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On star forest ascending subgraph decomposition

Josep M. Aroca and Anna Lladó

Department of Mathematics, Univ. Politècnica de Catalunya
Barcelona, Spain

josep.m.aroca@upc.edu,aina.llado@upc.edu

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Abstract

The Ascending Subgraph Decomposition (ASD) Conjecture asserts that every graph G with $\binom{n+1}{2}$ edges admits an edge decomposition $G = H_1 \oplus \cdots \oplus H_n$ such that H_i has i edges and it is isomorphic to a subgraph of H_{i+1}, $i = 1, \ldots, n-1$. We show that every bipartite graph G with $\binom{n+1}{2}$ edges such that the degree sequence $d_1, \ldots, d_k$ of one of the stable sets satisfies $d_{k-i} \geqslant n-i$, $0 \leqslant i \leqslant k-1$, admits an ascending subgraph decomposition with star forests. We also give a necessary condition on the degree sequence which is not far from the above sufficient one.

1 Introduction

A graph G with $\binom{n+1}{2}$ edges has an Ascending Subgraph Decomposition (ASD) if it admits an edge–decomposition $G = G_1 \oplus \cdots \oplus G_n$ such that G_i has i edges and it is isomorphic to a subgraph of G_{i+1}, $1 \leqslant i < n$. Throughout this paper we use the symbol $\oplus$ to denote edge–disjoint union of graphs. It was conjectured by Alavi, Boals, Chartrand, Erdős and Oellerman [1] that every graph of size $\binom{n+1}{2}$ admits an ASD. The conjecture has been proved for a number of particular cases, including forests [5], regular graphs [9], complete multipartite graphs [8] or graphs with maximum degree $\Delta \leqslant n/2$ [6].

In the same paper Alavi et al. [1] conjectured that every star forest of size $\binom{n+1}{2}$ in which each connected component has size at least n admits an ASD in which every graph in the decomposition is a star. This conjecture was proved by Ma, Zhou and Zhou [13], and the condition was later on weakened to the effect that the second smallest component of the star forest has size at least n by Chen, Fu, Wang and Zhou [4].

The class of bipartite graphs which admit a star forest ASD (an ascending decomposition in which every subgraph is a forest of stars) is clearly larger than the one which admit a star ASD, see Figure 1 for a simple example. This motivates the study of star forest ASD for bipartite graphs in terms of the degree sequence of one of the stable sets, which is the purpose of this paper.

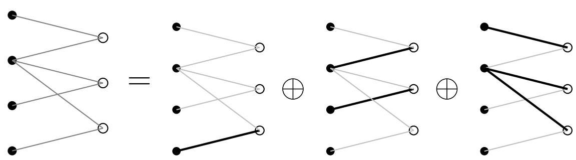


Figure 1: Bipartite graph with Star forest ASD but with no Star ASD.

Faudree, Gyárfás and Schelp [5] proved that every forest of stars admits a star forest ASD. These authors mention, in the same paper, that "Surprisingly [this result] is the most difficult to prove. This could indicate that the ASD conjecture (if true) is a difficult one to prove". In the same paper the authors propose the following question: Let G be a graph with $\binom{n+1}{2}$ edges. Does G have an ASD such that each member is a star forest?

In this paper we address this question for bipartite graphs when the centres of the stars in the star forest ASD belong to the same stable set. Throughout the paper we simply call stable sets the two maximum stable sets of a bipartite graph. The degree sequence of a stable set is the sequence of degrees of its vertices. Our main result is the following one.

Theorem 1. *Let G be a bipartite graph with $\binom{n+1}{2}$ edges and let $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_k$ be the degree sequence of one of the stable sets of G. If*

$$d_{k-i} \geqslant n - i, \quad 0 \leqslant i \leqslant k-1,$$

then there is a star forest ASD of G.

The proof of Theorem 1 is made in two steps. First we prove the result for a class of bipartite graphs which we call reduced graphs. For this we use a representation of a star forest decomposition by the so–called ascending matrices and certain multigraphs, and reduce the problem to the existence of a particular edge–coloring of these multigraphs. The terminology and the proof of Theorem 1 for reduced graphs is contained in Section 2.

