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A necessary and sufficient condition for total observability of discrete-time linear time-varying systems

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Abstract: This paper deals with the total observability problem of discrete-time linear time-varying systems. In particular, a review and suitable analysis of the state-of-the-art of this emerging area are provided. Subsequently, the total observability problem of discrete-time linear time-varying systems is transformed into the one of checking the rank of a convex sum of matrices. As a result, a new total observability test is proposed, along with a suitable computational strategy. The final part of this paper shows examples regarding observability analysis that clearly exhibit the benefits of using the proposed approach.

Keywords: linear time-varying systems, observability, block P-matrices, discrete-time systems

1. INTRODUCTION

Observability concerns the possibility of determining the initial state of a dynamic system from the past inputs and outputs, and is undoubtedly one of the most important properties of both linear and non-linear systems (Antsaklis and Michel, 2007). In fact, the constantly growing complexity of modern systems has led to a situation where it is not possible anymore to assume that the whole state is accessible through measurements. In such situations, the problem of designing state observers, whose existence is strongly related to the satisfaction of the observability property, becomes of paramount importance (Rajamani, 1998; Korbicz et al., 2007; Darouach and Boutat-Baddas, 2008; Farza et al., 2014; Witczak, 2014). For this reason, the last decades have witnessed a strong effort of the control community to assess the observability property (Gilbert, 1963; Zhirabok and Shumsky, 2012; Luo et al., 2015; Levin and Narendra, 1996).

Linear time-varying (LTV) systems have attracted a lot of interest, both as general models of linear system behavior and as linearized models of non-linear systems along a given trajectory (D'Angelo, 1970; Rugh, 1996; Schwaller et al., 2016). Several different approaches have been reported in the literature concerning the observer design problem for LTV systems, e.g. the Kalman–Bucy filter (Kalman and Bucy, 1961), the matrix differential Riccati equation approach (Kwakernaak and Sivan, 1972), the weighted observability Grammian approach (Rugh, 1996), the Luenberger observer (Lovass-Nagy et al., 1980) and the least squares with covariance reset approach (Chen and Yen, 1999).

However, when it comes to the assessment of the observability property in LTV systems, the results found in the literature are mostly devoted to continuous-time systems. For example, a thorough discussion of the observability concept from both theoretical and practical viewpoint is provided in Kreindler and Sarachik (1964). The observability characterized in terms of known system coefficient matrices was investigated in Silverman and Meadows (1967). An algebraic rank condition, which relies on expanding the time varying structure matrix in the generated Lie algebra, with respect to a basis was provided by Szigeti (1992). In particular, it is proven that, under a differential-algebraic condition for the time-dependent coefficients, observability is equivalent to a multivariable Kalman condition. Necessary and sufficient conditions for observability of LTV systems with coefficients in the form of time polynomials were proposed by Starkov (2002). These conditions rely on representing the solution of an LTV system as a product of matrix exponentials using the Wei–Norman formula (Wei and Norman, 1964). A necessary algebraic condition, which allows determining that a given continuous-time LTV system is not observable, is given by Leiva and Siegmund (2003).

However, one can verify that all works (Kreindler and Sarachik, 1964; Silverman and Meadows, 1967; Szigeti, 1992; Starkov, 2002; Wei and Norman, 1964; Leiva and Siegmund, 2003) deal with continuous-time systems. In fact, to the best of the authors' knowledge, the only works that deal with the assessment of the observability property for discrete-time LTV systems are Seo et al. (2005) and Reissig et al. (2014). However, the focus of both works is centered on the case where uncertainties

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are present in the model description. In particular, Seo et al. (2005) derived sufficient conditions, which ensure observability of discrete-time LTV systems subject to norm-bounded uncertainties, by using the observability Gramians of these systems. On the other hand, Reissig et al. (2014) studied the *strong structural observability*, which corresponds to the case when only the nonzero pattern of the system (i.e., the locations of the nonzero entries in its coefficient matrices) is known.

From the literature review, it seems that at this point there is a lack of a practical test capable of assessing the observability property of discrete-time LTV systems, and with the relevant feature of being not only a sufficient condition but a necessary one, too. The goal of this paper is to fill this gap, by providing a necessary and sufficient condition for the observability of a class of discrete-time LTV systems, i.e., those whose time-varying state space matrices admit a polytopic representation through a convex combination of local linear submodel matrices. This class of systems is quite wide, encompassing representations that have received much attention in the last decades, e.g., polytopic linear parameter-varying (LPV) systems (Mohammadpour and Scherer, 2012), Takagi–Sugeno (T-S) fuzzy systems (Tanaka and Wang, 2001) and switched systems (Liberzon, 2003). The provided necessary and sufficient condition is derived using the results on the rank characterization of convex combinations of matrices (Kolodziejczak and Szulc, 1999), and involves checking whether or not a matrix is a block P-one (Elsnerm and Szulc, 1998) with respect to some partition of an appropriate set of integers. Illustrative examples are used to show that the proposed condition is easy to check, such that it constitutes a practical test for assessing the total observability of discrete-time LTV systems.

