TRANSFORMATION AND DECOMPOSITION OF CLUTTERS INTO MATROIDS

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Abstract. A clutter is a family of mutually incomparable sets. The set of circuits of a matroid, its set of bases, and its set of hyperplanes are examples of clutters arising from matroids. In this paper we address the question of determining which are the matroidal clutters that best approximate an arbitrary clutter Λ. For this, we first define two orders under which to compare clutters, which give a total of four possibilities for approximating Λ (i.e., above or below with respect to each order); in fact, we actually consider the problem of approximating Λ with clutters from any collection of clutters Σ, not necessarily arising from matroids. We show that, under some mild conditions, there is a finite non-empty set of clutters from Σ that are the closest to Λ and, moreover, that Λ is uniquely determined by them, in the sense that it can be recovered using a suitable clutter operation. We then particularize these results to the case where Σ is a collection of matroidal clutters and give algorithmic procedures to compute these clutters.

1. Introduction

There are two general problems in mathematics that have been considered repeatedly: given a class of objects and a subclass of those satisfying certain conditions, there is the problem of approximating an arbitrary object with elements from the subclass, and also the related problem of decomposing the object in terms of objects in the subclass. Examples of either problem can be found in various areas of mathematics. For instance, examples of the first are the orthogonal projection in linear algebra and, in analysis, the theories concerning the approximation of functions of some kind by simpler functions. Integer factorization into primes fits into the second problem, as do the decomposition of a non-zero non-unit element as a product of prime elements in a unique factorization domain, the primary decomposition of ideals in noetherian rings and the decomposition of algebraic varieties into irreducible ones. In any case, many theoretical issues, properties and algorithms have been studied regarding both problems. This paper fits in this context and stems from considering these problems in a situation involving discrete objects. Namely, we study the approximation and decomposition of some families of subsets (clutters) into families of subsets arising from combinatorial objects with some specific structure (matroidal clutters).

A clutter on a finite set Ω is a family of mutually incomparable subsets of Ω; in other words, an antichain of the powerset 2^{Ω} ordered by set inclusion. Clutters are also known as simple hypergraphs or Sperner families, and they abound in combinatorics. Here we will be concerned in clutters arising from matroids, such as the collection of circuits, but also the collection of bases and the collection of hyperplanes. This paper is motivated by the following questions: given an arbitrary clutter Λ, which are the matroidal clutters that are *closest* to it, if any? If these exist, do they determine Λ? And also, how can we effectively find them?

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Our approach to the problem starts by specifying the meaning of *close* in the question above. For this we define two partial orders $\leqslant^+$ and $\leqslant^-$ on the set of clutters on a set Ω. For a clutter Λ and collection Σ of clutters, one can consider the set of all clutters that lie above or below Λ with respect to either order. We refer to elements of these four sets (that could be empty) as *completions* of Λ in Σ. We say that we make a choice of order ($\leqslant^+$ or $\leqslant^-$) and side (above or below). Among all completions, we call *optimal* the ones that are closest to Λ (minimal or maximal depending on the choice of side). We usually want Σ to be a family of clutters related to matroids, but one advantage of our approach is that we have general results that can be applied to any family Σ of interest. Loosely speaking, we show that for each choice of order and side, there is a family of clutters $\mathcal{F}$ such that if Σ contains $\mathcal{F}$ then the corresponding set of completions of Λ is non-empty and, moreover, that the optimal completions are enough to uniquely determine Λ. In fact, there is a clutter operation that allows us to express Λ as a combination of its optimal completions; we thus speak of a *decomposition* of Λ. We then specialize these results to the case where Σ is a family of clutters arising from matroids, showing that in most cases completions and decompositions exist and giving algorithms to find them.

The relationship between clutters and matroids has been explored before. Closest in spirit to our work is the paper by Dress and Wenzel [2], where they give a method to construct, from an arbitrary clutter Λ, another clutter that is the clutter of bases of a matroid (we compare this construction to our results in Section 4). Also, the paper by Martini and Wenzel [7] deals with a similar question of approximating arbitrary closure operators by closure operators arising from matroids. Other papers about clutters and matroids include [1, 10, 11, 12, 13], and focus on finding ways in which clutters behave like matroids and in characterizing matroidal clutters among all clutters. Another common theme is the behaviour of clutters under deletion and contraction, but we will not touch this topic either.

This paper is a wide generalization of the papers [5, 6]. In those only one of the four choices for completions was considered, namely, the ones above with respect to the order $\leqslant^+$; also, the only matroidal clutters considered were clutters of circuits. In [5] a method was given to construct such completions in circuit clutters (we review it in Section 5). In [6] the focus was on clutters of circuits of representable matroids (we point out the connections with this paper in Sections 3 and 4).

The paper is structured as follows. In Section 2 we give all the definitions needed about clutters and matroids. In Section 3 we present our results about completions in arbitrary clutters and in Section 4 we specialize these results to the cases of matroidal clutters. Finally, we give in Section 5 an algorithmic procedure to compute optimal matroidal completions, whenever they exist.

2. Clutters. Posets of clutters. Matroidal clutters

In this section we present the definitions and basic facts concerning families of subsets, clutters, matroids and matroidal clutters that are used in the paper. We omit all proofs that are a straightforward consequence of the definitions.

Throughout the paper Ω is a non-empty finite set. The power set of Ω is denoted 2^Ω. For $\Upsilon \subseteq 2^\Omega$, $\min(\Upsilon)$ and $\max(\Upsilon)$ respectively denote the sets of minimal and maximal elements of Υ with respect to set inclusion.

2.1. **Families of subsets and clutters.** A *monotone increasing* family of subsets Γ of Ω is a collection of subsets of Ω such that any superset of a set in Γ also belongs to Γ; that is, if $A \in \Gamma$ and $A \subseteq A' \subseteq \Omega$, then $A' \in \Gamma$. A *monotone decreasing* family of subsets Γ of Ω is a collection of subsets of Ω such that any subset of a set in Γ also belongs to Γ; that is, if $A \in \Gamma$ and $A' \subseteq A$, then $A' \in \Gamma$. A *clutter* Λ on Ω is a collection of subsets Λ of Ω none of which is a proper subset of another; that is, if $A, A' \in \Lambda$ and $A \subseteq A'$ then $A = A'$. Note that Λ being a clutter on Ω does not imply that every element of Ω belongs to some set in Λ; in particular, both $\{\} = \emptyset$ and $\{\emptyset\}$ are clutters on every finite set Ω. These two clutters are called *trivial clutters*. The set of all clutters on Ω is denoted by $\text{Clutt}(\Omega)$.

A clutter Λ determines a monotone increasing family Λ^+ and a monotone decreasing family Λ^- of subsets:

$$\Lambda^+ = \{A \subseteq \Omega \,:\, A_0 \subseteq A \text{ for some } A_0 \in \Lambda\},$$
$$\Lambda^- = \{A \subseteq \Omega \,:\, A \subseteq A_0 \text{ for some } A_0 \in \Lambda\}.$$

Conversely, if Γ is a monotone increasing family of subsets of Ω, the collection $\min(\Gamma)$ of its inclusion-minimal elements is a clutter, and clearly $\Gamma = (\min(\Gamma))^+$ and $\Lambda = \min(\Lambda^+)$. Similarly, if now Γ is a monotone decreasing family of subsets of Ω, the collection $\max(\Gamma)$ of its inclusion-maximal elements is also a clutter, and $\Gamma = (\max(\Gamma))^-$ and $\Lambda = \max(\Lambda^-)$. So a monotone increasing (decreasing) family of subsets Γ is uniquely determined by the clutter $\min(\Gamma)$ (respectively, $\max(\Gamma)$), while a clutter Λ is uniquely determined by either of the families Λ^+ and Λ^-. Note that $\{\}^+ = \{\}^- = \{\}$, $\{\emptyset\}^+ = 2^\Omega$ and $\{\emptyset\}^- = \{\emptyset\}$.

We next introduce an operation on families of subsets, related to taking complements. For $\Upsilon \subseteq 2^\Omega$, let $\Upsilon^c \subseteq 2^\Omega$ be the family

$$\Upsilon^c = \{B \subseteq \Omega \,:\, \Omega \setminus B \in \Upsilon\}.$$

We refer to Υ^c as the *complementary* family of Υ. The following lemma states some immediate properties of this operation.

Lemma 2.1. *Let $\Upsilon \subseteq 2^\Omega$. The following statements hold:*

(1) Υ is monotone increasing if and only if Υ^c is monotone decreasing.
(2) Υ is monotone decreasing if and only if Υ^c is monotone increasing.
(3) Υ is a clutter if and only if Υ^c is a clutter.
(4) $\left(\Upsilon^c\right)^c = \Upsilon$.

The complementary clutters of the trivial clutters $\{\}$ and $\{\emptyset\}$, that is, the clutters $\{\}$ and $\{\Omega\}$, are called *cotrivial clutters*. Observe that the empty clutter $\{\}$ is both trivial and cotrivial.

Another operation on set families that maps clutters to clutters is the blocker. For $\Upsilon \subseteq 2^\Omega$, its *blocker* (or *transversal*) is the clutter

$$b(\Upsilon) = \min\left(\{B \,:\, B \cap A \neq \emptyset \text{ for all } A \in \Upsilon\}\right).$$

The blocker of the empty clutter $\{\}$ is thus $\{\emptyset\}$, and $b(\{\emptyset\}) = \{\}$. This is no coincidence as the blocker map is involutive on clutters ([3]), that is:

Lemma 2.2. *Let Λ be a clutter on Ω. Then, $b(b(\Lambda)) = \Lambda$.*

2.2. **Comparison of clutters. Posets of clutters. Operations between clutters.** In order to endow $\text{Clutt}(\Omega)$ with a poset structure, one could simply consider clutter containment, but as we shall see later this is not fine enough for our purposes (see Subsection 2.4 and Remark 4.1 in Subsection 4.1). It turns out to be more convenient to compare clutters in terms of the monotone increasing or decreasing families to which they give rise. Although clearly $\Lambda_1 \subseteq \Lambda_2$ implies $\Lambda_1^+ \subseteq \Lambda_2^+$ and $\Lambda_1^- \subseteq \Lambda_2^-$, the converses are not true; for instance, the clutters $\Lambda_1 = \{\{1,2\},\{2,3\}\}$ and $\Lambda_2 = \{\{1\},\{2,3\}\}$ on $\Omega = \{1,2,3\}$ satisfy $\Lambda_1^+ \subseteq \Lambda_2^+$ and $\Lambda_2^- \subseteq \Lambda_1^-$ but $\Lambda_1 \not\subseteq \Lambda_2$ and $\Lambda_2 \not\subseteq \Lambda_1$.