In Section 3 we present an extension lemma, which uses a result of Häggkvist [10] on list edge–colorings, which allows one to extend the decomposition from reduced to all bipartite graphs with the same degree sequence on one of the stable sets, completing the proof of Theorem 1.

The sufficient condition on the degrees given in Theorem 1 is not far from being necessary as shown by the next Proposition which is proved in Section 4 (we again refer to Section 2 for undefined terminology.)

Proposition 2. *Let $G = (X \cup Y, E)$ be a reduced bipartite graph with degree sequence $d_X = (d_1 \leqslant \cdots \leqslant d_k)$. If G admits an edge–decomposition*

$$G = F_1 \oplus F_2 \oplus \cdots \oplus F_n,$$

where F_i is a star forest with i edges whose centers belong to X, then

$$\sum_{i=0}^{t-1} d_{k-i} \geqslant \sum_{i=0}^{t-1}(n-i) \quad \textit{for each} \;\; t=1,\ldots,k. \tag{1}$$

2 Star forest ASD for reduced bipartite graphs

Throughout the section $G = G(X, Y)$ denotes a bipartite graph with stable sets $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_n\}$. We denote by $d_X = (d_1 \leqslant \cdots \leqslant d_k)$, $d_i = d(x_i)$, the degree sequence of the vertices in X. We focus on star forest ASD with the stars of the decomposition centered at the vertices in X.

We first introduce some definitions.

Definition 3 (Reduced graph)**.** The reduced graph $G_R = G_R(X, Y')$ of $G(X, Y)$ has stable sets X and $Y' = \{y'_1, \ldots, y'_{d_k}\}$ and x_i is adjacent to the vertices $y'_1, \ldots, y'_{d_i}$, $i = 1, \ldots, k$.

We say that G is reduced if $G = G_R$.

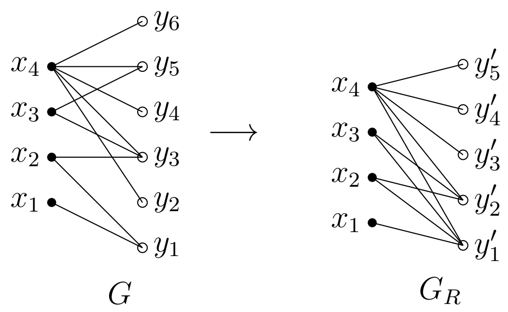


Figure 2: A bipartite graph and its reduced graph.

Given two vectors $c = (c_1, \ldots, c_k)$ and $c' = (c'_1, \ldots, c'_k)$, we say that $c \preceq c'$ if, after reordering the components of each vector in nondecreasing order, the i–th component of c is not larger than the i–th component of c'. This definition is motivated by the following remark.

Remark 4. Let F, F' be two forests of stars with centers $x_1, \ldots, x_k$ and $x'_1, \ldots, x'_k$ respectively. Then F is isomorphic to a subgraph of F' if and only if

$$(d_F(x_1), \ldots, d_F(x_k)) \preceq (d_{F'}(x'_1), \ldots, d_{F'}(x'_k)).$$

Given two sequences $d = (d_1 \leqslant \cdots \leqslant d_k)$ and $b = (b_1 \leqslant \cdots \leqslant b_n)$ of nonnegative integers with $\sum_i d_i = \sum_j b_j$, denote by $\mathcal{N}(d, b)$ the set of matrices $A = (a_{ij})_{k\times n}$ with nonnegative integer entries such that the row sums and the column sums satisfy respectively,

$$\sum_j a_{ij} = d_i, \;\; 1 \leqslant i \leqslant k, \qquad \sum_i a_{ij} = b_j; \;\; 1 \leqslant j \leqslant n.$$

Definition 5 (Ascending matrix)**.** We say that a matrix A is ascending if,

$$A_1 \preceq A_2 \preceq \cdots \preceq A_n,$$

where A_j denotes the j–th column of A.

We denote by $\mathcal{N}_a(d,b)$ the class of ascending matrices in $\mathcal{N}(d,b)$.