This paper is organized as follows. Section 2 recalls some basic notions and mathematical tools used throughout this paper. Section 3 introduces the main theoretical result. Section 4 provides numerical examples, showing the effectiveness of the proposed approach. Finally, Section 5 concludes the paper.

2. PRELIMINARIES

Before providing the main result of this paper, let us introduce mathematical tools that will be used for the observability analysis of a class of systems that will be defined in the subsequent part of this section.

2.1 LTV discrete-time systems and their observability

Let us consider a discrete-time LTV system

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_k \boldsymbol{x}_k + \boldsymbol{B}_k \boldsymbol{u}_k \tag{1}$$

$$\boldsymbol{y}_k = \boldsymbol{C}_k \boldsymbol{x}_k, \tag{2}$$

where $\boldsymbol{x}_k \in \mathbb{R}^n$ stands for the state, $\boldsymbol{y}_k \in \mathbb{R}^m$ is the output, and $\boldsymbol{u}_k \in \mathbb{R}^r$ denotes the nominal control input.

Since the class of systems is briefly portrayed, it is possible to provide the following suitable observability definitions (Houpis and Lamont, 1992)

Definition 1. The system (1)–(2) is completely observable if and only if, for any initial time k_0, any initial state $\boldsymbol{x}_{k_0}$ can be determined from the knowledge of output $\boldsymbol{y}_k$ and input $\boldsymbol{u}_k$ for $k_0 \leq k \leq k_N$, where k_N is some final finite time.

Definition 2. The system (1)–(2) is totally observable if and only if it is completely observable for every k_0 and $k_N > k_0$.

Like in the linear time-invariant (LTI) framework (Antsaklis and Michel, 2007), the observability of (1)–(2) can be formulated by the following condition

$$\operatorname{rank}(\boldsymbol{O}_{k_0}) = n, \tag{3}$$

where

$$\boldsymbol{O}_{k_0} = \begin{bmatrix} \boldsymbol{C}_{k_0} \\ \boldsymbol{C}_{k_0+1}\boldsymbol{A}_{k_0} \\ \boldsymbol{C}_{k_0+2}\boldsymbol{A}_{k_0+1}\boldsymbol{A}_{k_0} \\ \vdots \\ \boldsymbol{C}_{k_0+n-1}\boldsymbol{A}_{k_0+n-2}\cdot \ldots \cdot \boldsymbol{A}_{k_0+1}\boldsymbol{A}_{k_0} \end{bmatrix} \tag{4}$$

is the *observability matrix* with $k_N = k_0 + n - 1$. If one wants to verify the *total observability* of (1)–(2), then it is necessary to check (3) for any k_0, which is not a trivial task.

The main objective of this paper is to provide an easily checkable necessary and sufficient condition for the total observability of (1)–(2). Similarly as for Leiva and Siegmund (2003), to achieve this goal the time-dependence of the matrices $\boldsymbol{A}_k$, $\boldsymbol{B}_k$ and $\boldsymbol{C}_k$ is separated from their linear time-independent structure. However, this task is to be realised in the subsequent section along with a new proposal for the total observability test, but first the necessary mathematical tools have to be introduced.

2.2 Brief introduction to real and block P-matrices

Real P-matrices are well known in matrix theory because they play an important role in many applications (Elsner et al., 2002). In Johnson and Tsatsomeros (1995), it was shown that the P-property of a single matrix is equivalent to the nonsingularity of all matrices in a certain convex matrix set. This fact has motivated generalizing this notion, introducing block P-matrices (Elsnerm and Szulc, 1998) later used to study the Schur (Elsner and Szulc, 1998) and Hurwitz (Elsner and Szulc, 2000) stability of convex combinations of matrices.

In order to make the paper self-contained, let us recall some essential definitions and concepts that will be exploited further on. Let $\boldsymbol{X} \in \mathbb{R}^{n\times n}$. The $m \times m$ submatrix of $\boldsymbol{X}$ formed by deleting $n - m$ columns and the same $n - m$ rows from $\boldsymbol{X}$ is called an m-th order principal submatrix of $\boldsymbol{X}$.