This leads us to consider the following two binary relations $\leqslant^+$ and $\leqslant^-$ defined on $\text{Clutt}(\Omega)$. If Λ_1 and Λ_2 are clutters on Ω, then we say that

$$\Lambda_1 \leqslant^+ \Lambda_2 \text{ if and only if } \Lambda_1^+ \subseteq \Lambda_2^+;$$

and we say that

$$\Lambda_1 \leqslant^- \Lambda_2 \text{ if and only if } \Lambda_1^- \subseteq \Lambda_2^-.$$

Since $\{\}^+ = \{\}^- = \{\}$, the clutter $\{\}$ lies below any other clutter with respect to both $\leqslant^+$ and $\leqslant^-$. Similarly, $\Lambda \leqslant^+ \{\emptyset\}$ and $\Lambda \leqslant^- \{\Omega\}$ for every clutter Λ on Ω.

It is clear that both $\leqslant^+$ and $\leqslant^-$ are partial orders on the set of clutters on Ω. So, $(\mathrm{Clutt}(\Omega), \leqslant^+)$ and $(\mathrm{Clutt}(\Omega), \leqslant^-)$ are posets of clutters.

The following lemma rephrases the relations $\leqslant^+$ and $\leqslant^-$ solely in terms of the elements of Λ_1 and Λ_2; it will be used repeatedly throughout the paper.

Lemma 2.3. *Let Λ_1, Λ_2 be clutters on Ω. The following statements hold:*

(1) *$\Lambda_1 \leqslant^+ \Lambda_2$ if and only if $\Lambda_1 \subseteq \Lambda_2^+$. Therefore, $\Lambda_1 \leqslant^+ \Lambda_2$ if and only if for all $A_1 \in \Lambda_1$ there exists $A_2 \in \Lambda_2$ such that $A_2 \subseteq A_1$.*
(2) *$\Lambda_1 \leqslant^- \Lambda_2$ if and only if $\Lambda_1 \subseteq \Lambda_2^-$. Therefore, $\Lambda_1 \leqslant^- \Lambda_2$ if and only if for all $A_1 \in \Lambda_1$ there exists $A_2 \in \Lambda_2$ such that $A_1 \subseteq A_2$.*

We next look at the relationship between the orders $\leqslant^+, \leqslant^-$ and the operations on clutters introduced above.

Lemma 2.4. *Let Λ_1, Λ_2 be clutters on Ω. The following statements hold:*

(1) *$\Lambda_1 \leqslant^+ \Lambda_2$ if and only if $\Lambda_1^c \leqslant^- \Lambda_2^c$.*
(2) *$\Lambda_1 \leqslant^- \Lambda_2$ if and only if $\Lambda_1^c \leqslant^+ \Lambda_2^c$.*
(3) *$\Lambda_1 \leqslant^+ \Lambda_2$ if and only if $b(\Lambda_2) \leqslant^+ b(\Lambda_1)$.*
(4) *$\Lambda_1 \leqslant^- \Lambda_2$ if and only if $b(\Lambda_2^c)^c \leqslant^- b(\Lambda_1^c)^c$.*

Proof. The statements about the complementary follow easily from the definitions and Lemma 2.1.

For statement (3), assume $\Lambda_1 \leqslant^+ \Lambda_2$ and let $B \in b(\Lambda_2)$. If we show that $A \cap B \neq \emptyset$ for all $A \in \Lambda_1$, we will be done since this implies $b(\Lambda_2) \subseteq b(\Lambda_1)^+$. Now, the relation $\Lambda_1 \leqslant^+ \Lambda_2$ implies that there is $A' \in \Lambda_2$ such that $A' \subseteq A$. Since $B \in b(\Lambda_2)$, we have $\emptyset \neq A' \cap B \subseteq A \cap B$, as needed. The converse implication is immediate since the blocker map is involutive. Finally (4) follows from (2) and (3). □

We next introduce four operations between clutters that play a key role in Section 3.

Let $\Lambda_1, \dots, \Lambda_r$ be clutters on Ω. We define

$$\begin{cases} \Lambda_1 \sqcap^+ \cdots \sqcap^+ \Lambda_r = \min\left(\Lambda_1^+ \cap \cdots \cap \Lambda_r^+\right), \\ \Lambda_1 \sqcup^+ \cdots \sqcup^+ \Lambda_r = \min\left(\Lambda_1^+ \cup \cdots \cup \Lambda_r^+\right), \\ \Lambda_1 \sqcap^- \cdots \sqcap^- \Lambda_r = \max\left(\Lambda_1^- \cap \cdots \cap \Lambda_r^-\right), \\ \Lambda_1 \sqcup^- \cdots \sqcup^- \Lambda_r = \max\left(\Lambda_1^- \cup \cdots \cup \Lambda_r^-\right). \end{cases}$$

The following lemma provides characterizations and descriptions of these clutters.

Lemma 2.5. *Let $\Lambda_1, \dots, \Lambda_r$ be clutters on Ω. The following statements hold:*

(1) *$\Lambda_1 \sqcap^+ \cdots \sqcap^+ \Lambda_r = \min\left(\{A_1 \cup \cdots \cup A_r : A_i \in \Lambda_i\}\right)$, and it is the unique clutter Λ_0 satisfying the following two conditions:*
    (a) *$\Lambda_0 \leqslant^+ \Lambda_i$ for all i, and*
    (b) *if Λ' is a clutter such that $\Lambda' \leqslant^+ \Lambda_i$ for all i, then $\Lambda' \leqslant^+ \Lambda_0$.*
(2) *$\Lambda_1 \sqcup^+ \cdots \sqcup^+ \Lambda_r = \min(\Lambda_1 \cup \cdots \cup \Lambda_r)$, and it is the unique clutter Λ_0 satisfying the following two conditions:*
    (a) *$\Lambda_i \leqslant^+ \Lambda_0$ for all i, and*
    (b) *if Λ' is a clutter such that $\Lambda_i \leqslant^+ \Lambda'$ for all i, then $\Lambda_0 \leqslant^+ \Lambda'$.*

(3) $\Lambda_1 \sqcap^- \cdots \sqcap^- \Lambda_r = \max\left(\{A_1 \cap \cdots \cap A_r \,:\, A_i \in \Lambda_i\}\right)$, *and it is the unique clutter* Λ_0 *satisfying the following two conditions:*
   (a) $\Lambda_0 \leqslant^- \Lambda_i$ *for all* i, *and*
   (b) if Λ' *is a clutter such that* $\Lambda' \leqslant^- \Lambda_i$ *for all* i, *then* $\Lambda' \leqslant^- \Lambda_0$.

(4) $\Lambda_1 \sqcup^- \cdots \sqcup^- \Lambda_r = \max(\Lambda_1 \cup \cdots \cup \Lambda_r)$, *and it is the unique clutter* Λ_0 *satisfying the following two conditions:*
   (a) $\Lambda_i \leqslant^- \Lambda_0$ *for all* i, *and*
   (b) if Λ' *is a clutter such that* $\Lambda_i \leqslant^- \Lambda'$ *for all* i, *then* $\Lambda_0 \leqslant^- \Lambda'$.

Another way to state Lemma 2.5 is by saying that the poset $(\mathrm{Clutt}(\Omega), \leqslant^+)$ is a lattice with meet $\sqcap^+$ and join $\sqcup^+$, and that similarly $(\mathrm{Clutt}(\Omega), \leqslant^-)$ is a lattice with meet $\sqcap^-$ and join $\sqcup^-$.

We conclude this subsection with the following lemma concerning the behaviour of these operations between clutters with respect to the complementary and blocker maps.

Lemma 2.6. *Let* $\Lambda_1, \ldots, \Lambda_r$ *be clutters on* Ω. *The following statements hold:*

(1) $(\Lambda_1 \sqcap^+ \cdots \sqcap^+ \Lambda_r)^c = \Lambda_1^c \sqcap^- \cdots \sqcap^- \Lambda_r^c$.
(2) $(\Lambda_1 \sqcup^+ \cdots \sqcup^+ \Lambda_r)^c = \Lambda_1^c \sqcup^- \cdots \sqcup^- \Lambda_r^c$.
(3) $b(\Lambda_1 \sqcap^+ \cdots \sqcap^+ \Lambda_r) = b(\Lambda_1) \sqcup^+ \cdots \sqcup^+ b(\Lambda_r)$.
(4) $b(\Lambda_1 \sqcup^+ \cdots \sqcup^+ \Lambda_r) = b(\Lambda_1) \sqcap^+ \cdots \sqcap^+ b(\Lambda_r)$.

Proof. The proofs of these statements follow easily from Lemma 2.4 and the descriptions in Lemma 2.5. □

2.3. **Matroids and matroidal clutters.** The families of clutters most relevant to this paper are those that arise from matroids. Matroids are combinatorial objects that admit several equivalent axiomatizations, the most common ones being in terms of independent sets, bases, circuits, rank function, flats, or hyperplanes (the reader is referred to [8, 14] for general references on matroid theory). Here we present the definition in terms of circuits.

A *matroid* $\mathcal{M}$ is an ordered pair $\mathcal{M} = (\Omega, \mathcal{C})$ consisting of a finite non-empty set Ω, called the *ground set* of the matroid, and a clutter $\mathcal{C}$ of non-empty subsets of Ω that satisfies the *circuit elimination property*: if C_1 and C_2 are distinct members of $\mathcal{C}$ and $x \in C_1 \cap C_2$, then there is some member C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{x\}$.