We will use appropriate ascending matrices to define multigraphs which will lead to star forest ASD as stated in Proposition 7 below.

For convenience we use the following notation for sequences. The constant sequence with r entries equal to x is denoted by x^r and (x^r, y^s) denotes the concatenation of x^r and y^s. Also, for an integer x we denote by x^- the ascending sequence $x^- = (1,2,\dots,x-1,x)$. Sums and differences of sequences of the same length are understood to be componentwise.

We recall that the bipartite adjacency matrix of a bipartite multigraph H with stable sets $X = \{x_1,\dots,x_k\}$ and $Z = \{z_1,\dots,z_n\}$ is the matrix $A = (a_{ij})_{k\times n}$ where a_{ij} is the number of edges joining x_i with z_j.

We need a last definition which is borrowed from [10].

Definition 6 (Sequential coloring)**.** A bipartite multigraph $H = H(X,Z)$ with degree sequence $d_X = (d_1 \leqslant \cdots \leqslant d_k)$ has a *sequential coloring* for X if there is a proper edge coloring of H such that the edges incident with $x_i \in X$ receive colors $\{1,\dots,d_i\}$ for each i.

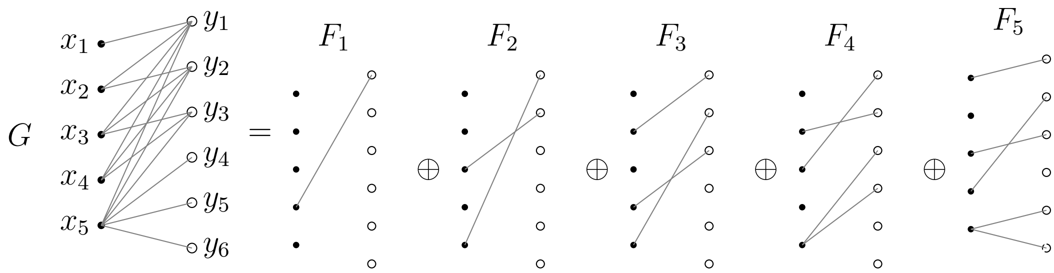
Proposition 7. *A reduced bipartite graph $G = G(X,Y)$ with $\binom{n+1}{2}$ edges and degree sequence $d_X = (d_1 \leqslant \cdots \leqslant d_k)$ has a star forest ASD with centers of stars in X if and only if there is an ascending matrix $A \in \mathcal{N}_a(d_X, n^-)$ such that the bipartite multigraph $H = H(X,Z)$ with bipartite incidence matrix A admits a sequential coloring.*

Proof. Let $X = \{x_1,\dots,x_k\}$ and $Y = \{y_1,\dots,y_{d_k}\}$.

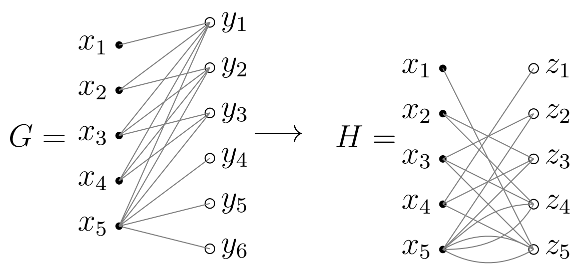
Assume first that G admits a star forest ASD with the centers of the stars in X,

$$G = F_1 \oplus \cdots \oplus F_n.$$

See an illustration with $d_X = (1,2,3,3,6)$ and $n = 5$,



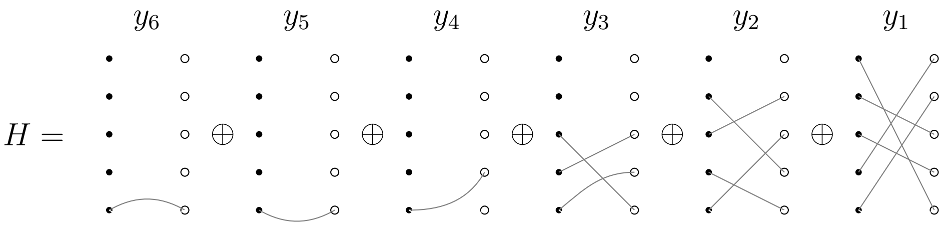
We define the multigraph $H = H(X,Z)$ with $Z = \{z_1,\dots,z_n\}$ by placing $d_{F_j}(x_i)$ parallel edges joining x_i with z_j, where $d_{F_j}(x_i)$ denotes the degree of x_i in the forest F_j.



