

Res. Lett. Inf. Math. Sci., (2001) **2**, 1–10

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On the q -analogue of Kummer's 24 solutions

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Abstract

The ${}_3\phi_2$ transformations are used to derive q -analogues of the relations amongst Kummer's 24 solutions.

1 Introduction

The purpose of this article is to give q -analogues of the 20 relations between Kummer's 24 hypergeometric functions.

Standard notation for q -series is used throughout - see, for example, [1, Chapter 10] or [4, Chapter 1].

An interesting feature of the results is the occurrence of divergent series, namely the ${}_3\phi_1$ and ${}_2\phi_0$ functions. It is still an open question to assign a meaningful interpretation to these divergent series. They formally reduce to hypergeometric functions when $q \rightarrow 1$.

The paper is organised as follows. In section 2, the basic transformation formulas for the ${}_2\phi_1$ function are given. Three fundamental transformation properties of ${}_3\phi_2$ functions are given in section 3. These formulas are written using Sears' [6] q -analogue of Whipple's notation. Only two of these formulas will be used; the third is mentioned for completeness. The 20 q -analogues of the relations between Kummer's 24 solutions - the main results of this paper - are given in section 4. A proof of one of the 20 results is given in detail in section 5 and the remaining proofs, which are similar, are summarized in a table. The key idea in the proofs is to use limiting properties of ${}_3\phi_2$ functions, for example:

$$\lim_{f=abcx/e, c \rightarrow 0} {}_3\phi_2 \left(\begin{matrix} a, b, c \\ e, f \end{matrix}; \frac{ef}{abc}; x \right) = {}_2\phi_1 \left(\begin{matrix} a, b \\ e \end{matrix}; x \right).$$

2 Basic properties of the ${}_2\phi_1$ function

The following formulas will be used throughout.

$${}_2\phi_1 \left(\begin{matrix} a, b \\ e \end{matrix}; \frac{e}{ab}; q \right) = \frac{(e/a, e/b; q)_\infty}{(e, e/ab; q)_\infty}; \quad (1)$$

$${}_2\phi_1 \left(\begin{matrix} a, b \\ e \end{matrix}; x; q \right) = \frac{(b, ax; q)_\infty}{(e, x; q)_\infty} {}_2\phi_1 \left(\begin{matrix} e/b, x \\ ax \end{matrix}; b; q \right) \quad (2)$$

$$= \frac{(e/b, bx; q)_\infty}{(e, x; q)_\infty} {}_2\phi_1 \left(\begin{matrix} abx/e, b \\ bx \end{matrix}; e/b; q \right) \quad (3)$$

$$= \frac{(abx/e; q)_\infty}{(x; q)_\infty} {}_2\phi_1 \left(\begin{matrix} e/a, e/b \\ e \end{matrix}; abx/e; q \right) \quad (4)$$

$$= \frac{(ax; q)_\infty}{(x; q)_\infty} {}_2\phi_2 \left(\begin{matrix} a, e/b \\ e, ax \end{matrix}; bx; q \right) \quad (5)$$

$$= \frac{(ax, bx; q)_\infty}{(e, x; q)_\infty} {}_2\phi_2 \left(\begin{matrix} x, abx/e \\ ax, bx \end{matrix}; e \right) \quad (6)$$

$$= \frac{(abx/e; q)_\infty}{(bx/e; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, e/b, 0 \\ e, eq/bx \end{matrix}; q \right). \quad (7)$$

Equation (1) is Heine's q -analogue of Gauss' theorem [1, p. 522], [2, p. 68], [4, p. 10]. Equations (2), (3) and (4) are Heine's transformation, the iterate of Heine's transformation, and the q -analogue of Euler's transformation, respectively [1, pp. 521–524], [4, pp. 9–10]. Equation (5) is Jackson's q -analogue of Pfaff's transformation [1, (10.10.12)], [4, (1.5.4)], and equation (6) appears in [6] as the function $Y(1, 6)$ in Table IIA. Equation (7) is due to Jackson [4, p. 241, (III.5)].

3 Transformation properties of ${}_3\phi_2$ functions

Let $r_0, r_1, r_2, r_3, r_4, r_5$ be six parameters such that $r_0 r_1 r_2 r_3 r_4 r_5 = 1$. With these parameters, associate numbers α_{lmn} and β_{mn} such that

$$\begin{aligned} \alpha_{lmn} &= q^{1/2} r_l r_m r_n \\ \beta_{mn} &= q r_m / r_n. \end{aligned}$$

Let $(i; j, k; l, m, n)$ be any permutation of $(0, 1, 2, 3, 4, 5)$. The semicolons indicate that permutations with the second and third entries interchanged are regarded as equivalent, as are permutations with the last three entries in any order. For example, $(2; 1, 4; 5, 0, 3)$ is equivalent to $(2; 4, 1; 5, 0, 3)$, and both are equivalent to $(2; 1, 4; 0, 3, 5)$. Consequently, it is only necessary to specify the first three entries, in which case $(i; j, k)$ can be used to represent $(i; j, k; l, m, n)$. Accordingly, let

$$F(i; j, k) = (\beta_{ji}, \beta_{ki}, \alpha_{lmn}; q)_\infty {}_3\phi_2 \left(\begin{matrix} \alpha_{jkl}, \alpha_{jkm}, \alpha_{jkn} \\ \beta_{ji}, \beta_{ki} \end{matrix}; \alpha_{lmn} \right). \quad (8)$$