The matrix $\boldsymbol{X} \in \mathbb{R}^{n\times n}$ is called a P-matrix if all its principal minors are positive (Elsner et al., 2002). On the other hand, a matrix $\boldsymbol{X} \in \mathbb{R}^{n\times n}$ is a block P-matrix with respect to a partition $N(\lambda)$ of $N = \{1, \ldots, n\}$ into $\lambda \in [1, n]$ pair-wise disjoint nonvoid subsets N_i of cardinality n_i, $i = 1, \ldots, \lambda$, if, for any $T \in \mathcal{T}_n^{(\lambda)}$,

$$\det\left(\boldsymbol{T}\boldsymbol{X} + (\boldsymbol{I} - \boldsymbol{T})\right) \neq 0, \tag{5}$$

where $\mathcal{T}_n^{\lambda}$ is the set of all diagonal matrices $\boldsymbol{T} \in \mathbb{R}^{n\times n}$ such that $\boldsymbol{T}[N_i] = t_i$, $t_i \in [0, 1]$, $i = 1, \ldots, \lambda$, with $\boldsymbol{T}[N_i]$ being the principal submatrix of T with row and column indices in N_i (Elsner et al., 2002). A P-matrix is also a block P-matrix with respect to any partition (Elsner et al., 2002).

Having a general description of P-matrices, it is possible to remind the following lemmas (Kolodziejczak and Szulc, 1999), which play an important role in the derivation of the main result of this paper.

Lemma 1. Let $\boldsymbol{M}^j \in \mathbb{C}^{n_r\times n_c}$, $j = 1, \ldots, J$, and let us define

$$\boldsymbol{Q}_{j,j} = \boldsymbol{M}^j \left(\boldsymbol{M}^j\right)^T, \qquad j = 1, \ldots, J \tag{6}$$

$$\boldsymbol{Q}_{j,a} = \boldsymbol{M}^j\left(\boldsymbol{M}^a\right)^T + \boldsymbol{M}^a\left(\boldsymbol{M}^j\right)^T - \boldsymbol{M}^a\left(\boldsymbol{M}^a\right)^T - \boldsymbol{M}^j\left(\boldsymbol{M}^j\right)^T, \quad j < a \tag{7}$$

and the matrices $\boldsymbol{R}_j$, $j = 1, \ldots, J$,

$$\boldsymbol{R}_j = (\boldsymbol{R}^j_{a,b})_{a,b\in[1,J]} = \begin{bmatrix} \boldsymbol{R}^j_{1,1} & \boldsymbol{R}^j_{1,2} & \cdots & \boldsymbol{R}^j_{1,J} \\ \boldsymbol{R}^j_{2,1} & \boldsymbol{R}^j_{2,2} & \cdots & \boldsymbol{R}^j_{2,J} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{R}^j_{J,1} & \boldsymbol{R}^j_{J,2} & \cdots & \boldsymbol{R}^j_{J,J} \end{bmatrix}, \tag{8}$$

with the generic block entry $\boldsymbol{R}^j_{a,b}$ defined as

$$\boldsymbol{R}^j_{a,b} = \begin{cases} \boldsymbol{Q}_{j,j} & \text{if } a = 1, b = 1 \\ \boldsymbol{Q}_{b-1,j} & \text{if } a = 1, b = 2, \ldots, j \\ \boldsymbol{I}_{n_r} & \text{if } a = b, 1 < b \leq J \\ -\boldsymbol{I}_{n_r} & \text{if } b = 1, a = j+1 \\ \boldsymbol{0}_{n_r} & \text{otherwise.} \end{cases} \tag{9}$$

Then, the following statements are equivalent:

(a) all convex combinations of matrices $\boldsymbol{M}^j$, $j = 1, \ldots, J$, are full row-rank;
(b) $\boldsymbol{M}^J$ (A.1) is full row-rank and the $(J-1)Jn_r \times (J-1)Jn_r$ matrix is a block P-matrix with respect to the partition $\{F_1, \ldots, F_{J-1}\}$ of $\{1, \ldots, (J-1)Jn_r\}$, with $F_i = \{(i-1)Jn_r + 1, \ldots, iJn_r\}$, $i = 1, \ldots, J-1$.

Proof: *See Theorem 2 in Kolodziejczak and Szulc (1999).* □

In the case of square matrices, the full row-rank condition corresponds to nonsingularity, and the following lemma can be used instead.