The members of the clutter $\mathcal{C}$ are the *circuits* of the matroid $\mathcal{M}$. We shall often write $\mathcal{C}(\mathcal{M})$ instead of $\mathcal{C}$. The *dependent sets* of the matroid are the supersets of the circuits, that is, the members of $\mathcal{C}(\mathcal{M})^+$.

Sets that are not dependent are called *independent*; the collection of independent sets of a matroid $\mathcal{M}$ is denoted by $\mathcal{I}(\mathcal{M})$. The family $\mathcal{I}(\mathcal{M})$ is monotone decreasing and its maximal elements are the *bases* of the matroid. The clutter of bases of $\mathcal{M}$ is denoted by $\mathcal{B}(\mathcal{M})$; thus, $\mathcal{B}(\mathcal{M})^- = \mathcal{I}(\mathcal{M})$.

We say that the clutter Λ on Ω is a *clutter of circuits* (or a *circuit clutter*) if there exists a matroid $\mathcal{M}$ with ground set Ω such that $\Lambda = \mathcal{C}(\mathcal{M})$. Similarly, Λ is a *clutter of bases* (or a *basis clutter*) if $\Lambda = \mathcal{B}(\mathcal{M})$ for some matroid $\mathcal{M}$ on Ω. Among all non-empty clutters Λ, basis clutters are precisely those that satisfy the following *basis exchange* property ([8, Thm. 1.2.3]): if B_1, B_2 are elements of Λ and $x \in B_1 \setminus B_2$, then there is $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \Lambda$.

The other family of clutters associated to matroids that will appear in this work are clutters of hyperplanes. The *rank* of a subset $A \subseteq \Omega$ is the size of the largest independent set included in A, and it is denoted by $r(A)$. The rank $r(\mathcal{M})$ of the matroid $\mathcal{M}$ is $r(\Omega)$. The set $H \subseteq \Omega$ is a *hyperplane* if H is maximal with the property $r(H) = r(\mathcal{M}) - 1$. The collection of hyperplanes of $\mathcal{M}$ forms a clutter $\mathcal{H}(\mathcal{M})$, and a clutter Λ is called a *clutter of hyperplanes* (or *hyperplane clutter*) if

$\Lambda = \mathcal{H}(\mathcal{M})$ for some matroid $\mathcal{M}$. As circuit and basis clutters, hyperplane clutters also have a characterization ([8, Prop. 2.1.21]): a clutter Λ is a hyperplane clutter if and only if $\Lambda \neq \{\Omega\}$ and if H_1, H_2 are distinct members of Λ and $x \in \Omega \setminus (H_1 \cup H_2)$, then there is $H_3 \in \Lambda$ such that $H_3 \supseteq (H_1 \cap H_2) \cup \{x\}$.

We use the generic term *matroidal clutter* to refer to any of the clutters arising from matroids. Thus, each matroid determines three matroidal clutters that are usually different.

Before stating some examples of matroidal clutters, next we look now at the behaviour of the operations introduced in Subsection 2.1 in the particular case of matroidal clutters.

It is a well-known fact ([8, Thm. 2.1.1]) that there is a matroid $\mathcal{M}^*$ on the ground set Ω such that $\mathcal{B}(\mathcal{M}^*) = \mathcal{B}(\mathcal{M})^c$; this matroid is called the *dual* of $\mathcal{M}$. Clearly $\mathcal{M}^{**} = \mathcal{M}$. Table 1 below includes this and some other less trivial relationships ([8, Sec. 2.1]). From these relationships we see that the complementary Λ^c of a matroidal clutter Λ is also a matroidal clutter, but the blocker $b(\Lambda)$ of a matroidal clutter might not be a matroidal clutter when Λ is a hyperplane clutter (note that the blocker of a hyperplane clutter is sometimes a matroidal clutter, as in the case $b(\{\emptyset\}) = \{\}$, but there is no general relationship).

Λ	Λ^c	$b(\Lambda)$
$\mathcal{B}(\mathcal{M})$	$\mathcal{B}(\mathcal{M}^*)$	$\mathcal{C}(\mathcal{M}^*)$
$\mathcal{C}(\mathcal{M})$	$\mathcal{H}(\mathcal{M}^*)$	$\mathcal{B}(\mathcal{M}^*)$
$\mathcal{H}(\mathcal{M})$	$\mathcal{C}(\mathcal{M}^*)$	–

Table 1. The effect of the complementary and blocker maps on matroidal clutters.

To conclude this subsection, we recall the definition of two well-known classes of matroids: uniform matroids and partition matroids. The circuit, basis and hyperplane clutters of these matroids will appear several times in this paper.

Let n be the size of the finite set Ω and let r be a non-negative integer such that $r \leq n$. The *uniform matroid* $\mathcal{U}_{r,n}$ is the matroid with ground set Ω and clutter of circuits $\mathcal{C}(\mathcal{U}_{r,n}) = \{C \subseteq \Omega : |C| = r + 1\}$. The clutter of basis of the uniform matroid is $\mathcal{B}(\mathcal{U}_{r,n}) = \{B \subseteq \Omega : |B| = r\}$, and its clutter of hyperplanes is $\mathcal{H}(\mathcal{U}_{r,n}) = \{H \subseteq \Omega : |H| = r - 1\}$. The dual of a uniform matroid is a uniform matroid, namely $\left(\mathcal{U}_{r,n}\right)^* = \mathcal{U}_{n-r,n}$.

Remark 2.7. By using matroidal clutters of uniform matroids we show that trivial and cotrivial clutters are matroidal clutters, and that there are clutters that belong to the three kinds of matroidal clutters. Indeed, the clutter $\{\}$ is not a basis clutter, but $\{\}$ is the circuit clutter $\mathcal{C}(\mathcal{U}_{n,n})$ and the hyperplane clutter $\mathcal{H}(\mathcal{U}_{0,n})$ (with $n = |\Omega|$). Similarly, the clutter $\{\emptyset\}$ is not a circuit clutter, but $\{\emptyset\}$ is the basis clutter $\mathcal{B}(\mathcal{U}_{0,n})$ and the hyperplane clutter $\mathcal{H}(\mathcal{U}_{1,n})$. Finally the clutter $\{\Omega\}$ is not a hyperplane clutter but $\{\Omega\}$ is the circuit clutter $\mathcal{C}(\mathcal{U}_{n-1,n})$ and the basis clutter $\mathcal{B}(\mathcal{U}_{n,n})$. A clutter that belongs to the three kinds of matroidal clutters is $\{\{\omega\} : \omega \in \Omega\}$ for $n \geq 2$. Actually, it is the circuit clutter $\mathcal{C}(\mathcal{U}_{0,n})$, it is the basis clutter $\mathcal{B}(\mathcal{U}_{1,n})$ and it is the hyperplane clutter $\mathcal{H}(\mathcal{U}_{2,n})$. (These results are consistent with Table 1 because the clutters $\{\}$ and $\{\emptyset\}$ are each one the blocker of the other, and the same occurs with the clutters $\{\Omega\}$ and $\{\{\omega\} : \omega \in \Omega\}$.)

Let $(\Omega_1, \ldots, \Omega_k)$ be a partition of the finite set Ω and let $(r_1, \ldots, r_k)$ be a sequence of integers with $0 \leq r_i \leq |\Omega_i|$ for all $1 \leq i \leq k$. The *partition matroid*

$\Pi(\Omega_1,\ldots,\Omega_k;r_1,\ldots,r_k)$ is the matroid with ground set Ω and clutter of circuits

$$\{C\subseteq\Omega : \exists\ i \text{ such that } |C\cap\Omega_i| = r_i+1 \text{ and } C\cap\Omega_j=\emptyset \text{ for } 1\le j\ne i\le k\}.$$

The corresponding clutter of bases of $\Pi(\Omega_1,\ldots,\Omega_k;r_1,\ldots,r_k)$ is

$$\{B\subseteq\Omega : |B\cap\Omega_i| = r_i \text{ for } 1\le i\le k\};$$

whereas its clutter of hyperplanes is

$$\{H\subseteq\Omega : \exists\ i \text{ such that } |H\cap\Omega_i| = r_i-1 \text{ and } \Omega_j\subseteq H \text{ for } 1\le j\ne i\le k\}.$$

The dual of a partition matroid is a partition matroid. More concretely, $\big(\Pi(\Omega_1,\ldots,\Omega_k;r_1,\ldots,r_k)\big)^* = \Pi(\Omega_1,\ldots,\Omega_k;|\Omega_1|-r_1,\ldots,|\Omega_k|-r_k)$.

Remark 2.8. Observe that uniform matroids are the partition matroids where $k=1$, that is, $\mathcal{U}_{r,n}=\Pi(\Omega;r)$ (where $n=|\Omega|$). An arbitrary partition matroid can be written as a direct sum of uniform matroids, namely, we have that $\Pi(\Omega_1,\ldots,\Omega_k;r_1,\ldots,r_k)=\mathcal{U}_{r_1,|\Omega_1|}\oplus\cdots\oplus\mathcal{U}_{r_k,|\Omega_k|}$ (see [8] for the definition of direct sum).

2.4. **Comparing matroidal clutters. Operations between matroidal clutters.** The orders and operations defined above for arbitrary clutters are naturally meaningful when restricted to matroidal clutters. Moreover, some of them coincide with, or are related to, well-studied matroid theoretic notions, as the already mentioned relation between the complementary and blocker operators and duality. Here we present some other such connections in order to give more concrete instances of our definitions, but the contents of this subsection are not actually used in the sequel.