Set $\alpha_{145} = a, \alpha_{245} = b, \alpha_{345} = c, \beta_{40} = e, \beta_{50} = f$. Equivalently,

$$\begin{aligned} r_0^3 &= \frac{q^{5/2} abc}{e^2 f^2} \\ r_1^3 &= \frac{aef}{q^{1/2} b^2 c^2} \\ r_2^3 &= \frac{bef}{q^{1/2} a^2 c^2} \\ r_3^3 &= \frac{cef}{q^{1/2} a^2 b^2} \\ r_4^3 &= \frac{abce}{q^{1/2} f^2} \\ r_5^3 &= \frac{abcf}{q^{1/2} e^2}. \end{aligned}$$

All of the α 's and β 's can be expressed in terms of a, b, c, d, e and q :

$\alpha_{012} = q/c$	$\alpha_{123} = ef/abc$	$\beta_{01} = q^2bc/ef$	$\beta_{20} = ef/ac$	$\beta_{40} = e$
$\alpha_{013} = q/b$	$\alpha_{124} = e/c$	$\beta_{02} = q^2ac/ef$	$\beta_{21} = qb/a$	$\beta_{41} = qbc/f$
$\alpha_{014} = qa/f$	$\alpha_{125} = f/c$	$\beta_{03} = q^2ab/ef$	$\beta_{23} = qb/c$	$\beta_{42} = qac/f$
$\alpha_{015} = qa/e$	$\alpha_{134} = e/b$	$\beta_{04} = q^2/e$	$\beta_{24} = qf/ac$	$\beta_{43} = qab/f$
$\alpha_{023} = q/a$	$\alpha_{135} = f/b$	$\beta_{05} = q^2/f$	$\beta_{25} = qe/ac$	$\beta_{45} = qe/f$
$\alpha_{024} = qb/f$	$\alpha_{145} = a$	$\beta_{10} = ef/bc$	$\beta_{30} = ef/ab$	$\beta_{50} = f$
$\alpha_{025} = qb/e$	$\alpha_{234} = e/a$	$\beta_{12} = qa/b$	$\beta_{31} = qc/a$	$\beta_{51} = qbc/e$
$\alpha_{034} = qc/f$	$\alpha_{235} = f/a$	$\beta_{13} = qa/c$	$\beta_{32} = qc/b$	$\beta_{52} = qac/e$
$\alpha_{035} = qc/e$	$\alpha_{245} = b$	$\beta_{14} = qf/bc$	$\beta_{34} = qf/ab$	$\beta_{53} = qab/e$
$\alpha_{045} = qabc/ef$	$\alpha_{345} = c$	$\beta_{15} = qe/bc$	$\beta_{35} = qe/ab$	$\beta_{54} = qf/e$

With this notation,

$$F(0; 4, 5) = (e, f, ef/abc; q)_{\infty} {}_3\phi_2 \left(\begin{matrix} a, b, c \\ e, f \end{matrix}; \frac{ef}{abc} \right). \quad (9)$$

Two term transformations

Hall's [1, (10.10.8)], [4, (4.2.10)], [5], [6, (p. 173, statement I)] formula

$$\begin{aligned} & (e, f, ef/abc; q)_{\infty} {}_3\phi_2 \left(\begin{matrix} a, b, c \\ e, f \end{matrix}; \frac{ef}{abc} \right) \\ &= (a, ef/ab, ef/ac; q)_{\infty} {}_3\phi_2 \left(\begin{matrix} e/a, f/a, ef/abc \\ ef/ab, ef/ac \end{matrix}; a \right) \end{aligned}$$

becomes in this notation simply

$$F(0; 4, 5) = F(0; 2, 3). \quad (10)$$

Interchanging r_4 and r_1 , we find that

$$F(0; 1, 5) = F(0; 2, 3)$$

and thus

$$F(0; 1, 5) = F(0; 4, 5). \quad (11)$$

Accordingly all of the permutations of the indices 1 to 5 are legitimate, and we see that all ten expressions $F(0; v, w)$ are equal, and may be denoted by $F(0)$. For every value $u = 1, \dots, 5$, the ten functions $F(u; v, w)$ are equal and may be denoted by $F(u)$.