Lemma 2. Let $\boldsymbol{M}^j \in \mathbb{C}^{n\times n}$, $j = 1, \ldots, J$. Then, the following statements are equivalent:

(a) all convex combinations of matrices $\boldsymbol{M}^j$, $j = 1, \ldots, J$, are nonsingular;
(b) $\boldsymbol{M}^J$ is nonsingular and the $(J-1)n \times (J-1)n$ matrix

$$\begin{bmatrix} \boldsymbol{M}^1(\boldsymbol{M}^J)^{-1} & \left(\boldsymbol{M}^2 - \boldsymbol{M}^1\right)(\boldsymbol{M}^J)^{-1} & \cdots & \left(\boldsymbol{M}^{J-1} - \boldsymbol{M}^{J-2}\right)(\boldsymbol{M}^J)^{-1} \\ -\boldsymbol{I}_n & \boldsymbol{I}_n & \cdots & \boldsymbol{0}_n \\ \boldsymbol{0}_n & -\boldsymbol{I}_n & \cdots & \boldsymbol{0}_n \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0}_n & \boldsymbol{0}_n & \cdots & \boldsymbol{0}_n \\ \boldsymbol{0}_n & \boldsymbol{0}_n & \cdots & \boldsymbol{I}_n \end{bmatrix} \tag{10}$$

is a block P-matrix with respect to the partition $\{F_1, \ldots, F_{J-1}\}$ of $\{1, \ldots, (J-1)n\}$, with $F_i = \{(i-1)n+1, \ldots, in\}$, $i = 1, \ldots, J-1$.

Proof: *See Theorem 4 in Elsner and Szulc (1998).* □

3. NEW OBSERVABILITY TEST

This section presents the main results of this paper. To achieve this goal, the system (1)–(2) is expressed in the following form:

$$\boldsymbol{x}_{k+1} = \sum_{i=1}^{M} h_{i,k}[\boldsymbol{A}^i\boldsymbol{x}_k + \boldsymbol{B}^i\boldsymbol{u}_k], \tag{11}$$

$$\boldsymbol{y}_k = \sum_{i=1}^{M} h_{i,k}\boldsymbol{C}^i\boldsymbol{x}_{k+1}, \tag{12}$$

where M stands for the number of local linear submodels, while $h_{i,k}$ $(i = 1, \ldots, M)$ are time-varying parameters satisfying

$$\begin{cases} \sum_{i=1}^{M} h_{i,k} = 1, \\ 0 \leqslant h_{i,k} \leqslant 1, \quad \forall i = 1, \ldots, M. \end{cases} \tag{13}$$

Note that there is no assumption about the knowledge of these parameters, the only assumption is that they must satisfy the constraint (13).

Theorem 1. Let us define indices $l_k \in \{1, \ldots, M\}$ (for $k = k_0, \ldots, k_0 + n - 1$) corresponding to $k_0, \ldots, k_0 + n - 1$, and let $N = M^n$ denote the total number of possible combinations of these indices. Moreover, let us define the following matrices:

$$\boldsymbol{M}^j = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} & \ldots & \ldots & \boldsymbol{0} & \boldsymbol{0} & \ldots & \ldots & \boldsymbol{0} & (\boldsymbol{C}^{l_{k_0+n-1}})^T \\ -(\boldsymbol{A}^{l_{k_0+n-2}})^T & \boldsymbol{I}_n & \ldots & \ldots & \boldsymbol{0} & \boldsymbol{0} & \ldots & \ldots & (\boldsymbol{C}^{l_{k_0+n-2}})^T & \boldsymbol{0} \\ \vdots & \ddots & \ddots & \ddots & \ldots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ldots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \ldots & \ldots & \boldsymbol{I}_n & \boldsymbol{0} & \ldots & \ldots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \ldots & \ldots & -(\boldsymbol{A}^{l_{k_0}})^T & (\boldsymbol{C}^{l_{k_0}})^T & \ldots & \ldots & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},$$
$$j = 1, \ldots, N. \tag{14}$$

The following statements are equivalent:

(a) the system (1)–(2), in the form (11)–(12), is totally observable;
(b) $\boldsymbol{M}^N$ has full row-rank and the $(N-1)Nn^2$-by-$(N-1)Nn^2$ block matrix $\boldsymbol{V}$ (A) with $\boldsymbol{R}_j$ defined by (8), is a block P-matrix with respect to the partition $\{F_1, \ldots, F_{N-1}\}$ of $\{1, \ldots, (N-1)Nn^2\}$, with $F_i = \{(i-1)Nn^2 + 1, \ldots, iNn^2\}$, $i = 1, \ldots, N-1$.