In this way, the bipartite adjacency matrix A of H belongs to $\mathcal{N}(d_X, n^-)$. Moreover, since F_j is isomorphic to a subgraph of F_{j+1}, the matrix A is ascending, $A \in \mathcal{N}_a(d_X, n^-)$.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 2 \end{pmatrix}.$$

Next we define an edge–coloring of H as follows. Denote by $N_{F_j}(x_i) = \{y_{i_1}, \ldots, y_{i_s}\}$ the set of neighbours of x_i in the forest F_j. Then x_i is joined in H to z_j with s parallel edges. We color these edges with the subscripts $\{i_1, \ldots, i_s\}$ of the neighbours of x_i in F_j, by assigning one of these colors to each parallel edge bijectively.



Since the original graph G is simple and the star forests $F_1, \ldots, F_n$ form a decomposition of G, no two edges in H incident to a vertex x_i receive the same color. On the other hand, since each star forest F_j has its stars centered in vertices in X, and therefore each vertex in Y has degree at most one in each F_j, by the bijections which define the edge–coloring, no two edges incident to z_j receive the same color. Hence, the coloring is proper. Moreover, since the graph G is reduced, the edges incident to x_i receive the colors $\{1, \ldots, d_i\}$ and the coloring is sequential. This completes the if part of the proof.

Reciprocally, assume that $A \in \mathcal{N}_a(d_X, n^-)$ and that the multigraph $H = H(X, Z)$ with bipartite adjacency matrix A has a sequential coloring.

Let $Z = \{z_1, \ldots, z_n\}$, where vertex z_i has degree i (the sum of entries of column i of A), in H, $1 \leqslant i \leqslant n$. Let $c : E \to \{1, 2, \ldots, d_k\}$ be a sequential coloring of H, so that the edges incident to x_i receive colors $\{1, \ldots, d_i\}$.

Each z_j will be associated to the subgraph F_j of G defined as follows. For each edge $x_i z_j$ of color h, we say that the edge $x_i y_h$ is in F_j. This way we obtain a subgraph of G because $h \leqslant d(x_i)$ (the coloring is sequential) and the graph G is reduced. Moreover, since the coloring is proper, the degree of every vertex y_h in F_j is at most one. Hence F_j is a forest of stars and it has j edges. Moreover, for $j \neq j'$, the subgraphs F_j and $F_{j'}$ are edge–disjoint again from the fact that the coloring is proper. Finally, since the matrix A is ascending, F_i is isomorphic to a subgraph of F_{i+1} for each $i = 1, \cdots, n-1$. Hence,

$$G = F_1 \oplus \cdots \oplus F_n$$

is a star forest ASD for G. This completes the proof. □

To prove the main result for reduced graphs, we will show the existence of an appropriate ascending matrix such that the multigraph associated to it admits a sequential coloring.

We recall that Faudree, Gyárfás and Schelp [5] have proved that every star forest with $\binom{n+1}{2}$ edges admit a star forest ASD. In view of Remark 4, this result can be reformulated as the following Lemma:

Lemma 8. *For every sequence $d = (d_1 \leqslant \cdots \leqslant d_k)$ with $d_1 > 0$ and $\sum_{i=1}^{k} d_i = \binom{n+1}{2}$ there is $C \in \mathcal{N}_a(d, n^-)$.*

Proof. Let $G = S_1 \cup \cdots \cup S_k$ be a forest of stars where S_i is a star with d_i edges and centre x_i, $1 \leqslant i \leqslant k$. If $G = F_1 \oplus \cdots \oplus F_n$ is a star forest ASD of G, let $C = (c_{ij})$ be the matrix where c_{ij} the number of edges in F_j incident with x_i. The matrix C clearly belongs to $\mathcal{N}(d, n^-)$ and Remark 4 shows that C is ascending as well. □

We next show that there exists an ascending matrix $A \in \mathcal{N}_a(d, n^-)$ of a particular shape that will be useful to prove the existence of star forest ASD for reduced graphs with that degree sequence.