Three term relations

The three term relation [4, (3.3.1)], [6, (p. 173 II(a))] in this notation is equivalent to

$$\begin{aligned} F(i) &= \frac{(\alpha_{klm}, \alpha_{kln}, \alpha_{kmn}, \beta_{ji}; q)_{\infty}}{(\beta_{kj}/q; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} \alpha_{jlm}, \alpha_{jln}, \alpha_{jmn} \\ \beta_{ji}, \beta_{jk} \end{matrix}; q \right) \\ &+ \frac{(\alpha_{jlm}, \alpha_{jln}, \alpha_{jmn}, \beta_{ki}; q)_{\infty}}{(\beta_{jk}/q; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} \alpha_{klm}, \alpha_{kln}, \alpha_{kmn} \\ \beta_{ki}, \beta_{kj} \end{matrix}; q \right), \end{aligned} \quad (12)$$

where (i, j, k, l, m, n) is any permutation of $(0, 1, 2, 3, 4, 5)$. Observe that the right hand side is symmetric in j and k , and also in l, m and n . Consequently, there are $6 \times \binom{5}{2} = 60$ equations of this type.

Although we shall not need it, the three term relation [4, (3.3.3)], [6, (p. 173 III(b))], written in this notation, is

$$\begin{aligned} 0 &= (r_k - r_j)(\beta_{kj}, \beta_{jk}, \alpha_{ilm}, \alpha_{iln}, \alpha_{imn}; q)_{\infty} F(i) \\ &+ (r_i - r_k)(\beta_{ik}, \beta_{ki}, \alpha_{jlm}, \alpha_{jln}, \alpha_{jmn}; q)_{\infty} F(j) \\ &+ (r_j - r_i)(\beta_{ji}, \beta_{ij}, \alpha_{klm}, \alpha_{kln}, \alpha_{kmn}; q)_{\infty} F(k). \end{aligned} \quad (13)$$

Any permutation of the indices $0, \dots, 5$ is legitimate, and so there are $\binom{6}{3} = 20$ equations of this type relating any three of the functions $F(0), \dots, F(5)$.

Equations (10) and (11) are q -analogues of [2, section 3.6, (1) and (2)], while (12) and (13) are q -analogues of [2, section 3.7, (2) and (3)], respectively.

4 The 20 relations

In this section, q -analogues of the 20 relations between Kummer's solutions are given. For the purpose of ease of reference, the same equation numbering system as [3, pp. 106 – 108] is used, i.e., our equation (25) is the q -analogue of equation (25) on page 106 of [3], etc.

$$\begin{aligned}
& {}_3\phi_1 \left(\begin{matrix} a, b, q/x \\ qab/e \end{matrix}; \frac{x}{e} \right) \\
&= \frac{(qa/e, e)_\infty}{(qab/e, e/b)_\infty} \frac{(x, q/x)_\infty}{(bx, q/bx)_\infty} {}_2\phi_0 \left(\begin{matrix} 1/b, bx/e \\ - \end{matrix}; \frac{e}{x} \right) {}_2\phi_1 \left(\begin{matrix} a, b \\ e \end{matrix}; x \right) \\
&+ \frac{(b, qa/b)_\infty}{(qab/e, e/b)_\infty} \frac{(abx/e, qe/abx)_\infty}{(bx/q, q^2/bx)_\infty} {}_2\phi_0 \left(\begin{matrix} q/e, x/q \\ - \end{matrix}; \frac{e}{x} \right) {}_2\phi_1 \left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx} \right)
\end{aligned} \tag{25}$$

$$\begin{aligned}
& {}_3\phi_1 \left(\begin{matrix} a, b, q/x \\ qab/e \end{matrix}; \frac{x}{e} \right) \\
&= \frac{(qb/e, e)_\infty}{(qab/e, e/a)_\infty} \frac{(x, q/x)_\infty}{(ax, q/ax)_\infty} {}_2\phi_0 \left(\begin{matrix} 1/a, ax/e \\ - \end{matrix}; \frac{e}{x} \right) {}_2\phi_1 \left(\begin{matrix} a, b \\ e \end{matrix}; x \right) \\
&+ \frac{(a, qb/a)_\infty}{(qab/e, e/a)_\infty} \frac{(abx/e, qe/abx)_\infty}{(ax/q, q^2/ax)_\infty} {}_2\phi_0 \left(\begin{matrix} q/e, x/q \\ - \end{matrix}; \frac{e}{x} \right) {}_2\phi_1 \left(\begin{matrix} b, qb/e \\ qb/a \end{matrix}; \frac{qe}{abx} \right)
\end{aligned} \tag{26}$$

$$\begin{aligned}
& \frac{(abx/e)_\infty}{(x)_\infty} {}_3\phi_1 \left(\begin{matrix} e/a, e/b, eq/abx \\ eq/ab \end{matrix}; \frac{abx}{e^2} \right) \\
&= \frac{(q/a, e)_\infty}{(qe/ab, b)_\infty} \frac{(abx/e, qe/abx)_\infty}{(ax, q/ax)_\infty} {}_2\phi_0 \left(\begin{matrix} b/e, ax/e \\ - \end{matrix}; \frac{e^2}{abx} \right) {}_2\phi_1 \left(\begin{matrix} a, b \\ e \end{matrix}; x \right) \\
&+ \frac{(qb/a, e/b)_\infty}{(qe/ab, b)_\infty} \frac{(abx/e, qe/abx)_\infty}{(ax/q, q^2/ax)_\infty} {}_2\phi_0 \left(\begin{matrix} q/e, abx/eq \\ - \end{matrix}; \frac{e^2}{abx} \right) {}_2\phi_1 \left(\begin{matrix} b, bq/e \\ bq/a \end{matrix}; \frac{qe}{abx} \right)
\end{aligned} \tag{27}$$

