Preserving T-Transitivity

G. Mayor [a] and J. Recasens [b]

[a] *Universitat de les Illes Balears*
Palma (Mallorca), Spain.
e-mail: gmayor@uib.es

[b] *ETS Arquitectura del Vallès*
Universitat Politècnica de Catalunya
Sant Cugat del Vallès, Spain.
e-mail: j.recasens@upc.edu

Abstract. This contribution deals with the problem of aggregating T-equivalence relations, in the sense that we are looking for functions that preserve reflexivity, symmetry and transitivity with respect to a given t-norm T. Under any extra condition on the t-norm, we obtain a complete description of those functions in terms of that we call T-triangular triplets.

Keywords. t-norm, additive generator, T-equivalence relation, preserving properties, triangular triplet.

Introduction

Fuzzy equivalence relations, together with T-preorders, are probably the most important kind of fuzzy relations since they measure the degree to which two points of an universe are indistinguishable, equal or equivalent, and generalize the concept of classical equivalence relations.
They were introduced in [13] under the name *similarity relations* (with respect to the minimum) although they are also present in [9] and in [5]. The generalization to t-norms was considered in [11]. Other names have been used for this concept in the literature (sometimes in connection with a specific t-norm), such as *likeness relation, indistinguishability relation, fuzzy equality, proximity relation*, etc. We shall use in the sequel the term T-equivalence relation which, in our opinion, reflects in the best way the mathematical motivation in the axioms we recall in Section 1. The term T-indistinguishability operator is also widely used in the literature [3,8,11,12].

In many situations, there can be more than one T-indistinguishabilities defined on a universe and, in these cases, we may need to aggregate them. The most common way to aggregate a collection of T-equivalence relations is calculating their minimum, which also is a T-equivalence relation. However, sometimes this way of aggregating fuzzy relations leads to undesirable results since the Minimum only takes the smaller value for every couple into account and disregards the information of the other values. Similar drawback occurs when the Minimum is replaced by the t-norm T, specially when it is non-strict Archimedean. Thus, more general procedures to aggregate T-indistinguishability are needed.

Several authors have dealt the problem of the aggregation of some classes of fuzzy relations. With the same spirit as in [12,8], we revisit this topic in order to give, whatever the t-norm T we use, a characterization of those functions that combine a collection of T-equivalence relations in a single one.

After a section of preliminaries containing the basic definitions related with t-norms and T-equivalence relations, Section 2 introduces the concept of T-triangular triplet that will be central in the study of the preservation of T-transitivity. Section 3 contains the main results of the paper characterizing the functions that aggregate T-equivalence relations and some examples of functions aggregating T-equivalence relations for continuous Archimedean t-norms and the minimum and drastic t-norms. The contribution ends with a section of Concluding Remarks.

1. Preliminaries

Despite the fact that triangular norms (t-norms, for short) were first introduced in the context of statistical metric spaces [6], they have become an important tool in many other fields: fuzzy sets, decision making, statistics, theories of non-additive measures, etc. Comprehensive monographs on t-norms are [1,4]. We use the set of axioms provided by Schweizer and Sklar [10]. Thus, our requirements on a t–norm $T : [0,1] \times [0,1] \to [0,1]$ for all a, b, c, d in $[0,1]$ are:

(i) $T(a,b) = T(b,a)$,
(ii) $T(T(a,b),c) = T(a,T(b,c))$,
(iii) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$,
(iv) $T(a,1) = a$.

The following are the four basic t–norms T_M, T_L, T_P and T_D:

- $T_M(a,b) = \min(a,b)$, (minimum)
- $T_L(a,b) = \max(a+b-1,0)$, (Łukasiewicz t–norm)
- $T_P(a,b) = ab$, (Product t-norm)
- $T_D(a,b) = \begin{cases} \min(a,b), & \text{if } a=1 \text{ or } b=1 \\ 0, & \text{otherwise} \end{cases}$ (drastic t–norm).