Proof. The statement (a) is true if and only if the $n^2 \times n(n+m-1)$ dimensional matrix

$$\boldsymbol{S}_k = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} & \ldots & \ldots & \boldsymbol{0} & \boldsymbol{0} & \ldots & \ldots & \boldsymbol{0} & \boldsymbol{C}^T_{k_0+n-1} \\ -\boldsymbol{A}^T_{k_0+n-2} & \boldsymbol{I}_n & \ldots & \ldots & \boldsymbol{0} & \boldsymbol{0} & \ldots & \ldots & \boldsymbol{C}^T_{k_0+n-2} & \boldsymbol{0} \\ \vdots & \ddots & \ddots & \ldots & \ldots & \ldots & \ddots & \ddots & \ldots & \vdots \\ \vdots & \ddots & \ddots & \ldots & \ldots & \ldots & \ddots & \ddots & \ldots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \ldots & \ldots & \boldsymbol{I}_n & \boldsymbol{0} & \ldots & \ldots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \ldots & \ldots & -\boldsymbol{A}^T_{k_0} & \boldsymbol{C}^T_{k_0} & \ldots & \ldots & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \tag{15}$$

has rank n^2 for all k_0. Indeed, the first block row of (15) can be multiplied by $\boldsymbol{A}^T_{k_0+n-2}$ and then the result is added to the second one. Similarly, the procedure continues by multiplying the second block row of the resulting matrix by $\boldsymbol{A}^T_{k_0+n-3}$ and then adding the result to the third one. Proceeding in a similar way with the remaining rows gives (note that the procedure resembles the one provided by Rosenbrock (1970) for the LTI systems)

$$\boldsymbol{S}'_k = \left[\begin{matrix} \boldsymbol{I}_n & \boldsymbol{0} & \dots & \dots & \boldsymbol{0} & \boldsymbol{0} & \dots \\ \boldsymbol{0} & \boldsymbol{I}_n & \dots & \dots & \boldsymbol{0} & \boldsymbol{0} & \dots \\ \dots & \dots & \ddots & \ddots & \dots & \vdots & \vdots \\ \dots & \dots & \ddots & \ddots & \dots & \vdots & \vdots & \dots \\ \boldsymbol{0} & \boldsymbol{0} & \dots & \dots & \boldsymbol{I}_n & \boldsymbol{0} & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \dots & \dots & \boldsymbol{0} & \boldsymbol{C}^T_{k_0} & \boldsymbol{A}^T_{k_0}\boldsymbol{C}^T_{k_0+1} \end{matrix}\right.$$

$$\left.\begin{matrix} \dots & \boldsymbol{0} & & \boldsymbol{C}^T_{k_0+n-1} \\ \dots & \boldsymbol{C}^T_{k_0+n-1} & & \boldsymbol{A}^T_{k_0+n-2}\boldsymbol{C}^T_{k_0+n-1} \\ \vdots & \vdots & & \vdots \\ \dots\ \vdots & \vdots & & \vdots \\ \dots & \vdots & \boldsymbol{A}^T_{k_0+1} & \cdots\cdot \boldsymbol{A}^T_{k_0+n-2}\boldsymbol{C}^T_{k_0+n-1} \\ \dots & \dots & \boldsymbol{A}^T_{k_0} & \cdots\cdot \boldsymbol{A}^T_{k_0+n-2}\boldsymbol{C}^T_{k_0+n-1} \end{matrix}\right]. \qquad (16)$$

Thus, it is straightforward to see that (3) is satisfied if and only if the matrix (16) has rank n^2.

Subsequently, it can be noted the matrix $\boldsymbol{S}_{k_0}$ can be written in an expanded form:

$$\boldsymbol{S}_{k_0} = \sum_{l_{k_0}=1}^{M} h_{l_{k_0}} \sum_{l_{k_0+1}=1}^{M} h_{l_{k_0+1}} \dots \sum_{l_{k_0+n-1}=1}^{M} h_{l_{k_0+n-1}} \boldsymbol{M}^{l_{k_0},\dots,l_{k_0+n-1}}, \qquad (17)$$

where the $N = M^n$ matrices $\boldsymbol{M}^{l_{k_0},\dots,l_{k_0+n-1}}$ are defined in the same way as (14), with each j associated with N combinations of $l_{k_0},\dots,l_{k_0+n-1}$. This means that $\boldsymbol{S}_{k_0}$ has rank n^2 if and only if (for all k_0) all convex combinations of $\boldsymbol{M}^j$ ($j = 1,\dots,N$) have rank n^2. Note that the full row-rank of $\boldsymbol{M}^j$, $j = 1,\dots,N$, corresponds to local observability condition of (11)–(12), which clearly justifies the full row rank property of $\boldsymbol{M}^N$ indicated in the statement (b). Finally, by applying Lemma 1, the statement (b) is obtained, which completes the proof.