$$\begin{aligned}
& {}_3\phi_1 \left(\begin{matrix} qa/e, qb/e, q/x \\ qab/e \end{matrix}; \frac{ex}{q^2} \right) \\
&= \frac{(a, q^2/e)_\infty}{(qab/e, q/b)_\infty} \frac{(x, q/x)_\infty}{(qbx/e, e/bx)_\infty} {}_2\phi_0 \left(\begin{matrix} e/qb, bx/q \\ - \end{matrix}; \frac{q^2}{ex} \right) {}_2\phi_1 \left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x \right) \\
&+ \frac{(qb/e, qa/b)_\infty}{(qab/e, q/b)_\infty} \frac{(abx/e, qe/abx)_\infty}{(bx/e, qe/bx)_\infty} {}_2\phi_0 \left(\begin{matrix} e/q, x/q \\ - \end{matrix}; \frac{q^2}{ex} \right) {}_2\phi_1 \left(\begin{matrix} qa/e, a \\ qa/b \end{matrix}; \frac{qe}{abx} \right)
\end{aligned} \tag{28}$$

$$\begin{aligned}
& \frac{(abx/e)_\infty}{(x)_\infty} {}_3\phi_1 \left(\begin{matrix} e/a, e/b, eq/abx \\ eq/ab \end{matrix}; \frac{abx}{e^2} \right) \\
&= \frac{(q/b, e)_\infty}{(qe/ab, a)_\infty} \frac{(abx/e, qe/abx)_\infty}{(bx, q/bx)_\infty} {}_2\phi_0 \left(\begin{matrix} a/e, bx/e \\ - \end{matrix}; \frac{e^2}{abx} \right) {}_2\phi_1 \left(\begin{matrix} a, b \\ e \end{matrix}; x \right) \\
&+ \frac{(qa/b, e/a)_\infty}{(qe/ab, a)_\infty} \frac{(abx/e, qe/abx)_\infty}{(bx/q, q^2/bx)_\infty} {}_2\phi_0 \left(\begin{matrix} q/e, abx/eq \\ - \end{matrix}; \frac{e^2}{abx} \right) {}_2\phi_1 \left(\begin{matrix} a, aq/e \\ aq/b \end{matrix}; \frac{qe}{abx} \right)
\end{aligned}$$

(29)

$$\begin{aligned}
& {}_3\phi_1 \left(\begin{matrix} qa/e, qb/e, q/x \\ qab/e \end{matrix}; \frac{ex}{q^2} \right) \\
&= \frac{(b, q^2/e)_\infty}{(qab/e, q/a)_\infty} \frac{(x, q/x)_\infty}{(qax/e, e/ax)_\infty} {}_2\phi_0 \left(\begin{matrix} e/qa, ax/q \\ - \end{matrix}; \frac{q^2}{ex} \right) {}_2\phi_1 \left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x \right) \\
&+ \frac{(qa/e, qb/a)_\infty}{(qab/e, q/a)_\infty} \frac{(abx/e, qe/abx)_\infty}{(ax/e, qe/ax)_\infty} {}_2\phi_0 \left(\begin{matrix} e/q, x/q \\ - \end{matrix}; \frac{q^2}{ex} \right) {}_2\phi_1 \left(\begin{matrix} qb/e, b \\ qb/a \end{matrix}; \frac{qe}{abx} \right)
\end{aligned} \tag{30}$$

$$\begin{aligned}
& {}_3\phi_1 \left(\begin{matrix} q/a, q/b, qe/abx \\ qe/ab \end{matrix}; \frac{abx}{q^2} \right) \\
&= \frac{(e/a, q^2/e)_\infty}{(qe/ab, qb/e)_\infty} \frac{(x, qe/abx)_\infty}{(qax/e, e/ax)_\infty} {}_2\phi_0 \left(\begin{matrix} b/q, ax/q \\ - \end{matrix}; \frac{q^2}{abx} \right) {}_2\phi_1 \left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x \right) \\
&+ \frac{(q/b, qb/a)_\infty}{(qe/ab, qb/e)_\infty} \frac{(x, qe/abx)_\infty}{(ax/e, qe/ax)_\infty} {}_2\phi_0 \left(\begin{matrix} e/q, abx/eq \\ - \end{matrix}; \frac{q^2}{abx} \right) {}_2\phi_1 \left(\begin{matrix} qb/e, b \\ qb/a \end{matrix}; \frac{qe}{abx} \right)
\end{aligned} \tag{31}$$