A t-norm T is called Archimedean if for each $a, b \in (0,1)^2$ there is $n \geq 1$ such that $T(\overbrace{a,\ldots,a}^{n}) < b$. One special property of a continuous Archimedean t-norm is that it is strictly increasing, except for the subset of $[0,1]^2$ where its value is 0. A remarkable fact is that any continuous Archimedean t-norm T can be expressed with the help of an additive generator [1]: $T(a,b) = g^{(-1)}(g(a)+g(b))$, where $g^{(-1)}$ is the pseudo-inverse[2] of g. Note that T_L and T_P are continuous Archimedean t-norms with additive generators $g(a) = 1-a$ and $g(a) = -\log a$ respectively.

[1] An additive generator is a continuous and strictly decreasing function $g : [0,1] \longrightarrow [0,\infty]$ such that $g(1) = 0$.
[2] $g^{(-1)}(t) = sup\{c \in [0,1]; g(c) > t\}$, $sup\ \emptyset = 0$.

Given a set X and a t-norm T, we say that a fuzzy relation $E : X \times X \longrightarrow [0,1]$ is a T-equivalence (or a T-indistinguishability) if for all x, y, z in X the following conditions hold:

(i) $E(x,x) = 1$ (reflexivity)
(ii) $E(x,y) = E(y,x)$ (symmetry)
(iii) $E(x,y) \geq T(E(x,z), E(z,y))$ (T-transitivity)

As it is known, $E(x,y)$ is interpreted as the degree of indistinguishability (or similarity) between x and y. The axioms of reflexivity, symmetry and T-transitivity fuzzify the ones of a crisp equivalence relation.

Given a left continuous t-norm T, we can define the function on $[0,1]^2$ defined by $\overrightarrow{T}(a,b) = \sup\{c \in [0,1]; T(a,c) \leq b\}$ that we call the residuation of T. It is easy to see that $\overrightarrow{T}$ is a T-preorder[3] on [0,1]. The biresiduation of T is the function on $[0,1]^2$ defined by $\overleftrightarrow{T}(a,b) = T(\overrightarrow{T}(a,b), \overrightarrow{T}(b,a)) = \min(\overrightarrow{T}(a,b), \overrightarrow{T}(b,a))$. It is an important example of T-equivalence on $[0,1]$ that usually is called the natural T-equivalence associated to T, denoted by E_T.

If T is a continuous Archimedean t-norm with additive generator g, then $E_T(x,y) = g^{(-1)}(\mid g(x) - g(y) \mid)$ for all $x, y \in [0,1]$. In particular, for the Łukasiewicz t-norm, $E_T(x,y) = 1 - |x-y|$ and for the Product t-norm, $E_T(x,y) = \min(\frac{x}{y}, \frac{y}{x})$.

Complete information on indistinguishability operators can be found in the recent monograph [8].

2. T-triangular triplets

Definition 1. *We say that a triplet $(a,b,c) \in [0,\infty]^3$ is triangular if and only if*

$$a \leq b + c, b \leq a + c \text{ and } c \leq a + b.$$

Being $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in [0,\infty]^m$, $m \geq 1$, we say that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a (m-dimensional) triangular triplet if (a_i, b_i, c_i) is triangular for all $i = 1, \ldots, m$, where $\boldsymbol{a} = (a_1, \ldots, a_m), \boldsymbol{b} = (b_1, \ldots, b_m), \boldsymbol{c} = (c_1, \ldots, c_m)$.