Even restricted to matroidal clutters, the orders $\leqslant^+$ and $\leqslant^-$ are much finer than clutter inclusion. For instance, let us consider two uniform matroids $\mathcal{U}_{r,n}$ and $\mathcal{U}_{s,n}$. It is clear that if $r\ne s$ then $\mathcal{C}(\mathcal{U}_{r,n})\not\subseteq\mathcal{C}(\mathcal{U}_{s,n})$, $\mathcal{B}(\mathcal{U}_{r,n})\not\subseteq\mathcal{B}(\mathcal{U}_{s,n})$, and $\mathcal{H}(\mathcal{U}_{r,n})\not\subseteq\mathcal{H}(\mathcal{U}_{s,n})$. However, observe that for $0<r\le s<n$, by Lemma 2.3 we get that:

$$\begin{aligned}
&\mathcal{C}(\mathcal{U}_{s,n})\leqslant^+\mathcal{C}(\mathcal{U}_{r,n}),\quad \mathcal{B}(\mathcal{U}_{s,n})\leqslant^+\mathcal{B}(\mathcal{U}_{r,n}),\quad \mathcal{H}(\mathcal{U}_{s,n})\leqslant^+\mathcal{H}(\mathcal{U}_{r,n}),\\
&\mathcal{C}(\mathcal{U}_{r,n})\leqslant^-\mathcal{C}(\mathcal{U}_{s,n}),\quad \mathcal{B}(\mathcal{U}_{r,n})\leqslant^-\mathcal{B}(\mathcal{U}_{s,n}),\quad \mathcal{H}(\mathcal{U}_{r,n})\leqslant^-\mathcal{H}(\mathcal{U}_{s,n}).
\end{aligned}$$

Therefore, the partial orders $\leqslant^+$ and $\leqslant^-$ seem more informative in order to compare matroidal clutters. It turns out that these orders are close to the well-known *weak order* $\leqslant_w$ on matroids: given two matroids $\mathcal{M}_1,\mathcal{M}_2$ on Ω, we have $\mathcal{M}_1\leqslant_w\mathcal{M}_2$ if every circuit of $\mathcal{M}_2$ contains a circuit of $\mathcal{M}_1$. (See [8, Sec. 7.3] for more details on the weak order.)

Lemma 2.9. *Let $\mathcal{M}_1,\mathcal{M}_2$ be matroids on Ω. The following statements hold:*

(1) *$\mathcal{C}(\mathcal{M}_1)\leqslant^+\mathcal{C}(\mathcal{M}_2)$ if and only if $\mathcal{M}_2\leqslant_w\mathcal{M}_1$.*
(2) *$\mathcal{B}(\mathcal{M}_1)\leqslant^+\mathcal{B}(\mathcal{M}_2)$ if and only if $\mathcal{M}_1^*\leqslant_w\mathcal{M}_2^*$.*
(3) *$\mathcal{B}(\mathcal{M}_1)\leqslant^-\mathcal{B}(\mathcal{M}_2)$ if and only if $\mathcal{M}_1\leqslant_w\mathcal{M}_2$.*
(4) *$\mathcal{H}(\mathcal{M}_1)\leqslant^-\mathcal{H}(\mathcal{M}_2)$ if and only if $\mathcal{M}_2^*\leqslant_w\mathcal{M}_1^*$.*

Proof. The first and third statement follow easily from Lemma 2.3. As for the second statement, Lemma 2.4 implies that $\mathcal{B}(\mathcal{M}_1)\leqslant^+\mathcal{B}(\mathcal{M}_2)$ is equivalent to $b(\mathcal{B}(\mathcal{M}_2))\leqslant^+ b(\mathcal{B}(\mathcal{M}_1))$; by duality (Table 1) this is equivalent to $\mathcal{C}(\mathcal{M}_2^*)\leqslant^+\mathcal{C}(\mathcal{M}_1^*)$, and finally we apply the first statement of this lemma. The fourth statement follows from the first one by duality. □

Remark 2.10. The relationships $\mathcal{C}(\mathcal{M}_1)\leqslant^-\mathcal{C}(\mathcal{M}_2)$ and $\mathcal{H}(\mathcal{M}_1)\leqslant^+\mathcal{H}(\mathcal{M}_2)$ are not expressible in terms of the weak order.

Next, we discuss some questions concerning the behaviour of the operations $\sqcup^+$, $\sqcap^+$, $\sqcup^-$ and $\sqcap^-$ between matroidal clutters. It will be clear from the results in Section 4 that these four operations are not closed when restricted to matroidal clutters. Let us show a concrete example.

Example 2.11. Let Λ_1 and Λ_2 be the clutters of circuits corresponding to the partition matroids $\mathcal{U}_{2,5}$ and $\Pi(\{1,2\},\{3,4,5\};1,2)$, both on the ground set $\Omega = \{1,2,3,4,5\}$. Then $\Lambda_1 \sqcap^+ \Lambda_2 = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{3,4,5\}\}$, which is not a circuit clutter since the sets $\{1,2,3\}$ and $\{1,2,4\}$ do not satisfy the circuit elimination property.

The operations $\sqcap^+, \sqcup^+, \sqcap^-$ and $\sqcup^-$ are somewhat reminiscent of matroid union and intersection. Recall that the *union* of two matroids $\mathcal{M}_1, \mathcal{M}_2$ on the ground set Ω is the matroid $\mathcal{M}_1 \vee \mathcal{M}_2$ on Ω whose independent sets are $\mathcal{I}(\mathcal{M}_1 \vee \mathcal{M}_2) = \{I_1 \cup I_2 : I_i \in \mathcal{I}(\mathcal{M}_i)\}$. The *intersection* of the matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ is the family of sets $\mathcal{I}(\mathcal{M}_1) \cap \mathcal{I}(\mathcal{M}_2)$. The intersection $\mathcal{I}(\mathcal{M}_1) \cap \mathcal{I}(\mathcal{M}_2)$ is a monotone decreasing family of subsets but it is not, in general, the set of independent sets of a matroid. We refer to [8, Sec. 11.3] for further details on matroid union and intersection.

In the following lemma we briefly point out some connections between the operations between clutters and matroid union and intersection.

Lemma 2.12. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be matroids on the same ground set Ω. The following statements hold:*

(1) $\mathcal{I}(\mathcal{M}_1) \cap \mathcal{I}(\mathcal{M}_2) = \big(\mathcal{B}(\mathcal{M}_1) \sqcap^- \mathcal{B}(\mathcal{M}_2)\big)^-$.
(2) $\mathcal{C}(\mathcal{M}_1 \vee \mathcal{M}_2) \leqslant^+ \mathcal{C}(\mathcal{M}_1) \sqcap^+ \mathcal{C}(\mathcal{M}_2)$*, and equality does not hold in general.*

Proof. The first statement is clear because, in our notation, $\mathcal{B}(\mathcal{M}_1) \sqcap^- \mathcal{B}(\mathcal{M}_2)$ are the maximal members of the monotone decreasing family of subsets $\mathcal{I}(\mathcal{M}_1) \cap \mathcal{I}(\mathcal{M}_2)$.

Let us prove that $\mathcal{C}(\mathcal{M}_1 \vee \mathcal{M}_2) \leqslant^+ \mathcal{C}(\mathcal{M}_1) \sqcap^+ \mathcal{C}(\mathcal{M}_2)$. Let C be a circuit of $\mathcal{M}_1 \vee \mathcal{M}_2$. For $i \in \{1,2\}$, let I_i be a subset of C that is maximum independent in $\mathcal{M}_i$. As C is not independent in $\mathcal{M}_1 \vee \mathcal{M}_2$, there is $x \in C \setminus (I_1 \cup I_2)$. Both $I_1 \cup \{x\}$ and $I_2 \cup \{x\}$ are dependent in $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, so there are circuits $C_1 \in \mathcal{C}(\mathcal{M}_1)$ and $C_2 \in \mathcal{C}(\mathcal{M}_2)$ such that $C \supseteq C_1 \cup C_2$, as needed.

To finish, let us give an example showing that the clutter $\mathcal{C}(\mathcal{M}_1) \sqcap^+ \mathcal{C}(\mathcal{M}_2)$ is different from the union of $\mathcal{M}_1$ and $\mathcal{M}_2$. Consider the matroids $\mathcal{M}_1 = \mathcal{U}_{2,3}$ and $\mathcal{M}_2 = \Pi(\{1,2\},\{3\};1,1)$. Its union $\mathcal{M}_1 \vee \mathcal{M}_2$ is $\mathcal{U}_{3,3}$, since the set $\{1,2,3\}$ is the union of $\{1,2\}$, independent in $\mathcal{M}_1$, and $\{3\}$, independent in $\mathcal{M}_2$. Thus $\mathcal{C}(\mathcal{M}_1 \vee \mathcal{M}_2) = \{\}$. Now, $\mathcal{C}(\mathcal{M}_1) \sqcap^+ \mathcal{C}(\mathcal{M}_2) = \{\{1,2,3\}\} \sqcap^+ \{\{1,2\}\} = \{\{1,2,3\}\}$. □

Remark 2.13. The problem of determining the size of the largest common independent set of two matroids is well-known in optimization and it can be solved in polynomial time, provided that checking independence in either matroid can be done in polynomial time (see, for instance, Chapter 41 of Schrijver's book [9]). For three matroids though, the problem of determining the size of the largest element in $\mathcal{I}(\mathcal{M}_1) \cap \mathcal{I}(\mathcal{M}_2) \cap \mathcal{I}(\mathcal{M}_3)$ is NP-complete. This implies that the complexity of the clutter operation $\sqcap^-$ is at least as hard, since otherwise we could use associativity of this operation to solve the intersection problem with 3 matroids.