The computational procedure for checking the observability of (1)–(2) can be summarized as follows:

(1) Obtain matrices $\boldsymbol{M}^i$, $i = 1,\dots,N$.
(2) Check if all matrices $\boldsymbol{M}^i$, $i = 1,\dots,N$, are full row-rank. If not, the system (1)–(2) is not totally observable. Otherwise, continue the algorithm.
(3) Calculate matrices $\boldsymbol{R}_i$ $i = 1,\dots,N$.
(4) Calculate matrix $\boldsymbol{V}$ according to (A.2).
(5) Calculate the principal minors of $\boldsymbol{V}$.
(6) If all principal minors of $\boldsymbol{V}$ are positive, then the system (1)–(2) is totally observable. Otherwise, continue the algorithm.
(7) Find $\boldsymbol{T}$ with $\boldsymbol{T}\left\{(i-1)Nn^2+1,\dots,iNn^2\right\} = t_i$, $i = 1,\dots,N-1$, such that
$$\det\left(\boldsymbol{T}\boldsymbol{V} + (\boldsymbol{I}-\boldsymbol{T})\right) = 0. \qquad (18)$$
If there exists a solution for (18), such that $t_i \in [0,1]$ $\forall i = 1,\dots,N-1$, the system (1)–(2) is not totally observable. Otherwise, it is totally observable.

Remark 1. The proposed strategy can be employed for a general class of multi-input multi-output systems, but when a multi-input single-output system is considered, then the above computational procedure can be simplified significantly. Indeed, in this case the matrices $\boldsymbol{M}^i$ are square ones, and hence we can formulate an alternative procedure to tackle this special case.

Theorem 2. The following statements are equivalent:

(a) the system (1)–(2), in the form (11)–(12), is totally observable;
(b) $\boldsymbol{M}^N$ is nonsingular and the $(N-1)n^2$-by-$(N-1)n^2$ matrix $\boldsymbol{W}$ (A.3) is a block P-matrix (Kolodziejczak and Szulc, 1999) with respect to the partition $\{F_1,\dots,F_{N-1}\}$ of $\{1,\dots,(N-1)n^2\}$, with $F_i = \{(i-1)n^2+1,\dots,in^2\}$, $i = 1,\dots,N-1$.

Proof. The proof can be derived by the same reasoning as the one used in the proof of *Theorem 1*. The only exception is the fact that *Lemma 2* should be used instead of *Lemma 1*.

The design procedure is also almost the same. The only exception is that the matrix $\boldsymbol{V}$ should be replaced by $\boldsymbol{W}$ calculated with (A.3).

4. ILLUSTRATIVE EXAMPLES

The objective of this section is to provide numerical examples, that will clearly and step-by-step illustrate the proposed strategy. It should be pointed out that for the sake of space limits and due to the possibly large size of matrices (A.2) (for large n), the selected examples concern single output second order systems, for which a simplified procedure can be applied that is based on *Theorem 2*. The selected examples correspond to the totally observable and unobservable case, respectively.

4.1 Example 1: the totally observable case

Let us consider the system (1)–(2) for $M = 2$ and with

$$\boldsymbol{A}^1 = \begin{bmatrix} 0 & 0.9293 \\ 0.2435 & 0.35 \end{bmatrix}, \quad \boldsymbol{A}^2 = \begin{bmatrix} 0 & 0.616 \\ 0.2511 & 0.4733 \end{bmatrix}, \qquad (19)$$

$$\boldsymbol{C}^1 = [0.9\ \ 0], \quad \boldsymbol{C}^2 = [1.1\ \ 0]. \qquad (20)$$