We note that, if $b = (b_1 \leqslant \cdots \leqslant b_n)$, each matrix $A \in \mathcal{N}(a, b)$ with $(0, 1)$ entries is ascending. The support of a matrix B is the set of positions with nonzero entries.

We observe that if $B \in \mathcal{N}_a(a, b)$ and $T \in \mathcal{N}_a(a', b')$ have disjoint support and the same dimensions, then $B + T \in \mathcal{N}_a(a + a', b + b')$. The last sentence also holds if the support of T and B intersect in a square submatrix and T has constant entries in this submatrix. The above observations will be used in the proof of the next Lemma.

Lemma 9. *Let $d = (d_1 \leqslant \cdots \leqslant d_k)$ be a sequence satisfying $\sum_i d_i = \binom{n+1}{2}$ and*

$$d_{k-i} \geqslant n - i, \; i = 0, \ldots, k-1.$$

Then, there is $A = (a_{ij})_{k \times n}$ in $\mathcal{N}_a(d, n^-)$ such that $a_{ij} \geqslant 1$ for each (i, j) with $i + j \geqslant k + 1$.

Proof. Consider the $(k \times n)$ matrix T with $t_{ij} = 1$ for $i + j \geqslant k + 1$ and $t_{ij} = 0$ otherwise. Let $d'_k = d_k - n, \;\; d'_{k-1} = d_{k-1} - (n-1), \ldots, d'_1 = d_1 - (n - k + 1)$. Since

$$\sum_i d'_i = (n-k) + (n-k-1) + \cdots + 2 + 1,$$

by Lemma 8 for $d' = (d'_1, \ldots, d'_k)$ there is $A \in \mathcal{N}_a(d', (n-k)^-)$. Extend this matrix to a $k \times n$ matrix A' by adding zero columns to the left. Since the last $(n-k)$ columns of T

are the all–one vectors, the matrix $A = A' + T$ still has the ascending column property and, by construction, it is in $\mathcal{N}_a(d, n^-)$ with nonzero entries in the positions (i, j) with $i + j \geqslant k + 1$. $\square$

Here is an illustration of Lemma 9 with $d_X = (4, 6, 9, 9)$ and $n = 7$.

$$T = \begin{pmatrix} 0 & 0 & 0 & 1 & \vdots & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & \vdots & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & \vdots & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \vdots & 1 & 1 & 1 \end{pmatrix}; \quad A' = \begin{pmatrix} 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & \vdots & 0 & 1 & 1 \end{pmatrix}$$

$$A = A' + T = \begin{pmatrix} 0 & 0 & 0 & 1 & \vdots & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & \vdots & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & \vdots & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & \vdots & 1 & 2 & 2 \end{pmatrix}$$

Next Lemma gives a sufficient condition for a degree sequence of a reduced graph to admit a star forest ASD.

Lemma 10. *Let $d = (d_1 \leqslant \cdots \leqslant d_k)$ with $d_i \in \mathbb{N}$ and $\sum_i d_i = \binom{n+1}{2}$. If*

$$d_{k-i} \geqslant n - i, \quad i = 0, 1, \ldots, k - 1,$$

the reduced graph $G = (X, Y)$ with degree sequence $d_X = d$ admits a star forest ASD.

Proof. Let $A = (a_{ij}) \in \mathcal{N}_a(d, n^-)$ such that $a_{ij} \geqslant 1$ for each (i, j) with $i + j \geqslant k + 1$, whose existence is ensured by Lemma 9.

Let H be the bipartite multigraph with stable sets $X = \{x_1, \ldots, x_k\}$ and $Z = \{z_1, \ldots, z_n\}$ whose bipartite adjacency matrix is A. We next show that H admits a sequential coloring. The result will follow by Proposition 7.