3. Completions and decompositions of clutters

Let $\Sigma \subseteq \mathrm{Clutt}\,(\Omega)$ be a collection of clutters on Ω and let Λ be a clutter on Ω. We wish to identify clutters in Σ that are in some sense close to the clutter Λ. For this, we consider the subset of those clutters from Σ that lie above or below Λ with respect to either $\leqslant^+$ or $\leqslant^-$. That is, we can associate to Λ four different subsets of Σ, one for each choice of order ($\leqslant^+$ or $\leqslant^-$) and side (above or below). We thus define

$$\begin{aligned}\Sigma_u^+(\Lambda) &= \{\Lambda' \in \Sigma \,:\, \Lambda \leqslant^+ \Lambda'\},\\ \Sigma_\ell^+(\Lambda) &= \{\Lambda' \in \Sigma \,:\, \Lambda' \leqslant^+ \Lambda\},\\ \Sigma_u^-(\Lambda) &= \{\Lambda' \in \Sigma \,:\, \Lambda \leqslant^- \Lambda'\},\\ \Sigma_\ell^-(\Lambda) &= \{\Lambda' \in \Sigma \,:\, \Lambda' \leqslant^- \Lambda\},\end{aligned}$$

and for $*_1 \in \{+,-\}$ and for $*_2 \in \{u,\ell\}$ we say that a clutter Λ' in $\Sigma_{*_2}^{*_1}(\Lambda)$ is a $\Sigma_{*_2}^{*_1}$-*completion* of Λ. The completions in $\Sigma_u^{*_1}(\Lambda)$ (respectively, in $\Sigma_\ell^{*_1}(\Lambda)$) will be called *upper* (respectively, *lower*) completions.

The goal of this section is to prove that, under certain mild assumptions, these four families of clutters $\Sigma_{*_2}^{*_1}(\Lambda)$ are non-empty and, moreover, that they uniquely determine the clutter Λ.

Table 2 gives a quick overview of our results (Theorems 3.1, 3.2, 3.4 and 3.5); it has to be interpreted as follows. For any choice of order $*_1 \in \{+,-\}$ and side $*_2 \in \{u,\ell\}$, the table gives a family of clutters $\mathcal{F}_{*_2}^{*_1}$ and an operation $\square_{*_2}^{*_1} \in \{\sqcap^+,\sqcup^+,\sqcap^-,\sqcup^-\}$. The corresponding theorem asserts that if $\mathcal{F}_{*_2}^{*_1} \subseteq \Sigma$, then for any clutter Λ the set of completions $\Sigma_{*_2}^{*_1}(\Lambda)$ is non-empty and, moreover, there exist some completions $\Lambda_1,\dots,\Lambda_r \in \Sigma_{*_2}^{*_1}(\Lambda)$ such that

$$\Lambda = \Lambda_1 \square_{*_2}^{*_1} \cdots \square_{*_2}^{*_1} \Lambda_r.$$

We refer to such an expression as a *decomposition* of the clutter Λ. The collection $\{\Lambda_1,\dots,\Lambda_r\}$ of $\Sigma_{*_2}^{*_1}$-completions that appears in the decomposition is denoted by $\Phi_{*_2}^{*_1}(\Lambda)$ in the table, and it corresponds to those completions that are "closest" to Λ; that is, either minimal or maximal elements of the poset $\left(\Sigma_{*_2}^{*_1}(\Lambda), \leqslant^{*_1}\right)$, depending on whether $*_2$ is u or ℓ. Therefore, from our results we conclude that the clutter Λ is univocally determined by the family $\Phi_{*_2}^{*_1}(\Lambda)$ of $\Sigma_{*_2}^{*_1}$-completions.

In Table 2, and from now on, for a family $\Sigma \subseteq \mathrm{Clutt}(\Omega)$ of clutters, $\min(\Sigma,\leqslant^+)$ denotes the set of minimal elements of Σ with respect to the order $\leqslant^+$. The sets $\min(\Sigma,\leqslant^-)$, $\max(\Sigma,\leqslant^+)$ and $\max(\Sigma,\leqslant^-)$ are defined analogously. Also, for a non-empty subset $X \subseteq \Omega$ we denote by Λ_X the clutter on Ω defined by $\Lambda_X = \{\{x\} : x \in X\}$. We finally remark that the actual statements of the theorems add some hypothesis to exclude degenerate cases, but we have not added this information to the table in order to keep readability.

The following two subsections are devoted to proving these four theorems on completions and decomposition of clutters, and to study the relationship between them (which is summarized in Table 3). In Section 4 we particularize all these theorems in the case of matroidal clutters and, furthermore, we analyze the necessity of the hypothesis on the collection of clutters Σ in Theorems 3.1, 3.2, 3.4 and 3.5.

3.1. Completions and decompositions with respect to the order $\leqslant^+$.

In this subsection we study the upper and lower completions $\Sigma_u^+(\Lambda)$ and $\Sigma_\ell^+(\Lambda)$ of a clutter Λ (Theorems 3.1 and 3.2). The first theorem, that deals with upper completions, was already given in [6], but here we present a different, shorter proof.

Theorem 3.1. *Let $\Sigma \subseteq \mathrm{Clutt}\,(\Omega)$ be such that $\Lambda_X \in \Sigma$ for all non-empty subsets X of Ω. If Λ is a non-trivial clutter on Ω, then the set $\Sigma_u^+(\Lambda)$ of upper Σ-completions of Λ with respect to the order $\leqslant^+$ is non-empty, and*

$$\Lambda = \Lambda_1 \sqcap^+ \cdots \sqcap^+ \Lambda_r\,,$$

where $\Lambda_1,\dots,\Lambda_r$ are the minimal elements of the poset $\left(\Sigma_u^+(\Lambda), \leqslant^+\right)$. In particular, the following statements hold:

	$\leqslant^+$	$\leqslant^-$
upper	Theorem 3.1 (order $\leqslant^+$ and upper completion) $\mathcal{F}_u^+ = \{\Lambda_X : \emptyset \subsetneq X \subseteq \Omega\}$ $\square_u^+ = \sqcap^+$ $\Phi_u^+(\Lambda) = \min\left(\Sigma_u^+(\Lambda), \leqslant^+\right)$	Theorem 3.4 (order $\leqslant^-$ and upper completion) $\mathcal{F}_u^- = \{\left(\Lambda_X\right)^c : \emptyset \subsetneq X \subseteq \Omega\}$ $\square_u^- = \sqcap^-$ $\Phi_u^-(\Lambda) = \min\left(\Sigma_u^-(\Lambda), \leqslant^-\right)$
lower	Theorem 3.2 (order $\leqslant^+$ and lower completion) $\mathcal{F}_\ell^+ = \{\{X\} : \emptyset \subsetneq X \subseteq \Omega\}$ $\square_\ell^+ = \sqcup^+$ $\Phi_\ell^+(\Lambda) = \max\left(\Sigma_\ell^+(\Lambda), \leqslant^+\right)$	Theorem 3.5 (order $\leqslant^-$ and lower completion) $\mathcal{F}_\ell^- = \{\{X\} : \emptyset \subseteq X \subsetneq \Omega\}$ $\square_\ell^- = \sqcup^-$ $\Phi_\ell^-(\Lambda) = \max\left(\Sigma_\ell^-(\Lambda), \leqslant^-\right)$

TABLE 2. Summary of Theorems 3.1, 3.2, 3.4 and 3.5: the corresponding family $\mathcal{F}_{*_2}^{*_1}$, the operation $\square_{*_2}^{*_1}$ and the family $\Phi_{*_2}^{*_1}(\Lambda)$ of completions providing a decomposition of Λ.

(a) A non-trivial clutter Λ on Ω belongs to the family Σ if and only if the poset $\left(\Sigma_u^+(\Lambda), \leqslant^+\right)$ has a unique minimal element.
(b) If Λ, Λ' are non-trivial clutters on Ω, then $\Lambda = \Lambda'$ if and only if the posets $\left(\Sigma_u^+(\Lambda), \leqslant^+\right)$ and $\left(\Sigma_u^+(\Lambda'), \leqslant^+\right)$ have the same minimal elements.

Proof. Let $\Omega = \{x_1, \ldots, x_n\}$. By Lemma 2.3, the fact that $\Lambda \neq \{\emptyset\}$ ensures that $\Lambda \leqslant^+ \Lambda_\Omega = \{\{x_1\}, \ldots, \{x_n\}\}$. This together with the assumption that $\Lambda_\Omega \in \Sigma$ gives $\Sigma_u^+(\Lambda) \neq \emptyset$.

Let $\Lambda_1, \ldots, \Lambda_r$ be the minimal elements of the poset $\left(\Sigma_u^+(\Lambda), \leqslant^+\right)$ and $\Lambda_0 = \Lambda_1 \sqcap^+ \cdots \sqcap^+ \Lambda_r$. Note the hypothesis $\Lambda \neq \{\}$ guarantees that none of these minimal clutters is $\{\}$. The proof of the theorem will then be completed by showing the equality $\Lambda = \Lambda_0$ (observe that statements (a) and (b) are a direct consequence of this equality).

Since $\Lambda_i \in \Sigma_u^+(\Lambda)$, the inequality $\Lambda \leqslant^+ \Lambda_i$ holds and so, from Lemma 2.5 we conclude that $\Lambda \leqslant^+ \Lambda_0$. Therefore, since the binary relation $\leqslant^+$ is a partial order, it is enough to show that $\Lambda_0 \leqslant^+ \Lambda$; that is, we must prove that if $A_0 \in \Lambda_0$, then there exists $A \in \Lambda$ such that $A \subseteq A_0$. This clearly holds if $A_0 = \Omega$, so we may assume that $A_0 \neq \Omega$.