Following the computational scheme given in the previous section, the square matrices $\boldsymbol{M}^i$, $i = 1,\dots,4$, are obtained:

$$\boldsymbol{M}^1 = \begin{bmatrix} 1 & 0 & 0 & 0.9 \\ 0 & 1 & 0 & 0 \\ 0 & -0.2435 & 0.9 & 0 \\ -0.9293 & -0.35 & 0 & 0 \end{bmatrix}, \qquad (21)$$

$$\boldsymbol{M}^2 = \begin{bmatrix} 1 & 0 & 0 & 0.9 \\ 0 & 1 & 0 & 0 \\ 0 & -0.2511 & 1.1 & 0 \\ -0.616 & -0.4733 & 0 & 0 \end{bmatrix}, \qquad (22)$$

$$\boldsymbol{M}^3 = \begin{bmatrix} 1 & 0 & 0 & 1.1 \\ 0 & 1 & 0 & 0 \\ 0 & -0.2511 & 1.1 & 0 \\ -0.616 & -0.4733 & 0 & 0 \end{bmatrix}, \qquad (23)$$

$$\boldsymbol{M}^4 = \begin{bmatrix} 1 & 0 & 0 & 1.1 \\ 0 & 1 & 0 & 0 \\ 0 & -0.2435 & 0.9 & 0 \\ -0.9293 & -0.35 & 0 & 0 \end{bmatrix}. \quad (24)$$

It can be easily verified that each of the matrices $\boldsymbol{M}^i$, $i = 1, \ldots, 4$, has rank $n^2 = 4$, which means that the system considered is locally observable. Finally, it can be verified that all principal minors are positive, which means that the system considered is totally observable, i.e., the condition (3) holds for all k_0, and hence for all values of $h_{i,k}$ satisfying (13).

4.2 Example 2: the unobservable case

Let us now consider a system (1)–(2) for $M = 2$ and with the following parameters:

$$\boldsymbol{A}^1 = \begin{bmatrix} 0 & 0.9293 \\ 0.2435 & 0.35 \end{bmatrix}, \quad \boldsymbol{A}^2 = \begin{bmatrix} 0 & 0.616 \\ 0.2511 & 0.4733 \end{bmatrix}, \quad (25)$$

$$\boldsymbol{C}^1 = [0.9 \ 0], \quad \boldsymbol{C}^2 = [-0.9 \ 0]. \quad (26)$$

In this case, the matrices $\boldsymbol{M}^i$, $i = 1, \ldots, 4$, are obtained as

$$\boldsymbol{M}^1 = \begin{bmatrix} 1 & 0 & 0 & 0.9 \\ 0 & 1 & 0 & 0 \\ 0 & -0.2435 & 0.9 & 0 \\ -0.9293 & -0.35 & 0 & 0 \end{bmatrix}, \quad (27)$$

$$\boldsymbol{M}^2 = \begin{bmatrix} 1 & 0 & 0 & 0.9 \\ 0 & 1 & 0 & 0 \\ 0 & -0.2511 & -0.9 & 0 \\ -0.616 & -0.4733 & 0 & 0 \end{bmatrix}, \quad (28)$$

$$\boldsymbol{M}^3 = \begin{bmatrix} 1 & 0 & 0 & -0.9 \\ 0 & 1 & 0 & 0 \\ 0 & -0.2511 & -0.9 & 0 \\ -0.616 & -0.4733 & 0 & 0 \end{bmatrix}, \quad (29)$$

$$\boldsymbol{M}^4 = \begin{bmatrix} 1 & 0 & 0 & -0.9 \\ 0 & 1 & 0 & 0 \\ 0 & -0.2435 & 0.9 & 0 \\ -0.9293 & -0.35 & 0 & 0 \end{bmatrix}. \quad (30)$$

Similarly as in the previous section, it can be easily verified that each of the square matrices $\boldsymbol{M}^i$, $i = 1, \ldots, 4$, has rank $n^2 = 4$, which means that the system considered is locally observable. It can be easily verified that some principal minors are negative. This means that the following equation has to be solved:

$$\det\left(\boldsymbol{T}\boldsymbol{W} + (\boldsymbol{I} - \boldsymbol{T})\right) = 0, \quad (31)$$

with $\boldsymbol{T}\left\{(i-1)n^2 + 1, \ldots, in^2\right\} = t_i$, $i = 1, \ldots, N-1$, which leads to

$$\det\left(\boldsymbol{T}\boldsymbol{W} + (\boldsymbol{I} - \boldsymbol{T})\right) = 1.0 - 2t_1 + 4.674 t_2 {t_1}^2 - 1.348 {t_2}^2 {t_1}^3 - 2.337 t_2 t_1 + 0.674 {t_2}^2 {t_1}^2 + 2 t_2 t_3 t_1 - 4.674 {t_2}^2 {t_1}^2 t_3 + 1.348 {t_2}^3 t_1^3 t_3 = 0. \quad (32)$$

One of the possible solutions of (32) is

$$t_1 = 0.5, \quad t_2 = 1, \quad t_3 = t_3,$$

where $t_3 = t_3$ means that t_3 may have an arbitrary value. The above result indicates that the system considered is not observable, due to the fact that it is a solution such that $t_i \in [0, 1] \quad \forall i = 1, 2, 3$. Indeed, for all k, when $h_{1,k} = h_{2,k} = 0.5$, the matrix $\boldsymbol{C}_{k+1}$ in (2) becomes

$$\boldsymbol{C}_{k+1} = h_{1,k}\boldsymbol{C}^1 + h_{2,k}\boldsymbol{C}^2 = [0 \ 0], \quad (33)$$

which clearly corresponds to an operating condition for which the system (1)–(2) is not observable.