Consider the k matchings of H defined by

$$\begin{aligned} M_1' =& \{x_1 z_n\} \quad\quad \cup \{x_2 z_{n-1}\} \cup \cdots \cup \{x_k z_{n-k+1}\} \\ M_2' =& \{x_1 z_{n-k+1}\} \cup \{x_2 z_n\} \quad \cup \cdots \cup \{x_k z_{n-k+2}\} \\ &\vdots \\ M_k' =& \{x_1 z_{n-1}\} \quad \cup \{x_2 z_{n-2}\} \cup \cdots \cup \{x_k z_n\}. \end{aligned}$$

In general, $M_i' = \{x_r z_s : r + s \in \{n + i,\ n - k + i\}, n - k + 1 \leqslant s \leqslant n\},\ i = 1, \ldots, k.$

Such matchings exist in H by the condition $a_{ij} \geqslant 1$ for each pair (i, j) with $i+j \geqslant k+1$, and they are pairwise edge–disjoint.

Let $\alpha_i = d_i - (n - k)$, $1 \leqslant i \leqslant k$. For each $j = 1, 2, \ldots, k$, let $M_j \subseteq M_j'$ be obtained by selecting from M_j' the edge incident to x_i whenever $\alpha_i \geqslant j$. In this way each x_i is incident with the matchings $M_1, \dots, M_{t(i)}$ with $t(i) = \min\{k, \alpha_i\}$. On the other hand, since $\alpha_{k-i} \geqslant k-i$ (by the condition on the degree sequence d_X), the vertex z_{n-i} is incident to at least $k-i$ edges in $M_1 \oplus \cdots \oplus M_k$.

Let H' the bipartite multigraph obtained from H by removing the edges in $M_1 \oplus \cdots \oplus M_k$. Let $d'_X = (d'_1 \leqslant \cdots \leqslant d'_k)$ be the degree sequence of X in H', where $d'_i = d_i - t(i) \geqslant n-k$. Moreover, each vertex z_{n-i}, $0 \leqslant i \leqslant n-1$ has degree at most $n-k$ in H' (because it has degree $n-i$ in H and at least $k-i$ of its incident edges in $M_1 \oplus \cdots \oplus M_k$.

An example of the matchings M_i (depicted by the incidence matrices) for $d_X = (4, 6, 9, 9)$ and $n = 7$ is given below.

$$\overset{H}{\begin{pmatrix} 0&0&0&1&1&1&1\\ 0&0&1&1&2&1&1\\ 0&1&1&1&1&2&3\\ 1&1&1&1&1&2&2 \end{pmatrix}} = \overset{M_1}{\begin{pmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0 \end{pmatrix}} + \overset{M_2}{\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0 \end{pmatrix}}$$

$$\overset{M_3}{\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0 \end{pmatrix}} + \overset{M_4}{\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}} + \overset{H'}{\begin{pmatrix} 0&0&0&1&1&1&0\\ 0&0&1&0&2&0&0\\ 0&1&1&0&0&1&2\\ 1&1&1&0&0&1&1 \end{pmatrix}}$$

We next define a sequential coloring in H as follows. Let $\Delta'(X)$ be the maximum degree in H' of the vertices in X. If $\Delta'(X) > n-k$ then there is a matching $M'_{\Delta'(X)}$ in H' from the vertices of maximum degree in X to Z. Color the edges of this matching with $\Delta'(X)$.

By removing this matching from H' we obtain a bipartite multigraph in which the maximum degree of vertices in X is $\Delta'(X) - 1$. By iterating this process we eventually reach a bipartite multigraph H'' in which every vertex in X has degree $n-k$ while the maximum degree of the vertices in Z still satisfies $\Delta''(Z) \leqslant n-k$. By König's theorem, the edge–chromatic number of H'' is $n-k$. By combining an edge–coloring of H'' with $n-k$ colors with the ones obtained in the process of reducing the maximum degree of H', the multigraph H' can be properly edge–colored in such a way that vertex x_i is incident in H' with colors $\{1, \dots, d'_i\}$, $1 \leqslant i \leqslant k$. Let $H'_1, H'_2, \dots, H'_{d'_k}$ be the matchings in H' induced by these colors.