Let $A_0 \in \Lambda_0$ with $A_0 \neq \Omega$, and let us assume for a contradiction that $A \not\subseteq A_0$ for all $A \in \Lambda$. By Lemma 2.3 it follows that $\Lambda \leqslant^+ \Lambda_{\Omega \setminus A_0}$, and so $\Lambda_{\Omega \setminus A_0} \in \Sigma_u^+(\Lambda)$ (because $\Omega \setminus A_0 \neq \emptyset$ and hence $\Lambda_{\Omega \setminus A_0} \in \Sigma$). Therefore, there exists $i_0 \in \{1, \ldots, r\}$ such that $\Lambda_{i_0} \leqslant^+ \Lambda_{\Omega \setminus A_0}$. Since $\Lambda_0 \leqslant^+ \Lambda_i$ for $1 \leq i \leq r$, we conclude that $\Lambda_0 \leqslant^+ \Lambda_{\Omega \setminus A_0}$ because the binary relation $\leqslant^+$ is a partial order. Now we have that $A_0 \in \Lambda_0 \leqslant^+ \Lambda_{\Omega \setminus A_0}$. Therefore, there exists $A_0' \in \Lambda_{\Omega \setminus A_0}$ such that $A_0' \subseteq A_0$, which is a contradiction because $\Lambda_{\Omega \setminus A_0} = \{\{x\} : x \in \Omega \setminus A_0\}$. This completes the proof of the theorem. □

Theorem 3.2. *Let $\Sigma \subseteq \mathrm{Clutt}(\Omega)$ be such that $\{X\} \in \Sigma$ for all non-empty subsets X of Ω. If Λ is a non-trivial clutter on Ω, then the set $\Sigma^+_\ell(\Lambda)$ of lower Σ-completions of Λ with respect to the order $\leqslant^+$ is non-empty, and*

$$\Lambda = \Lambda_1 \sqcup^+ \cdots \sqcup^+ \Lambda_r ,$$

where $\Lambda_1, \ldots, \Lambda_r$ are the maximal elements of the poset $\big(\Sigma^+_\ell(\Lambda), \leqslant^+\big)$. In particular, the following statements hold:

(a) A non-trivial clutter Λ on Ω belongs to the family Σ if and only if the poset $\big(\Sigma^+_\ell(\Lambda), \leqslant^+\big)$ has a unique maximal element.
(b) If Λ, Λ' are non-trivial clutters on Ω, then $\Lambda = \Lambda'$ if and only if the posets $\big(\Sigma^+_\ell(\Lambda), \leqslant^+\big)$ and $\big(\Sigma^+_\ell(\Lambda'), \leqslant^+\big)$ have the same maximal elements.

Proof. Since Λ is not trivial, it is clear that for every $A \in \Lambda$ we have $\{A\} \leqslant^+ \Lambda$. By assumption $\{A\} \in \Sigma$ if A is non-empty, hence we conclude that $\{A\} \in \Sigma^+_\ell(\Lambda)$ and thus $\Sigma^+_\ell(\Lambda)$ is non-empty.

Let $\Lambda_1, \ldots, \Lambda_r$ be the maximal elements of the poset $\big(\Sigma^+_\ell(\Lambda), \leqslant^+\big)$ and $\Lambda_0 = \Lambda_1 \sqcup^+ \cdots \sqcup^+ \Lambda_r$. The proof of the theorem will be completed by showing the equality $\Lambda = \Lambda_0$ (observe that statements (a) and (b) follow from this equality). Since $\Lambda_i \in \Sigma^+_\ell(\Lambda)$, the inequality $\Lambda_i \leqslant^+ \Lambda$ holds and so, from Lemma 2.5 we conclude that $\Lambda_0 \leqslant^+ \Lambda$. Therefore, it only remains to show that $\Lambda \leqslant^+ \Lambda_0$; that is, we must prove that if $A \in \Lambda$, then there exists $A_0 \in \Lambda_0$ such that $A_0 \subseteq A$.

Let $A \in \Lambda$. Since $\{A\} \in \Sigma^+_\ell(\Lambda)$, there exists $i_0 \in \{1, \ldots, r\}$ such that $\{A\} \leqslant^+ \Lambda_{i_0}$. As $\Lambda_{i_0} \leqslant^+ \Lambda_0$ (see Lemma 2.5), it follows that $\{A\} \leqslant^+ \Lambda_0$ (because $\leqslant^+$ is a partial order). Hence there exists $A_0 \in \Lambda_0$ such that $A_0 \subseteq A$, as we wanted to prove. □

The relationship between Theorem 3.1 and Theorem 3.2 is stated in the following remark. Specifically, it is showed that one theorem can be obtained from the other by considering blockers, thus giving in particular an alternative proof of Theorem 3.2.

Remark 3.3. Let $\Sigma \subseteq \mathrm{Clutt}(\Omega)$ and let $\mathcal{E} \subseteq \mathrm{Clutt}(\Omega)$ be the family of clutters $\mathcal{E} = \{b(\Lambda') : \Lambda' \in \Sigma\}$. Observe that from Lemma 2.4 it follows that $\Sigma^+_\ell(\Lambda) = \{b(\widetilde{\Lambda}) : \widetilde{\Lambda} \in \mathcal{E}^+_u(b(\Lambda))\}$ and that $\Sigma^+_u(\Lambda) = \{b(\widetilde{\Lambda}) : \widetilde{\Lambda} \in \mathcal{E}^+_\ell(b(\Lambda))\}$; that is, $\Sigma^+_\ell(\Lambda)$ and $\Sigma^+_u(\Lambda)$ are the sets of blockers of the clutters in $\mathcal{E}^+_u(b(\Lambda))$ and $\mathcal{E}^+_\ell(b(\Lambda))$, respectively. In addition, on one hand it is clear that for a non-empty subset $X \subseteq \Omega$ we have that $b(\{X\}) = \Lambda_X$ and that $b(\Lambda_X) = \{X\}$; whereas, on the other hand, from Lemma 2.6 we get the equality $b(\Lambda_1) \sqcap^+ \cdots \sqcap^+ b(\Lambda_r) = b\big(\Lambda_1 \sqcup^+ \cdots \sqcup^+ \Lambda_r\big)$ and the equality $b(\Lambda_1) \sqcup^+ \cdots \sqcup^+ b(\Lambda_r) = b\big(\Lambda_1 \sqcap^+ \cdots \sqcap^+ \Lambda_r\big)$. Therefore, the statements of Theorem 3.1 and Theorem 3.2 are one the blocker of the other.

3.2. **Completions and decompositions with order $\leqslant^-$.** In this subsection we study the family of upper and lower Σ-completions of the clutter Λ with the partial order $\leqslant^-$; that is, the families of clutters $\Sigma^-_u(\Lambda)$ and $\Sigma^-_\ell(\Lambda)$. The existence of Σ-completions and Σ-decompositions for the upper case is stated in Theorem 3.4, whereas Theorem 3.5 deals with the lower case. The proofs we present are in the same spirit as Remark 3.3 above, but one can also give direct proofs, similar to the proofs of Theorems 3.1 and 3.2.

Theorem 3.4. *Let $\Sigma \subseteq \mathrm{Clutt}(\Omega)$ be such that $\big(\Lambda_X\big)^c \in \Sigma$ for all non-empty subsets X of Ω. If Λ is a non-cotrivial clutter on Ω, then the set $\Sigma^-_u(\Lambda)$ of upper Σ-completions of Λ with respect to the order $\leqslant^-$ is non-empty, and*

$$\Lambda = \Lambda_1 \sqcap^- \cdots \sqcap^- \Lambda_r ,$$

where $\Lambda_1, \ldots, \Lambda_r$ *are the minimal elements of the poset* $\big(\Sigma_u^-(\Lambda), \leqslant^-\big)$*. In particular, the following statements hold:*

(a) A non-cotrivial clutter Λ *on* Ω *belongs to the family* Σ *if and only if the poset* $\big(\Sigma_u^-(\Lambda), \leqslant^-\big)$ *has a unique minimal element.*

(b) If Λ, Λ' *are non-cotrivial clutters on* Ω*, then* $\Lambda = \Lambda'$ *if and only if the posets* $\big(\Sigma_u^-(\Lambda), \leqslant^-\big)$ *and* $\big(\Sigma_u^-(\Lambda'), \leqslant^-\big)$ *have the same minimal elements.*

Proof. Let $\mathcal{E} = \{\Upsilon^c : \Upsilon \in \Sigma\} \subseteq \mathrm{Clutt}\,(\Omega)$. From our assumption we have that $\Lambda_X \in \mathcal{E}$ for all non-empty subsets X of Ω (because $\Lambda_X = \big(\big(\Lambda_X\big)^c\big)^c$ and $\big(\Lambda_X\big)^c \in \Sigma$). Moreover, since Λ is non-cotrivial, the clutter Λ^c is non-trivial. Therefore, by applying Theorem 3.1 to the clutter Λ^c it follows that $\mathcal{E}_u^+(\Lambda^c)$ is non-empty and $\Lambda^c = \Gamma_1 \sqcap^+ \cdots \sqcap^+ \Gamma_r$, where $\Gamma_1, \ldots, \Gamma_r$ are the minimal elements of the poset $\big(\mathcal{E}_u^+(\Lambda), \leqslant^+\big)$. On one hand, by Lemma 2.4 we get that $\Sigma_u^-(\Lambda) = \{(\widetilde{\Lambda})^c : \widetilde{\Lambda} \in \mathcal{E}_u^+(\Lambda^c)\}$ and that $\Gamma_1^c, \ldots, \Gamma_r^c$ are the minimal elements of the poset $\big(\Sigma_u^-(\Lambda), \leqslant^-\big)$. On the other hand, by Lemma 2.6, the equality $\big(\Gamma_1 \sqcap^+ \cdots \sqcap^+ \Gamma_r\big)^c = \Gamma_1^c \sqcap^- \cdots \sqcap^- \Gamma_r^c$ holds. Therefore we conclude that $\Sigma_u^-(\Lambda) \neq \emptyset$ and $\Lambda = \Lambda_1 \sqcap^- \cdots \sqcap^- \Lambda_r$ where $\Lambda_1, \ldots, \Lambda_r$ are the minimal elements of the poset $\big(\Sigma_u^-(\Lambda), \leqslant^-\big)$. $\square$

Theorem 3.5. *Let* $\Sigma \subseteq \mathrm{Clutt}\,(\Omega)$ *be such that* $\{X\} \in \Sigma$ *for all proper subsets* X *of* Ω*. If* Λ *is a non-cotrivial clutter on* Ω*, then the set* $\Sigma_\ell^-(\Lambda)$ *of lower* Σ*-completions of* Λ *with respect to the order* $\leqslant^-$ *is non-empty, and*

$$\Lambda = \Lambda_1 \sqcup^- \cdots \sqcup^- \Lambda_r\,,$$

where $\Lambda_1, \ldots, \Lambda_r$ *are the maximal elements of the poset* $\big(\Sigma_\ell^-(\Lambda), \leqslant^-\big)$*. In particular, the following statements hold:*