$$\begin{aligned}
& {}_3\phi_1 \left(\begin{matrix} q/a, q/b, qe/abx \\ qe/ab \end{matrix}; \frac{abx}{q^2} \right) \\
&= \frac{(e/b, q^2/e)_\infty}{(qe/ab, qa/e)_\infty} \frac{(x, qe/abx)_\infty}{(qbx/e, e/bx)_\infty} {}_2\phi_0 \left(\begin{matrix} a/q, bx/q \\ - \end{matrix}; \frac{q^2}{abx} \right) {}_2\phi_1 \left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x \right) \\
&+ \frac{(q/a, qa/b)_\infty}{(qe/ab, qa/e)_\infty} \frac{(x, qe/abx)_\infty}{(bx/e, qe/bx)_\infty} {}_2\phi_0 \left(\begin{matrix} e/q, abx/eq \\ - \end{matrix}; \frac{q^2}{abx} \right) {}_2\phi_1 \left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx} \right)
\end{aligned} \tag{32}$$

$$\begin{aligned}
& {}_2\phi_1 \left(\begin{matrix} a, b \\ e \end{matrix}; x \right) \\
&= \frac{(e/a, e/b)_\infty}{(e, e/ab)_\infty} {}_3\phi_2 \left(\begin{matrix} a, b, abx/e \\ 0, qab/e \end{matrix}; q \right) \\
&+ \frac{(a, b)_\infty}{(e, ab/e)_\infty} \frac{(abx/e)_\infty}{(x)_\infty} {}_3\phi_2 \left(\begin{matrix} e/a, e/b, x \\ 0, eq/ab \end{matrix}; q \right)
\end{aligned} \tag{33}$$

$$\begin{aligned}
& {}_2\phi_1 \left(\begin{matrix} a, b \\ e \end{matrix}; x \right) \\
&= \frac{(e/a, b)_\infty}{(e, b/a)_\infty} \frac{(ax, q/ax)_\infty}{(x, q/x)_\infty} {}_2\phi_1 \left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx} \right) \\
&+ \frac{(a, e/b)_\infty}{(e, a/b)_\infty} \frac{(bx, q/bx)_\infty}{(x, q/x)_\infty} {}_2\phi_1 \left(\begin{matrix} b, qb/e \\ qb/a \end{matrix}; \frac{qe}{abx} \right)
\end{aligned} \tag{34}$$

$$\begin{aligned}
& {}_3\phi_1 \left(\begin{matrix} a, b, q/x \\ qab/e \end{matrix}; \frac{x}{e} \right) \\
&= \frac{(qa/e, qb/e)_\infty}{(q/e, qab/e)_\infty} {}_2\phi_1 \left(\begin{matrix} a, b \\ e \end{matrix}; x \right) \\
&+ \frac{(a, b)_\infty}{(qab/e, e/q)_\infty} {}_2\phi_0 \left(\begin{matrix} e/q, q/x \\ - \end{matrix}; \frac{x}{e} \right) {}_2\phi_1 \left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x \right)
\end{aligned} \tag{35}$$

$$\begin{aligned}
& {}_3\phi_1 \left(\begin{matrix} a, b, q/x \\ qab/e \end{matrix}; \frac{x}{e} \right) \\
&= \frac{(qb/e, b)_\infty}{(b/a, qab/e)_\infty} {}_2\phi_0 \left(\begin{matrix} 1/a, ax/e \\ - \end{matrix}; \frac{e}{x} \right) {}_2\phi_1 \left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx} \right) \\
&+ \frac{(a, qa/e)_\infty}{(a/b, qab/e)_\infty} {}_2\phi_0 \left(\begin{matrix} 1/b, bx/e \\ - \end{matrix}; \frac{e}{x} \right) {}_2\phi_1 \left(\begin{matrix} b, qb/e \\ qb/a \end{matrix}; \frac{qe}{abx} \right)
\end{aligned} \tag{36}$$

$$\begin{aligned}
& {}_2\phi_1 \left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx} \right) \\
&= \frac{(q/b, qa/e)_\infty}{(q/e, qa/b)_\infty} \frac{(bx/e, qe/bx)_\infty}{(abx/e, qe/abx)_\infty} {}_2\phi_1 \left(\begin{matrix} a, b \\ e \end{matrix}; x \right) \\
&+ \frac{(a, e/b)_\infty}{(e/q, qa/b)_\infty} \frac{(bx/q, q^2/bx)_\infty}{(abx/e, qe/abx)_\infty} {}_2\phi_1 \left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x \right)
\end{aligned} \tag{37}$$

$$\begin{aligned}
& {}_2\phi_1 \left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx} \right) \\
&= \frac{(a, qa/e)_\infty}{(ab/e, qa/b)_\infty} \frac{(q/x)_\infty}{(qe/abx)_\infty} {}_3\phi_2 \left(\begin{matrix} q/b, e/b, qe/abx \\ 0, qe/ab \end{matrix}; q \right) \\
&+ \frac{(q/b, e/b)_\infty}{(qa/b, e/ab)_\infty} {}_3\phi_2 \left(\begin{matrix} a, qa/e, q/x \\ qab/e, 0 \end{matrix}; q \right)
\end{aligned} \tag{38}$$