We finally obtain a sequential coloring of H by combining the sequential coloring of H' with the matchings $M_1, \dots, M_k$ as follows. Recall that each vertex x_i is incident with the matchings $M_1, \dots, M_{t(i)}$, where $t(1) \leqslant t(2) \leqslant \cdots \leqslant t(k) = k$, and the matchings $H'_1, \dots, H'_{d'_i}$ in H', where $d'_1 \leqslant d'_2 \leqslant \cdots \leqslant d'_k$ and $d'_i + t(i) = d_i$. A sequential coloring of H can be obtained by combining the two sets of matchings in an ordered list such that the first d_i matchings in the list contain precisely $M_1, \dots, M_{t(i)}$ and $H'_1, \dots, H'_{d'_i}$.

In the above example, one may choose the following decomposition of H' (described by the adjacency matrices)

$$\overset{H'}{\begin{pmatrix} 0&0&0&1&1&1&0\\ 0&0&1&0&2&0&0\\ 0&1&1&0&0&1&2\\ 1&1&1&0&0&1&1 \end{pmatrix}} = \overset{H'_1}{\begin{pmatrix} 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0 \end{pmatrix}} + \overset{H'_2}{\begin{pmatrix} 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0 \end{pmatrix}}$$

$$\overset{H'_3}{\begin{pmatrix} 0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0 \end{pmatrix}} + \overset{H'_4}{\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}} + \overset{H'_5}{\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0 \end{pmatrix}}$$

A sequential coloring of H can be obtained, for instance, with the ordered sequence

$$H'_1, H'_2, H'_3, M_1, M_2, M_3, H'_4, H'_5, M_4.$$

Therefore, all conditions of Proposition 7 hold. Hence, G has a star forest ASD. This completes the proof. $\square$

3 An extension lemma

In this section we prove an extension lemma which shows that, if G_R admits a star forest ASD then so does G. This reduces the problem of giving sufficient conditions on the degree sequence of one stable set to ensure the existence of a star forest ASD to bipartite reduced graphs. For the proof of our extension lemma we use the following result by Häggkvist [10] on edge list–colorings of bipartite multigraphs.

Theorem 11. *[10] Let H be a bipartite multigraph with stable sets X and Z. If H admits a sequential coloring, then H can be properly edge–colored for an arbitrary assignment of lists $\{L(a): a \in X\}$ such that $|L(a)| = d(a)$ for each $a \in X$.*

In other words, Theorem 11 states that if H can be properly edge–colored in such a way that every vertex $a \in X$ is incident with colors $\{1, 2, \dots, d(a)\}$ then there is an edge–coloring of H by prescribing an arbitrary list $L(a) = \{c_1, c_2, \dots, c_{d(a)}\}$ of colors to each $a \in X$.

Lemma 12 (Extension Lemma)**.** *Let G be a bipartite graph with bipartition*

$$X = \{a_1, \dots, a_k\}$$

and Y and degree sequence $d_X = (d_1 \geqslant \cdots \geqslant d_k)$, $d_i = d(a_i)$. If the reduced graph G_R of G admits a decomposition

$$G_R = F'_1 \oplus \cdots \oplus F'_t,$$

where each F'_i is a star forest, then G has an edge decomposition

$$G = F_1 \oplus \cdots \oplus F_t$$

with $F_i \cong F'_i$ for each $i = 1, \dots, t$.

Proof. Let C be the $(k \times t)$ matrix whose entry c_{ij} is the number of edges incident to a_i in the star forest F'_j of the edge decomposition of G_R.

As done in Proposition 7, we consider the bipartite multigraph H with stable sets X and $U = \{u_1, \ldots, u_t\}$, where a_i is joined with u_j with c_{ij} parallel edges. Now, for each pair (i, j), color the c_{ij} parallel edges of H with the neighbors of a_i in the forest F'_j bijectively. Note that in this way we get a proper edge–coloring of H: two edges incident with a vertex a_i receive different colors since the bipartite graph G_R has no multiple edges, and two edges incident to a vertex u_j receive different colors since F'_j is a star forest.