(a) A non-cotrivial clutter Λ *on* Ω *belongs to the family* Σ *if and only if the poset* $\big(\Sigma_\ell^-(\Lambda), \leqslant^-\big)$ *has a unique maximal element.*

(b) If Λ, Λ' *are non-cotrivial clutters on* Ω*, then* $\Lambda = \Lambda'$ *if and only if the posets* $\big(\Sigma_\ell^-(\Lambda), \leqslant^-\big)$ *and* $\big(\Sigma_\ell^-(\Lambda'), \leqslant^-\big)$ *have the same maximal elements.*

Proof. The proof is analogous to the previous one. Namely, we consider the family of clutters $\mathcal{E} = \{\Upsilon^c : \Upsilon \in \Sigma\} \subseteq \mathrm{Clutt}\,(\Omega)$ and we apply Theorem 3.4 to the clutter Λ^c. Since $\Sigma_\ell^-(\Lambda) = \{(\widetilde{\Lambda})^c : \widetilde{\Lambda} \in \mathcal{E}_\ell^+(\Lambda^c)\}$ and $\big(\Gamma_1 \sqcup^+ \cdots \sqcup^+ \Gamma_r\big)^c = \Gamma_1^c \sqcup^- \cdots \sqcup^- \Gamma_r^c$, the result follows. $\square$

The relationship between Theorems 3.1, 3.2, 3.4 and 3.5 is stated in the following remark and it is summarized in Table 3.

Remark 3.6. From the proofs of Theorems 3.4 and 3.5 we get that these theorems can be obtained from Theorems 3.1 and 3.2 by considering complementary families. Moreover, recall that in Remark 3.3 it was showed that the statements of Theorem 3.1 and Theorem 3.2 are one the blocker of the other. Thereby we conclude that Theorem 3.4 can be obtained from Theorem 3.5 by considering the complementary of the blockers of the complementary families of clutters.

4. Matroidal completions and decompositions of clutters

We now specialize the results of the previous section to matroidal clutters. That is, the collection of clutters Σ in Theorems 3.1, 3.2, 3.4 and 3.5 will be a collection of circuit, basis, or hyperplane clutters. So, given a clutter Λ, in principle we could complete it to a matroidal clutter in twelve possible ways, choosing one of circuits, bases or hyperplanes, one of the two orders $\leqslant^+$ or $\leqslant^-$, and one of the two sides, upper or lower. We shall see that all of these possibilities always give non-empty completions and that, except in two cases, they yield the corresponding

Theorem 3.1 upper completion, order $\leqslant^+$	$\overset{(\cdot)^c}{\longleftrightarrow}$	Theorem 3.4 upper completion, order $\leqslant^-$
$b(\cdot)\ \updownarrow$		$\updownarrow\ \Big(b\big((\cdot)^c\big)\Big)^c$
Theorem 3.2 lower completion, order $\leqslant^+$	$\overset{(\cdot)^c}{\longleftrightarrow}$	Theorem 3.5 lower completion, order $\leqslant^-$

Table 3. The relationship between Theorems 3.1, 3.2, 3.4 and 3.5.

decompositions. The problem of actually finding the completions involved in these decompositions is treated in Section 5.

Let us remark that there are plenty of families Σ of clutters related to matroids to which we could apply the theorems from Section 3. For instance, rather than just taking all circuit clutters (or all basis clutters, or all hyperplane clutters), we could consider only those arising from a particular class of matroids, such as representable, graphic, binary or transversal matroids. In that setting, we could start with a matroid outside the class and complete and decompose it using matroids of the class. This was done in [6] for circuit clutters of representable matroids, but only for upper completions with respect to the order $\leqslant^+$; we refer to that paper for results and examples.

From now on, $\mathrm{Mat}\,(\Omega)$ denotes the set whose elements are the matroids with ground set Ω. So each element $\mathcal{M} \in \mathrm{Mat}\,(\Omega)$ determines the three clutters $\mathcal{C}(\mathcal{M}), \mathcal{B}(\mathcal{M})$ and $\mathcal{H}(\mathcal{M})$.

4.1. **Circuit completions and decompositions of clutters.** For a clutter Λ on Ω we can consider the following four sets of *circuit completions*:

$$\mathcal{C}_u^+(\Lambda) = \{\mathcal{C}(\mathcal{M}) \,:\, \Lambda \leqslant^+ \mathcal{C}(\mathcal{M}) \text{ and } \mathcal{M} \in \mathrm{Mat}\,(\Omega)\},$$
$$\mathcal{C}_\ell^+(\Lambda) = \{\mathcal{C}(\mathcal{M}) \,:\, \mathcal{C}(\mathcal{M}) \leqslant^+ \Lambda \text{ and } \mathcal{M} \in \mathrm{Mat}\,(\Omega)\},$$
$$\mathcal{C}_u^-(\Lambda) = \{\mathcal{C}(\mathcal{M}) \,:\, \Lambda \leqslant^- \mathcal{C}(\mathcal{M}) \text{ and } \mathcal{M} \in \mathrm{Mat}\,(\Omega)\},$$
$$\mathcal{C}_\ell^-(\Lambda) = \{\mathcal{C}(\mathcal{M}) \,:\, \mathcal{C}(\mathcal{M}) \leqslant^- \Lambda \text{ and } \mathcal{M} \in \mathrm{Mat}\,(\Omega)\}.$$

Remark 4.1. Let us show that clutter inclusion is rather limited as a criterion to compare arbitrary clutters with circuit clutters; in other words, let us justify that clutter inclusion is not useful as a guide to seek circuit completions of clutters. Indeed, consider the clutter $\Lambda = \{\{1,2\},\{1,3\},\{2,3,4\}\}$ on $\Omega = \{1,2,3,4\}$. It is not the clutter of circuits of any matroid, as the first two sets do not satisfy the circuit elimination property. Moreover, there is no matroid $\mathcal{M}$ such that $\Lambda \subseteq \mathcal{C}(\mathcal{M})$; indeed, if $\{1,2\}$ and $\{1,3\}$ are circuits, then circuit elimination forces $\{2,3\}$ to be a circuit as well, and thus $\{2,3,4\}$ cannot be a circuit. Thus, just looking for matroids that have the elements of Λ as circuits does not give any circuit completion of Λ. However, there are several matroids $\mathcal{M}$ whose dependent sets include Λ, that is, there are matroids $\mathcal{M}$ such that $\Lambda \leqslant^+ \mathcal{C}(\mathcal{M})$, among which the ones with clutters of circuits $\Lambda_1 = \{\{1,2\},\{1,3\},\{2,3\}\}$ or $\Lambda_2 = \{\{1\},\{2,3,4\}\}$. In other words, we get that $\Lambda_1, \Lambda_2 \in \mathcal{C}_u^+(\Lambda)$. So, there are several such upper circuit completions of the clutter Λ.

Below we show that the four circuit completions of Λ are non-empty, except for one of the completions of $\{\emptyset\}$. First, $\{\}$ is a circuit clutter, so it lies in its four circuit completions. In general, if $\Lambda = \{A_1, \ldots, A_k\}$ is a clutter on an n-element set and $s \geq 0$ and $S \leq n$ are the minimum and the maximum of the cardinalities of the sets A_i, respectively, then

$$\mathcal{C}(\mathcal{U}_{r,n}) \in \mathcal{C}_u^+(\Lambda) \text{ if } 0 \leq r \leq s-1;$$
$$\mathcal{C}(\mathcal{U}_{r,n}) \in \mathcal{C}_\ell^+(\Lambda) \text{ if } n-1 \leq r \leq n;$$
$$\mathcal{C}(\mathcal{U}_{r,n}) \in \mathcal{C}_u^-(\Lambda) \text{ if } S-1 \leq r \leq n-1;$$
$$\mathcal{C}(\mathcal{U}_{r,n}) \in \mathcal{C}_\ell^-(\Lambda) \text{ if } r = n.$$

Thus, all circuit completions are non-empty, except $\mathcal{C}_u^+(\{\emptyset\})$. Admittedly, these completions do not tell us much about the clutter Λ itself, the purpose here being just to show that completions exist.

Example 4.2. As for a concrete example that in general uniform matroids do not give the "closest" circuit completions of a clutter, consider $\Lambda = \{\{1,2,3\},\{1,2,4\},\{1,5\},\{4,5\}\}$ on $\Omega = \{1,2,3,4,5\}$. Then, $\Lambda \leqslant^+ \mathcal{C}(\mathcal{U}_{r,5})$ if and only if $r = 0, 1$. However we have that the clutter $\Lambda' = \{\{1,4\},\{1,5\},\{4,5\},\{1,2,3\},\{2,3,4\},\{2,3,5\}\}$ is a circuit clutter such that $\Lambda \leqslant^+ \Lambda' \leqslant^+ \mathcal{C}(\mathcal{U}_{1,5}) \leqslant^+ \mathcal{C}(\mathcal{U}_{0,5})$. Therefore, $\Lambda' \in \mathcal{C}_u^+(\Lambda)$ and it is a circuit completion of the clutter Λ closer than the circuit completions obtained from uniform matroids.

However, the existence of non-empty circuit completions does not guarantee that the corresponding decompositions exist. In fact there are only three different circuit decompositions of a clutter. These decompositions are presented in Theorem 4.3 and they correspond to the non-empty circuit completions $\mathcal{C}_u^+(\Lambda)$, $\mathcal{C}_\ell^+(\Lambda)$ and $\mathcal{C}_\ell^-(\Lambda)$. Moreover, in Remark 4.4 we point at examples showing that the non-empty circuit completions $\mathcal{C}_u^-(\Lambda)$ may not provide a circuit decomposition of Λ. In the theorem below, we treat the general case of Λ being non-trivial and non-cotrivial, since it avoids some wordiness and, if needed, the interested reader can easily work out these excluded cases.