Note that if $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is triangular then so is any reordering of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Proposition 1. *A triplet $(a,b,c) \in [0,\infty]^3$ is triangular if and only if it is of one of the following forms:*

(i) (∞, ∞, c) , $c \in [0,\infty]$

(ii) $c = \sqrt{a^2 + b^2 + \lambda ab}$, $\quad 0 \leq a, b < \infty$, $-2 \leq \lambda \leq 2$

Definition 2. *Let T be a t-norm. We say that $(a,b,c) \in [0,1]^3$ is T-triangular if and only if*

$$a \geq T(b,c), b \geq T(a,c) \text{ and} c \geq T(a,b).$$

[3] Reflexive and T-Transitive

Being $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in [0,1]^m$*,* $m \geq 1$*, we say that* $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ *is a (*m*-dimensional)* T*-triangular triplet if* (a_i, b_i, c_i) *is* T*-triangular for all* $i = 1, \ldots, m$ *, where* $\boldsymbol{a} = (a_1, \ldots, a_m), \boldsymbol{b} = (b_1, \ldots, b_m), \boldsymbol{c} = (c_1, \ldots, c_m)$*.*

Proposition 2. *Let* T *be a left continuous t-norm. A triplet* $(a, b, c) \in [0,1]^3$ *is* T*-triangular if and only if* $T(a, b) \leq c \leq E_T(a, b)$*.*

Proof:

$\Rightarrow$) Suppose that (a, b, c) is T-triangular. From $T(a, c) \leq b$ and $T(b, c) \leq a$ we deduce $c \leq \overrightarrow{T}(a, b)$ and $c \leq \overrightarrow{T}(b, a)$, hence $c \leq \min(\overrightarrow{T}(a, b), \overrightarrow{T}(b, a)) = E_T(a, b)$. Then $T(a, b) \leq c \leq E_T(a, b)$.

$\Leftarrow$) Reciprocally, assuming $T(a, b) \leq c \leq E_T(a, b)$ we have to prove that (a, b, c) is T-triangular. From $c \leq \overrightarrow{T}(a, b)$ and, applying left continuity and monotonicity of T, we obtain $T(a, c) \leq b$. Similarly, from $c \leq \overrightarrow{T}(b, a)$ we obtain $T(b, c) \leq a$. Thus, the triplet (a, b, c) is T-triangular.

Example 1.

- *A triplet is* T_M*-triangular if and only if there exists a reordering* (a, b, c) *such that* $a = b$ *and* $c \geq a$*.*
- *A triplet is* T_L*-triangular if and only if there exists a reordering* (a, b, c) *such that* $max(a + b - 1, 0) \leq c \leq 1- \mid a - b \mid$*.*
- *A triplet is* T_P*-triangular if and only if it is* $(0, 0, 0)$ *or there exists a reordering* (a, b, c) *with* $a, b, c > 0$*, such that* $ab \leq c \leq min(\frac{a}{b}, \frac{b}{a})$*.*

For the drastic t-norm, which is not left continuous, we have the following result.

Example 2. *A triplet* (a, b, c) *is* T_D*-triangular if and only if* a*,* b *and* c *are different from* 1 *or one of them is* 1 *and the other two coincide.*

Remark 1. *Denoting by* $\Delta(T)$ *the set of* T*-triangular triplets, observe that* $T_1 \leq T_2$ *implies* $\Delta(T_1) \supset \Delta(T_2)$*. Thus we have* $[0,1]^3 \supset \Delta(T_D) \supset \Delta(T_L) \supset \Delta(T_P) \supset \Delta(T_M) \supset \{(a, a, a); a \in [0,1]\}$*.*

3. Aggregating T-equivalence relations

Definition 3. *We say that a function* $F : [0,1]^m \longrightarrow [0,1], m \geq 1$*, aggregates* T*-equivalence relations if for any set X and any collection of* T*-equivalence relations on X,* $(E_1, \ldots, E_m)$*, then* $F(E_1, \ldots, E_m)$ *is also a* T*-equivalence relation on X, where* $F(E_1, \ldots, E_m)$ *is the fuzzy binary relation* $F(E_1, \ldots, E_m)(x, y)$ *=* $F(E_1(x, y), \ldots, E_m(x, y))$*.*

Example 3. *Any t-norm* T *aggregates* T*-equivalence relations (for any* $m \geq 1$*).*

In [3] an aggregation method with respect to E_T is introduced. Being g an additive generator of T (continuous Archimedean), then the corresponding aggregation function coincides with the quasi-arithmetic mean generated by g. Next proposition states that this function aggregates T-equivalence relations.

