampling frequency was 22050 Hz. Uncorrelated measurement noise of variance 0.2 was added to the output which gave a signal-to-noise (SNR) ratio of –7dB. An FFT of length 2048 were used to estimate the kepstrum coefficients. The kepstrum coefficients were averaged over time and the results of the simulation are shown in *Fig.* 3.

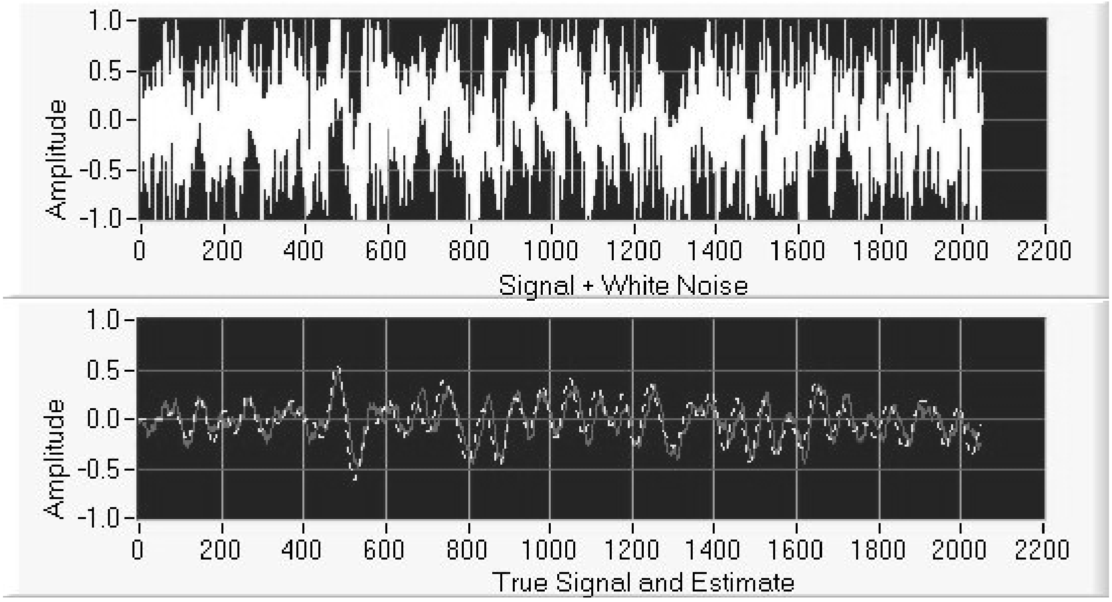

***Fig.* 3. Performance of Smoothing filter. (True Signal shown with broken line.)**

The smoothing lag was chosen to be d=5 steps and the cut-off point for the kepstrum coefficients was chosen as $\ell = 40$ terms. The true signal and the estimate are shown together in *Fig.* 3 and are quite close to one another other than the time-delay introduced by the smoother lag.

The same signal and noise statistics were used for a one step-ahead predictor and the results are shown in *Fig.* 4.

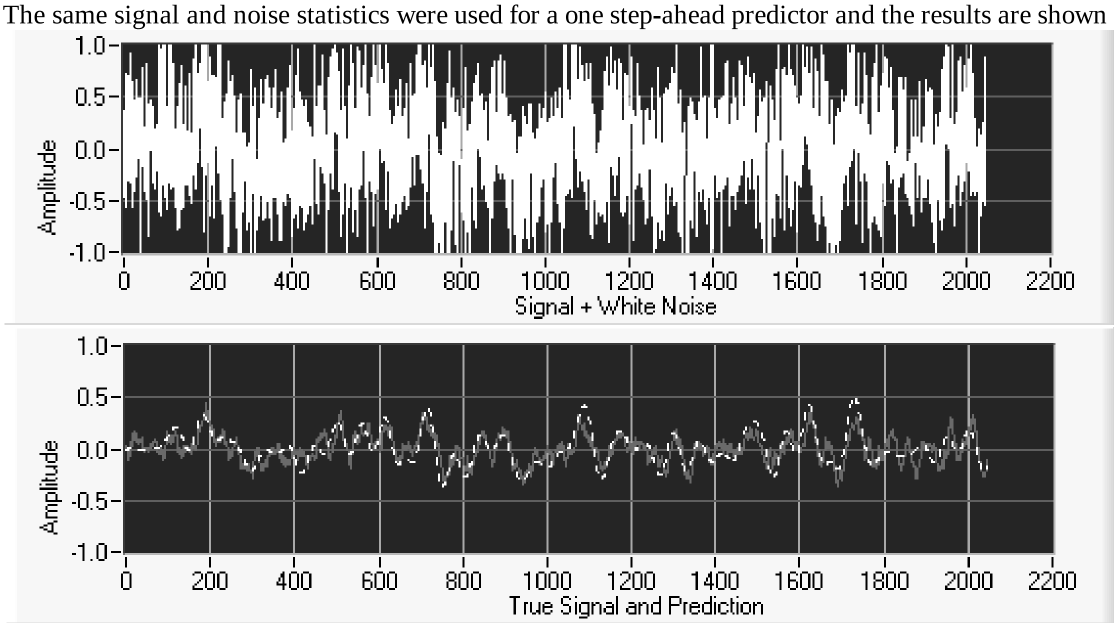

***Fig.* 4 Performance of a one step-ahead predictor. (True Signal shown with broken line)**

The predictor output is more noisy than the smoother as less information is available. In the case of the predictor a further approximation is made via (70) which is limited to $\ell = 40$ terms for this example. However, it is well known that by making such approximations too large that errors can often be added rather than reduced due to statistical inaccuracies. (Akaike [26]) Finally to show the possible applications of this method a speech signal was processed by the algorithm after adding computer generated white-noise. The sampling rate is 22050Hz using 8 bits/sample and an FFT length of 1024 points was used with a delay d=5. The kepstrum coefficients were truncated at $\ell = 40$ terms and were not averaged as the signal is assumed non-stationary from FFT frame to frame. The SNR was measured as 0.71dB for the whole run. The SNR after smoothing was measured as 14.2dB which is an improvement of 13.49dB. The resulting composite speech waveform sounded audibly improved with notably less hiss. The complete speech signal for the whole run is shown in *Fig. 5* together with

***Fig. 5*. Original signal (a) plus noise (b) and smoothed estimate (c)**

the signal plus noise and the smoothed estimate. They are each shown using the same scaling. The SNR was measured after smoothing by measuring the noise power (between utterances) and the signal plus noise power during an utterance. From this information it is straight forward to evaluate SNR piecewise and average for the whole waveform. This of course assumes that the noise remains stationary throughout the duration of the speech, which for this example is true. It must be pointed out however that for this example the variance of the white noise was assume known and although the results look impressive the smoothed signal did suffer from something similar to 'musical noise' which is a well known spectral subtraction phenomena.

6. Conclusions

The kepstrum method has been applied to the problems of filtering, smoothing and prediction of random signals with white additive measurement noise. The approach does not require any ARMA or state-space models. For prediction the impulse response of part of the predictor (the spectral factor) is approximated by truncation and this could be said to be equivalent to other approaches which use FIR approximations. The technique was applied to several examples with promising results. It remains to solve the more general coloured-noise problem.

7. References

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