$$\begin{aligned}
& {}_2\phi_1 \left(\begin{matrix} b, qb/e \\ qb/a \end{matrix}; \frac{qe}{abx} \right) \\
&= \frac{(q/a, qb/e)_\infty}{(q/e, qb/a)_\infty} \frac{(ax/e, qe/ax)_\infty}{(abx/e, qe/abx)_\infty} {}_2\phi_1 \left(\begin{matrix} a, b \\ e \end{matrix}; x \right) \\
&+ \frac{(b, e/a)_\infty}{(e/q, qb/a)_\infty} \frac{(ax/q, q^2/ax)_\infty}{(abx/e, qe/abx)_\infty} {}_2\phi_1 \left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x \right)
\end{aligned} \tag{39}$$

$$\begin{aligned}
& {}_2\phi_1 \left(\begin{matrix} b, qb/e \\ qb/a \end{matrix}; \frac{qe}{abx} \right) \\
&= \frac{(q/a, e/a)_\infty}{(qb/a, e/ab)_\infty} {}_3\phi_2 \left(\begin{matrix} b, qb/e, q/x \\ qab/e, 0 \end{matrix}; q \right) \\
&+ \frac{(b, qb/e)_\infty}{(ab/e, qb/a)_\infty} \frac{(q/x)_\infty}{(qe/abx)_\infty} {}_3\phi_2 \left(\begin{matrix} q/a, e/a, qe/abx \\ 0, qe/ab \end{matrix}; q \right)
\end{aligned} \tag{40}$$

$$\begin{aligned}
& {}_2\phi_1 \left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x \right) \\
&= \frac{(q/a, q/b)_\infty}{(q^2/e, e/ab)_\infty} {}_3\phi_2 \left(\begin{matrix} aq/e, bq/e, abx/e \\ 0, qab/e \end{matrix}; q \right) \\
&+ \frac{(qa/e, qb/e)_\infty}{(q^2/e, ab/e)_\infty} \frac{(abx/e)_\infty}{(x)_\infty} {}_3\phi_2 \left(\begin{matrix} q/a, q/b, x \\ 0, eq/ab \end{matrix}; q \right)
\end{aligned} \tag{41}$$

$${}_2\phi_1 \left(\begin{matrix} qa/e, qb/e \\ q^2/e \end{matrix}; x \right)$$

$$\begin{aligned}
&= \frac{(q/a, qb/e)_\infty (qax/e, e/ax)_\infty}{(q^2/e, b/a)_\infty (x, q/x)_\infty} {}_2\phi_1 \left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx} \right) \\
&+ \frac{(q/b, qa/e)_\infty (qbx/e, e/bx)_\infty}{(q^2/e, a/b)_\infty (x, q/x)_\infty} {}_2\phi_1 \left(\begin{matrix} b, qb/e \\ qb/a \end{matrix}; \frac{qe}{abx} \right)
\end{aligned} \tag{42}$$

$$\begin{aligned}
&\frac{(abx/e)_\infty}{(x)_\infty} {}_3\phi_1 \left(\begin{matrix} e/a, e/b, eq/abx \\ eq/ab \end{matrix}; \frac{abx}{e^2} \right) \\
&= \frac{(q/a, q/b)_\infty}{(q/e, qe/ab)_\infty} {}_2\phi_1 \left(\begin{matrix} a, b \\ e \end{matrix}; x \right) \\
&+ \frac{(e/a, e/b)_\infty}{(qe/ab, e/q)_\infty} {}_2\phi_0 \left(\begin{matrix} e/q, eq/abx \\ - \end{matrix}; \frac{abx}{e^2} \right) {}_2\phi_1 \left(\begin{matrix} aq/e, bq/e \\ q^2/e \end{matrix}; x \right)
\end{aligned} \tag{43}$$

$$\begin{aligned}
&\frac{(q/x)_\infty}{(eq/abx)_\infty} {}_3\phi_1 \left(\begin{matrix} e/a, e/b, eq/abx \\ eq/ab \end{matrix}; \frac{abx}{e^2} \right) \\
&= \frac{(e/a, q/a)_\infty}{(qe/ab, b/a)_\infty} {}_2\phi_0 \left(\begin{matrix} b/e, ax/e \\ - \end{matrix}; \frac{e^2}{abx} \right) {}_2\phi_1 \left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx} \right) \\
&+ \frac{(q/b, e/b)_\infty}{(a/b, qe/ab)_\infty} {}_2\phi_0 \left(\begin{matrix} a/e, bx/e \\ - \end{matrix}; \frac{e^2}{abx} \right) {}_2\phi_1 \left(\begin{matrix} b, qb/e \\ qb/a \end{matrix}; \frac{eq}{abx} \right)
\end{aligned} \tag{44}$$

5 Proofs

Proof of (25)