Proposition 3. *Let T be a continuous Archimedean t-norm with g as additive generator. The quasi-arithmetic mean generated by g, $M_g(a_1,\ldots,a_m) = g^{-1}(\frac{g(a_1)+\ldots+g(a_m)}{m})$, aggregates T-equivalence relations.*

The main result in this contribution is collected in the following proposition.

Proposition 4. *A function $F : [0,1]^m \longrightarrow [0,1], m \geq 1$, aggregates T-equivalence relations if and only if the following conditions hold:*

(i) $F(\overbrace{1,\ldots,1}^{m}) = 1$.

(ii) *F transforms m-dimensional T-triangular triplets into 1-dimensional T-triangular triplets*[4].

Proof:

$\Leftarrow$) Assuming that F satisfies (i) and (ii), we must prove that $F(E_1,\ldots,E_m)$ is a T-equivalence relation for all T-equivalence relations E_1,..., E_m.
We know that, for each $i = 1,\ldots,m$, it is $E_i(x,y) \geq T(E_i(x,z),E_i(z,y))$. Thus, the triplet $(\mathbf{a},\mathbf{b},\mathbf{c})$, where $a_i = E_i(x,y), b_i = E_i(x,z), c_i = E_i(z,y), i = 1,\ldots,m,$ is T-triangular, and from (ii) we have that $(F(\mathbf{a}),F(\mathbf{b}),F(\mathbf{c}))$ so is, and, consequently, we can write:

$$F(E_1,\ldots,E_m)(x,y) =$$
$$F(E_1(x,y),\ldots,E_m(x,y)) \geq$$
$$T(F(E_1(x,z),\ldots,E_m(x,z)),F(E_1(z,y),\ldots,E_m(z,y))) =$$
$$T(F(E_1,\ldots,E_m)(x,z)),F(E_1,\ldots,E_m)(z,y)).$$

Hence, $F(E_1,\ldots,E_m)$ is T-transitive. Reflexivity and symmetry follow immediately from (i) and symmetry of T.

$\Rightarrow$) Reciprocally, let us suppose that F aggregates T-equivalence relations. We have to prove that it satisfies conditions (i) and (ii).
First, it is $F(1,\ldots,1) = 1$ because $F(1,\ldots,1) = F(E(x,x),\ldots,E(x,x)) = F(E,\ldots,E)(x,x) = 1$, where E is a T-equivalence relation on a set X and $x \in X$.
Now, let us prove that $(F(\mathbf{a}),F(\mathbf{b}),F(\mathbf{c}))$ is T-triangular whenever $(\mathbf{a},\mathbf{b},\mathbf{c})$ also is.
There exist a set X , T-equivalence relations $E_1,\ldots,E_m$ on X and $x,y,z \in X$ such that $E_i(x,y) = a_i, E_i(x,z) = b_i$ and $E_i(z,y) = c_i$ for all $i = 1,\ldots,m$[5]. Then we can write

[4] If $(\mathbf{a},\mathbf{b},\mathbf{c})$ is a T-triangular triplet in $[0,1]^m$ then $(F(\mathbf{a}),F(\mathbf{b}),F(\mathbf{c}))$ is a T-triangular triplet in $[0,1]$.