5. CONCLUSIONS

In this paper, the problem of developing a practical observability test for time-varying linear systems has been tackled. The proposed solution is based on checking if all principal minors associated to an appropriate matrix are positive definite. If this condition holds, then the rank of the observability matrix associated to the LTV system is full, and thus the system is totally observable. On the other hand, if this condition is not satisfied, then a symbolic computation test is applied in order to conclude about the non-observability of the LTV system. The application of the proposed observability test has been demonstrated through two illustrative examples, which clearly exhibit its performance.

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Appendix A. LARGE MATRICES

Here are placed all matrices used in the paper, which due to their size did not fit to two-column style

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{R}_1\boldsymbol{R}_J^{-1} & (\boldsymbol{R}_2-\boldsymbol{R}_1)\boldsymbol{R}_J^{-1} & \cdots & (\boldsymbol{R}_{J-2}-\boldsymbol{R}_{J-3})\boldsymbol{R}_J^{-1} & (\boldsymbol{R}_{J-1}-\boldsymbol{R}_{J-2})\boldsymbol{R}_J^{-1} \\ -\boldsymbol{I}_{Jn_r} & \boldsymbol{I}_{Jn_r} & \cdots & \boldsymbol{0}_{Jn_r} & \boldsymbol{0}_{Jn_r} \\ \boldsymbol{0}_{Jn_r} & -\boldsymbol{I}_{Jn_r} & \cdots & \boldsymbol{0}_{Jn_r} & \boldsymbol{0}_{Jn_r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0}_{Jn_r} & \boldsymbol{0}_{Jn_r} & \cdots & \boldsymbol{I}_{Jn_r} & \boldsymbol{0}_{Jn_r} \\ \boldsymbol{0}_{Jn_r} & \boldsymbol{0}_{Jn_r} & \cdots & -\boldsymbol{I}_{Jn_r} & \boldsymbol{I}_{Jn_r} \end{bmatrix} \tag{A.1}$$

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{R}_1\boldsymbol{R}_N^{-1} & (\boldsymbol{R}_2-\boldsymbol{R}_1)\boldsymbol{R}_N^{-1} & (\boldsymbol{R}_3-\boldsymbol{R}_2)\boldsymbol{R}_N^{-1} & \dots & (\boldsymbol{R}_{N-1}-\boldsymbol{R}_{N-2})\boldsymbol{R}_N^{-1} \\ -\boldsymbol{I}_{Nn} & \boldsymbol{I}_{Nn} & \boldsymbol{0}_{Nn} & \dots & \boldsymbol{0}_{Nn} \\ \boldsymbol{0}_{Nn} & -\boldsymbol{I}_{Nn} & \boldsymbol{I}_{Nn} & \dots & \boldsymbol{0}_{Nn} \\ \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{0}_{Nn} & \dots & \boldsymbol{0}_{Nn} & -\boldsymbol{I}_{Nn} & \boldsymbol{I}_{Nn} \end{bmatrix}, \tag{A.2}$$

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{M}^1(\boldsymbol{M}^N)^{-1} & (\boldsymbol{M}^2-\boldsymbol{M}^1)(\boldsymbol{M}^N)^{-1} & (\boldsymbol{M}^3-\boldsymbol{M}^2)(\boldsymbol{M}^N)^{-1} & \dots & (\boldsymbol{M}^{N-1}-\boldsymbol{M}^{N-2})(\boldsymbol{M}^N)^{-1} \\ -\boldsymbol{I}_{n^2} & \boldsymbol{I}_{n^2} & \boldsymbol{0}_{n^2} & \dots & \boldsymbol{0}_{n^2} \\ \boldsymbol{0}_{n^2} & -\boldsymbol{I}_{n^2} & \boldsymbol{I}_{n^2} & \dots & \boldsymbol{0}_{n^2} \\ \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{0}_{n^2} & \dots & \boldsymbol{0}_{n^2} & -\boldsymbol{I}_{n^2} & \boldsymbol{I}_{n^2} \end{bmatrix} \tag{A.3}$$