Take $(i; j, k; l, m, n) = (3; 0, 2; 1, 4, 5)$ in (12) and use $F(3) = F(3; 4, 5)$, to get

$$\begin{aligned}
&(\beta_{43}, \beta_{53}, \alpha_{012}; q)_\infty {}_3\phi_2 \left(\begin{matrix} \alpha_{045}, \alpha_{145}, \alpha_{245} \\ \beta_{43}, \beta_{53} \end{matrix}; \alpha_{012} \right) \\
&= \frac{(\alpha_{124}, \alpha_{125}, \alpha_{245}, \beta_{03}; q)_\infty}{(\beta_{20}/q; q)_\infty} {}_3\phi_2 \left(\begin{matrix} \alpha_{014}, \alpha_{015}, \alpha_{045} \\ \beta_{03}, \beta_{02} \end{matrix}; q \right) \\
&+ \frac{(\alpha_{014}, \alpha_{015}, \alpha_{045}, \beta_{23}; q)_\infty}{(\beta_{02}/q; q)_\infty} {}_3\phi_2 \left(\begin{matrix} \alpha_{124}, \alpha_{125}, \alpha_{245} \\ \beta_{23}, \beta_{20} \end{matrix}; q \right).
\end{aligned}$$

In terms of the parameters a, b, c, e, f and q , this is

$$\begin{aligned}
&{}_3\phi_2 \left(\begin{matrix} qabc/ef, a, b \\ qab/f, qab/e \end{matrix}; \frac{q}{c} \right) \\
&= \frac{(e/c, f/c, b, q^2ab/ef; q)_\infty}{(q/c, qab/f, qab/e, ef/qac; q)_\infty} {}_3\phi_2 \left(\begin{matrix} qa/f, qa/e, qabc/ef \\ q^2ab/ef, q^2ac/ef \end{matrix}; q \right) \\
&+ \frac{(qa/f, qa/e, qabc/ef, qb/c; q)_\infty}{(q/c, qab/f, qab/e, qac/ef; q)_\infty} {}_3\phi_2 \left(\begin{matrix} e/c, f/c, b \\ qb/c, ef/ac \end{matrix}; q \right).
\end{aligned}$$

Set $f = abcx/e$ to get

$$\begin{aligned}
&{}_3\phi_2 \left(\begin{matrix} q/x, a, b \\ qe/cx, qab/e \end{matrix}; \frac{q}{c} \right) \\
&= \frac{(b, abx/e; q)_\infty (e/c, q^2/cx; q)_\infty}{(qab/e, bx/q; q)_\infty (q/c, qe/cx; q)_\infty} {}_3\phi_2 \left(\begin{matrix} qe/bcx, qa/e, q/x \\ q^2/cx, q^2/bx \end{matrix}; q \right) \\
&+ \frac{(qa/e, q/x; q)_\infty (qb/c, qe/bcx; q)_\infty}{(qab/e, q/bx; q)_\infty (q/c, qe/cx; q)_\infty} {}_3\phi_2 \left(\begin{matrix} e/c, abx/e, b \\ qb/c, bx \end{matrix}; q \right).
\end{aligned} \tag{45}$$

Next, observe that as $c \rightarrow 0$,

$${}_3\phi_2 \left(\begin{matrix} q/x, a, b \\ qe/cx, qab/e \end{matrix}; \frac{q}{c} \right) \rightarrow {}_3\phi_1 \left(\begin{matrix} q/x, a, b \\ qab/e \end{matrix}; \frac{x}{e} \right) \quad (46)$$

$$\begin{aligned} {}_3\phi_2 \left(\begin{matrix} qe/bcx, qa/e, q/x \\ q^2/cx, q^2/bx \end{matrix}; q \right) &\rightarrow {}_2\phi_1 \left(\begin{matrix} qa/e, q/x \\ q^2/bx \end{matrix}; \frac{e}{b} \right) \\ &= \frac{(qe/abx, qa/b; q)_\infty}{(q^2/bx, e/b; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, qa/e \\ qa/b \end{matrix}; \frac{qe}{abx} \right) \end{aligned} \quad (47)$$

$$\begin{aligned} {}_3\phi_2 \left(\begin{matrix} e/c, abx/e, b \\ qb/c, bx \end{matrix}; q \right) &\rightarrow {}_2\phi_1 \left(\begin{matrix} abx/e, b \\ bx \end{matrix}; \frac{e}{b} \right) \\ &= \frac{(x, e; q)_\infty}{(bx, e/b; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, b \\ e \end{matrix}; x \right). \end{aligned} \quad (48)$$

The iterate of Heine's transformation (3) has been used to transform the ${}_2\phi_1$ functions in (47) and (48). Furthermore, using (1) we have

$$\begin{aligned} \frac{(e/c, q^2/cx; q)_\infty}{(q/c, qe/cx; q)_\infty} &= {}_2\phi_1 \left(\begin{matrix} q/e, x/q \\ q/c \end{matrix}; \frac{qe}{cx} \right) \\ &\rightarrow {}_2\phi_0 \left(\begin{matrix} q/e, x/q \\ - \end{matrix}; \frac{e}{x} \right) \end{aligned} \quad (49)$$

and similarly

$$\begin{aligned} \frac{(qb/c, qe/bcx; q)_\infty}{(q/c, qe/cx; q)_\infty} &= {}_2\phi_1 \left(\begin{matrix} 1/b, bx/e \\ q/c \end{matrix}; \frac{qe}{cx} \right) \\ &\rightarrow {}_2\phi_0 \left(\begin{matrix} 1/b, bx/e \\ - \end{matrix}; \frac{e}{x} \right). \end{aligned} \quad (50)$$

Equation (25) now follows by taking the limit as $c \rightarrow 0$ in (45) and using equations (46) – (50).