[5] It is sufficient we consider a 3-element set $X = \{x,y,z\}$ and define $E_i(x,x) = E_i(y,y) = E_i(z,z) = 1, E_i(x,y) = E_i(y,x) = a_i, E_i(x,z) = E_i(z,x) = b_i, E_i(z,y) = E_i(y,z) = c_i, i = 1,\ldots,m$. Note that each E_i is a T-equivalence relation on X.

$$\begin{aligned} F(\mathbf{a}) &= F(E_1(x,y),\ldots,E_m(x,y)) \\ &= F(E_1,\ldots,E_m)(x,y) \\ &\geq T(F(E_1,\ldots,E_m)(x,z),F(E_1,\ldots,E_m)(y,z)) \\ &= T(F(\mathbf{b}),F(\mathbf{c})). \end{aligned}$$

Similarly, we obtain $F(\mathbf{b}) \geq T(F(\mathbf{a}, F(\mathbf{c}))$ and $F(\mathbf{c}) \geq T(F(\mathbf{a}, F(\mathbf{b}))$ and we have proved that $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c}))$ is T-triangular.

Next, an immediate consequence is shown.

Proposition 5. *A function $F : [0,1]^m \longrightarrow [0,1], m \geq 1$, aggregates T_M-equivalence relations (similarity relations) if and only if it is increasing in each variable and $F(1,\ldots,1) = 1$.*

Proof: Obvious, because F transforms m-dimensional T_M-triangular triplets into 1-dimensional T_M-triangular triplets if and only if it is increasing in each variable.

When T is a continuous Archimedean t-norm, a characterization of those functions that aggregate T-equivalence relations can be formulated in terms of an additive generator of T as follows.

Proposition 6. *If T is a continuous Archimedean t-norm with additive generator g, then $F : [0,1]^m \longrightarrow [0,1]$ aggregates T-equivalence relations if and only if the function $G = gFg^{(-1)}$[6] transforms (ordinary) triangular triplets of $[0,\infty]^m$ (with elements in $[0,g(0)]^m$) into (ordinary) triangle triplets of $[0,\infty]$ (with elements in $[0,g(0)]$) and $G(0,\ldots,0) = 0$.*

Proof: Straightforward.

Example 4. *A function $F : [0,1]^m \longrightarrow [0,1], m \geq 1$, aggregates T_L-equivalence relations if and only if $G(a_1,\ldots,a_m) = 1 - F(\max(1-a_1,0),\ldots,\max(1-a_m,0))$ transforms triangular triplets of $[0,\infty]^m$ (with elements in $[0,1]^m$) into triangle triplets of $[0,\infty]$ (with elements in $[0,1]$) and $G(0,\ldots,0) = 0$.*

Under increasingness, subadditivity[7] is equivalent to the property of transforming triangular triplets into triangle triplets.

Proposition 7. *Consider $G : [0,\infty]^m \longrightarrow [0,\infty]$. Then:*

(i) If G transforms triangular triplets of $[0,\infty]^m$ into triangular triplets of $[0,\infty]$ then it is subadditive.

(ii) If G is increasing and subadditive then it transforms triangular triplets of $[0,\infty]^m$ into triangular triplets of $[0,\infty]$.

[6] $g^{(-1)}(a_1,\ldots,a_m) = (g^{(-1)}(a_1),\ldots,g^{(-1)}(a_m))$.
[7] $G(\mathbf{a}+\mathbf{b}) \leq G(\mathbf{a}) + G(\mathbf{b})$.

Thus, from the two previous propositions, we can enunciate the following result.

Proposition 8. *An increasing function $F : [0,1]^m \longrightarrow [0,1]$, with $F(1,\dots,1) = 1$, aggregates T-equivalence relations (T is a continuous Archimedean t-norm with additive generator g) if and only if the function $G = gFg^{(-1)}$ is subadditive.*

Consequences of the previous propositions are two known results concerning the role of weighted arithmetic means and ordered weighted arithmetic means (OWA operators) in this approach. More details on these classes of aggregation functions can be found in the recent monograph [2].