By the definition of the bipartite graph G_R, each vertex $a_i \in X$ is incident in the bipartite multigraph H with edges colored $1, 2, \ldots, d_i$.

Let $L(a_i)$ be the list of neighbours of a_i in the original bipartite graph G. By Theorem 11, there is a proper edge–coloring χ' of H in which the edges incident to vertex $a_i \in X$ receive the colors from the list $L(a_i)$ for each $i = 1, \ldots, k$.

Now construct F_s by letting the edge $a_i b_j$ be in F_s whenever the edge $a_i u_s$ is colored b_j in the latter edge–coloring of H. In this way, F_s has the same number of edges than F'_s and the degree of a_i in F_s is c_{is}, the same as in F'_s. Moreover, since the coloring is proper, F_s is a star forest, so that $F_s \cong F'_s$, and two forests $F_s, F_{s'}$ are edge–disjoint whenever $s \neq s'$. Hence we have obtained a star forest decomposition of G. This concludes the proof. $\square$

We are now ready to prove our main result.

Proof of Theorem 1: Let $G = G(X, Y)$ be a bipartite graph with $d_X = (d_1 \leqslant \cdots \leqslant d_k)$ satisfying $\sum_i d_i = \binom{n+1}{2}$ and $d_i \geqslant n - i$, $i = 0, \ldots, k-1$.

By Lemma 10 there is a star forest ASD, $G_R = F'_1 \oplus \cdots \oplus F'_n$ of the reduced graph G_R with the same degree sequence d. By Lemma 12 there is also a star forest decomposition $G = F_1 \oplus \cdots \oplus F_n$ for G. Since $F_i \cong F'_i$ for each i, the latter is also an ascending subgraph decomposition. $\square$

4 A necessary condition for star forest decompositions

As mentioned in the Introduction, the sufficient condition in Theorem 1 is not far from being necessary. This is the statement of Proposition 2 which shows that, if a reduced bipartite graph with $\binom{n+1}{2}$ edges can be decomposed into n star forests with sizes $1, 2, \ldots, n$, regardless of the fact that it is ascending, then the degree sequence of the graph satisfies condition (1), which is necessary for the existence of a star forest ASD. We next give the proof of this Proposition.

Proof of Proposition 2: Let $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_{d_k}\}$ be the bipartition of G where x_i is adjacent to $y_1, \ldots, y_{d_i}$ for each i. Since the graph is reduced, the last $d_k - d_{k-1}$ vertices of Y have degree one, the preceding $d_{k-1} - d_{k-2}$ have degree 2 and, in general, the consecutive $d_{k-j+1} - d_{k-j}$ vertices $y_{d_{k-(j-1)}+1}, \ldots, y_{d_{k-j}}$ have degree j, $j = 1, \ldots, k-1$. The first d_1 vertices in Y have degree k.

Since F_n has n leaves in Y we clearly have $|Y| = d_k \geqslant n$. Thus (1) is satisfied for $t = 1$. Moreover, since G has $\binom{n+1}{2}$ edges, (1) is also trivially satisfied for $t = k$.

Let $1 \leqslant t \leqslant k-1$. Consider the subgraph G_t of G induced by $F_n \oplus F_{n-1} \oplus \cdots \oplus F_{n-(t-1)}$. Since each vertex in Y has degree at most one in each forest, it has degree at most t in G_t. By combining this remark with the former upper bound on the degrees of vertices in Y we have

$$\begin{aligned}
\sum_{i=0}^{t-1}(n-i) = \sum_{i=1}^{d_k} d_{G_t}(y_i) &\leqslant \sum_{i=1}^{d_{k-(t-1)}} d_{G_t}(y_i) + \sum_{i=d_{k-(t-1)}+1}^{d_k} d_G(y_i) \\
&\leqslant t d_{k-(t-1)} + (t-1)(d_{k-t} - d_{k-(t-1)}) + \cdots + 2(d_{k-1} - d_{k-2}) + (d_k - d_{k-1}) \\
&= \sum_{i=0}^{t-1} d_{k-i},
\end{aligned}$$

and (1) is satisfied for t. This concludes the proof. □

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