Remaining proofs

The remaining equations can be proved in the same way. Equations (26) – (32) can also be obtained from (25) just by change of variable. A summary of the proofs is as follows.

Eqn.	$(i; j, k; l, m, n)$	Alternative proof by change of variable
(25)	(3; 0, 2; 1, 4, 5)	
(26)	(3; 0, 1; 2, 4, 5)	$a \leftrightarrow b$ in (25)
(27)	(5; 0, 1; 2, 3, 4)	$a \rightarrow e/a, b \rightarrow e/b, x \rightarrow abx/e$ in (25)
(28)	(3; 2, 4; 0, 1, 5)	$a \rightarrow qa/e, b \rightarrow qb/e, e \rightarrow q^2/e$ in (25)
(29)	(5; 0, 2; 1, 3, 4)	$a \rightarrow e/b, b \rightarrow e/a, x \rightarrow abx/e$ in (25)
(30)	(3; 1, 4; 0, 2, 5)	$a \rightarrow qb/e, b \rightarrow qa/e, e \rightarrow q^2/e$ in (25)
(31)	(5; 1, 4; 0, 2, 3)	$a \rightarrow q/a, b \rightarrow q/b, e \rightarrow q^2/e, x \rightarrow abx/e$ in (25)
(32)	(5; 2, 4; 0, 1, 3)	$a \rightarrow q/b, b \rightarrow q/a, e \rightarrow q^2/e, x \rightarrow abx/e$ in (25)
(33)	(0; 3, 5; 1, 2, 4)	
(34)	(0; 1, 2; 3, 4, 5)	
(35)	(3; 0, 4; 1, 2, 5)	
(36)	(3; 1, 2; 0, 4, 5)	
(37)	(2; 0, 4; 1, 3, 5)	
(38)	(2; 3, 5; 0, 1, 4)	
(39)	(1; 0, 4; 2, 3, 5)	$a \leftrightarrow b$ in (37)
(40)	(1; 3, 5; 0, 2, 4)	$a \leftrightarrow b$ in (38)
(41)	(4; 3, 5; 0, 1, 2)	$a \rightarrow qa/e, b \rightarrow qb/e, e \rightarrow q^2/e$ in (33)
(42)	(4; 1, 2; 0, 3, 5)	$a \rightarrow qa/e, b \rightarrow qb/e, e \rightarrow q^2/e$ in (34)
(43)	(5; 0, 4; 1, 2, 3)	$a \rightarrow e/a, b \rightarrow e/b, x \rightarrow abx/e$ in (35)
(44)	(5; 1, 2; 0, 3, 4)	$a \rightarrow e/a, b \rightarrow e/b, x \rightarrow abx/e$ in (36)

6 Remarks

Equation (33) is due to Sears [6, (p. 178 II(a))]. It also can be obtained by taking $\theta_r = (abx/ep)^r$ in [6, Theorem 4] and then replacing all occurrences of p with q .

Equation (34) is due to Watson [7, (p. 285 eq. (7))]. It also appears in [6, (p. 178 III(d))] and [4, (p. 106, eq. (4.3.2))]. Equivalent forms of (34) are given in [6, (p. 178 II(c))] and [4, (p. 92, ex. 3.8)]; the equivalence follows immediately using the transformation formula (7).

Equation (35) is due to Sears [6, (p. 178 III(c))]. An equivalent form (use Heine's transformation (2)) is given in [4, (p. 64, eq. (3.3.5))].

In equation (39), first interchange a and b , then replace x with x/b and let $b \rightarrow \infty$, to get

$$\begin{aligned}
 & \frac{(qa/e)_\infty (x/e, qe/x)_\infty}{(q/e)_\infty (ax/e, qe/ax)_\infty} {}_1\phi_1 \left(\begin{matrix} a \\ e \end{matrix}; x \right) \\
 & + \frac{(a)_\infty (x/q, q^2/x)_\infty}{(e/q)_\infty (ax/e, qe/ax)_\infty} {}_1\phi_1 \left(\begin{matrix} qa/e \\ q^2/e \end{matrix}; x \right) \\
 & = {}_2\phi_1 \left(\begin{matrix} a, qa/e \\ 0 \end{matrix}; \frac{qe}{ax} \right)
 \end{aligned} \tag{51}$$

This formula is a q -analogue of [1, (p. 192, eq. (4.1.13))]. Note however that the right hand side of equation (4.1.13) in [1] is an asymptotic expansion which does not converge, while the q -analogue (51) is an equality among convergent series.

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