Proposition 9. *Any weighted quasi-arithmetic mean $F(a_1,\dots,a_m) = g^{-1}(\Sigma w_i g(a_i))$ where the components of the weighting list $(w_1,\dots,w_m)$ are non-negative real numbers satisfying $\Sigma w_i = 1$ and g is an additive generator of a given continuous Archimedean t-norm T, aggregates T-equivalence relations.*

Proposition 10. *An ordered weighted quasi-arithmetic mean $F(a_1,\dots,a_m) = g^{-1}(\Sigma w_i g(a_{(m-i)}))$ where $a_{(k)}$ denotes the k-largest input in the list $(a_1,\dots,a_m)$ and g is an additive generator of a given continuous Archimedean t-norm T, aggregates T-equivalence relations.*

Example 5. *Related to Proposition 2 and recalling that the fuzzy relation E^T on $[0,1]$ defined by $E^T(a,b) = \begin{cases} T(a,b), & \text{if } a \neq b \\ 1, & \text{otherwise} \end{cases}$ is a (decomposable) T-equivalence relation [7], and calculating the weighted mean of E^T and E_T,*

(i) For the Łukasiewicz t-norm T_L, the fuzzy relations

$$E(a,b) = \begin{cases} (2\lambda-1)(\max(a,b)-1)+\min(a,b), & \text{if } a+b \geq 1 \text{ and } a \neq b \\ (1-\lambda)(1-\max(a,b)+\min(a,b)), & \text{if } a+b < 1 \text{ and } a \neq b \\ 1, & \text{if } a = b \end{cases}$$

are T_L-equivalence relations between E^{T_L} and E_{T_L} for any $\lambda \in [0,1]$.

(ii) For the Product t-norm T_P, the fuzzy relations

$$E(a,b) = \begin{cases} \min(a,b)\max(a,b)^{2\lambda-1}, & \text{if } a \neq b \\ 1, & \text{if } a = b \end{cases}$$

are T_P-equivalence relations between E^{T_P} and E_{T_P} for any $\lambda \in [0,1]$.

In this way we have a way to go smoothly from E_T to E^T.

Let us end this section by characterizing the functions that aggregate T_D-equivalence relations.

Proposition 11. *A function $F : [0,1]^m \longrightarrow [0,1]$, $m \geq 1$, aggregates T_D-equivalence relations if and only if* *(i)* $F(1, 1, \ldots, 1) = 1$

(ii) If there exists $a_i \in [0, 1]$, $a_i \neq 1$, with $F(a_1, a_2, \ldots, a_i, \ldots, a_m) = 1$, then $F(a_1, a_2, \ldots, a_{i-1}, x, a_{i+1} \ldots, a_m)$, $x \neq a_i$, $x \neq 1$ is a constant function.

Proof:

$\Leftarrow$) Straigtforward.

$\Rightarrow$) If there exists $a_i \in [0, 1)$ with $F(a_1, a_2, \ldots, a_i, \ldots, a_m) = 1$ and (a_i, b_i, c_i) is a T_D-triangular triplet with $b_i \neq 1$ and $c_i \neq 1$, then according to Example 2. $F(a_1, a_2, \ldots, a_{i-1}, b_i, a_{i+1} \ldots, a_m) = F(a_1, a_2, \ldots, a_{i-1}, c_i, a_{i+1} \ldots, a_m)$.

For example, every $F : [0, 1]^m \longrightarrow [0, 1]$, $m \geq 1$ satisfying $F(a_1, a_2, \ldots, a_m) = 1$ if and only if $(a_1, a_2, \ldots, a_m) = (1, 1, \ldots, 1)$ aggregates T_D-equivalence relations.

4. Conclusions

In this contribution we revisit the problem of the aggregation of T-equivalence relations. After introducing the concept of T-triangular triplet, we characterize those functions that transform any collection of T-equivalence relations into a single one. The interest of this characterization is that we do not assume any extra condition on the t-norm T.

Considering Functions $F : [0, 1]^m \longrightarrow [0, 1]$, $m \geq 1$ transforming m-dimensional T-triangular triplets into 1-dimensional T-triangular triplets but not satisfying necessarily the property $F(1, 1, \ldots, 1) = 1$, we obtain the functions that preserve T-transitivity and hence more general T-transitive relations than T-equivalence relations, in particular T-preorders.

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