Theorem 4.3. *Let Λ be a non-trivial and non-cotrivial clutter on Ω. The following statements hold:*

(1) $\Lambda = \mathcal{C}(\mathcal{M}_{1,1}) \sqcap^+ \cdots \sqcap^+ \mathcal{C}(\mathcal{M}_{1,r_1})$, where $\mathcal{C}(\mathcal{M}_{1,1}), \ldots, \mathcal{C}(\mathcal{M}_{1,r_1})$ are the minimal elements of the poset $\big(\mathcal{C}_u^+(\Lambda), \leqslant^+\big)$.

(2) $\Lambda = \mathcal{C}(\mathcal{M}_{2,1}) \sqcup^+ \cdots \sqcup^+ \mathcal{C}(\mathcal{M}_{2,r_2})$, where $\mathcal{C}(\mathcal{M}_{2,1}), \ldots, \mathcal{C}(\mathcal{M}_{2,r_2})$ are the maximal elements of the poset $\big(\mathcal{C}_\ell^+(\Lambda), \leqslant^+\big)$.

(3) $\Lambda = \mathcal{C}(\mathcal{M}_{3,1}) \sqcup^- \cdots \sqcup^- \mathcal{C}(\mathcal{M}_{3,r_3})$, where $\mathcal{C}(\mathcal{M}_{3,1}), \ldots, \mathcal{C}(\mathcal{M}_{3,r_3})$ are the maximal elements of the poset $\big(\mathcal{C}_\ell^-(\Lambda), \leqslant^-\big)$.

In particular, the clutter Λ is a circuit clutter if and only if $r_{i_0} = 1$ for some $i_0 \in \{1,2,3\}$, if and only if $r_i = 1$ for all $i \in \{1,2,3\}$.

Proof. Let $n = |\Omega|$. We apply Theorems 3.1, 3.2 and 3.5 to the case where Σ is the collection of all circuit clutters on Ω.

To apply the first two theorems, it is enough to show that the clutters Λ_X and $\{X\}$ are circuit clutters on Ω for all non-empty subsets $X \subseteq \Omega$. If $X = \Omega$ then $\Lambda_\Omega = \mathcal{C}(\mathcal{U}_{1,n})$, while $\{\Omega\} = \mathcal{C}(\mathcal{U}_{n-1,n})$. For $\emptyset \neq X \subsetneq \Omega$, we show using partition matroids that Λ_X and $\{X\}$ are circuit clutters. Namely, if $|X| = r > 0$ then, the clutter $\Lambda_X = \{\{x\} : x \in X\}$ is the set of circuits of the partition matroid $\Pi(X, \Omega \setminus X; 0, n-r)$ and the clutter $\{X\}$ is the set of circuits of the partition matroid $\Pi(X, \Omega \setminus X; r-1, n-r)$. This gives statements 1 and 2.

As for applying Theorem 3.5, we need in principle that $\{X\}$ is a circuit clutter for all subsets $\emptyset \subseteq X \subsetneq \Omega$, which is not the case for $X = \emptyset$ since $\{\emptyset\}$ is not a circuit clutter. However, an analysis of the proof of Theorem 3.5 shows that the hypothesis of this clutter $\{\emptyset\}$ being in the collection Σ is only needed when dealing with completions of the clutter $\Lambda = \{\emptyset\}$, which is excluded here. Thus, statement 3 follows. □

Remark 4.4. Observe that if $\Sigma \subseteq \mathrm{Clutt}\,(\Omega)$ is the collection of all circuit clutters on Ω, then the condition "$\big(\Lambda_X\big)^c \in \Sigma$ *for all non-empty subsets* X *of* Ω" from Theorem 3.4 is not satisfied (for instance, if $X = \{1, 2\} \subseteq \{1, 2, 3, \ldots, n\} = \Omega$, then $\big(\Lambda_X\big)^c = \{\{2, 3, \ldots, n\}, \{1, 3, \ldots, n\}\}$ is not the clutter of circuits of a matroid $\mathcal{M}$ with ground set Ω). Therefore, we cannot apply Theorem 3.4 in order to obtain a decomposition of Λ with the minimal elements of the poset $\big(\Sigma_u^-(\Lambda), \leqslant^-\big) = \big(\mathcal{C}_u^-(\Lambda), \leqslant^-\big)$. Actually there is no statement analogous to those in Theorem 4.3 for the set of circuit completions $\mathcal{C}_u^-(\Lambda)$. Specifically, we have that:

(1) There are clutters Λ such that the poset $\big(\mathcal{C}_u^-(\Lambda), \leqslant^-\big)$ has a unique minimal element, but Λ is not a circuit clutter (see Example 4.5).
(2) There are clutters Λ such that the poset $\big(\mathcal{C}_u^-(\Lambda), \leqslant^-\big)$ has $r \geq 2$ minimal elements $\mathcal{C}(\mathcal{M}_1), \ldots, \mathcal{C}(\mathcal{M}_r)$, but $\Lambda \neq \mathcal{C}(\mathcal{M}_1) \sqcap^- \cdots \sqcap^- \mathcal{C}(\mathcal{M}_r)$ (see Example 4.6).

Example 4.5. Let us consider the clutter $\Lambda = \{\{1, 2\}, \{1, 3\}\}$ on the set $\Omega = \{1, 2, 3\}$. It is not hard to check that $\mathcal{C}_u^-(\Lambda) = \{\mathcal{C}(\mathcal{U}_{1,3}), \mathcal{C}(\mathcal{U}_{2,3})\}$. Since $\mathcal{C}(\mathcal{U}_{1,3}) \leqslant^- \mathcal{C}(\mathcal{U}_{2,3})$, the poset $\big(\mathcal{C}_u^-(\Lambda), \leqslant^-\big)$ has a unique minimal element $\mathcal{C}(\mathcal{U}_{1,3})$. However, Λ is not a circuit clutter.

Example 4.6. Let Λ be the clutter $\Lambda = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$ on the set $\Omega = \{1, 2, 3, 4\}$. It is not hard to check that $\mathcal{C}_u^-(\Lambda) = \{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5\}$, where

$$\begin{aligned}
\Lambda_1 &= \{\{1, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\},\\
\Lambda_2 &= \{\{1, 2\}, \{1, 3, 4\}, \{2, 3, 4\}\},\\
\Lambda_3 &= \{\{1, 3\}, \{1, 2, 4\}, \{2, 3, 4\}\},\\
\Lambda_4 &= \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\},\\
\Lambda_5 &= \{\{1, 2, 3, 4\}\}.
\end{aligned}$$

In this case, the poset $\big(\mathcal{C}_u^-(\Lambda), \leqslant^-\big)$ has three minimal elements, namely $\Lambda_1, \Lambda_2, \Lambda_3$. However, $\Lambda_1 \sqcap^- \Lambda_2 \sqcap^- \Lambda_3 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\} \neq \Lambda$.

4.2. **Basis completions and decompositions of clutters.** Next let us consider the sets of *basis completions* of a clutter Λ on a finite set Ω defined as follows:

$$\begin{aligned}
\mathcal{B}_u^+(\Lambda) &= \{\mathcal{B}(\mathcal{M}) \,:\, \Lambda \leqslant^+ \mathcal{B}(\mathcal{M}) \text{ and } \mathcal{M} \in \mathrm{Mat}\,(\Omega)\},\\
\mathcal{B}_\ell^+(\Lambda) &= \{\mathcal{B}(\mathcal{M}) \,:\, \mathcal{B}(\mathcal{M}) \leqslant^+ \Lambda \text{ and } \mathcal{M} \in \mathrm{Mat}\,(\Omega)\},\\
\mathcal{B}_u^-(\Lambda) &= \{\mathcal{B}(\mathcal{M}) \,:\, \Lambda \leqslant^- \mathcal{B}(\mathcal{M}) \text{ and } \mathcal{M} \in \mathrm{Mat}\,(\Omega)\},\\
\mathcal{B}_\ell^-(\Lambda) &= \{\mathcal{B}(\mathcal{M}) \,:\, \mathcal{B}(\mathcal{M}) \leqslant^- \Lambda \text{ and } \mathcal{M} \in \mathrm{Mat}\,(\Omega)\}.
\end{aligned}$$

As with circuit completions, basis clutters of uniform matroids show that basis completions exist in almost all cases. Note that $\{\}$ is not a basis clutter and it lies below all clutters in both orders $\leqslant^+$ and $\leqslant^-$; thus, it does not have lower basis completions and it is trivially upper completed by all basis clutters. In general, if $\Lambda = \{A_1, \ldots, A_k\}$ is a clutter on an n-element set and $s \geq 0$ and $S \leq n$ are the minimum and the maximum of the cardinalities of the sets A_i, respectively, then

$$\begin{aligned}
&\mathcal{B}(\mathcal{U}_{r,n}) \in \mathcal{B}_u^+(\Lambda) \text{ if } 0 \leq r \leq s;\\
&\mathcal{B}(\mathcal{U}_{r,n}) \in \mathcal{B}_\ell^+(\Lambda) \text{ if } r = n;\\
&\mathcal{B}(\mathcal{U}_{r,n}) \in \mathcal{B}_u^-(\Lambda) \text{ if } S \leq r \leq n;\\
&\mathcal{B}(\mathcal{U}_{r,n}) \in \mathcal{B}_\ell^-(\Lambda) \text{ if } r = 0.
\end{aligned}$$

Remark 4.7. Dress and Wenzel [2] did already show, in a different language, that every clutter has some lower basis completion with respect to the order $\leqslant^+$. Indeed, they give a construction, based on closure operators, that for each clutter Λ gives a matroid $\mathcal{M}$ such that $\mathcal{B}(\mathcal{M}) \leqslant^+ \Lambda$. As pointed out in [7, Ex. 2.13], this construction quite often gives $\mathcal{M} = \mathcal{U}_{n,n}$.

As for basis decompositions, they exist for all four possibilities.

Theorem 4.8. *Let Λ be a non-trivial and non-cotrivial clutter on Ω. The following statements hold:*

(1) $\Lambda = \mathcal{B}(\mathcal{M}_{1,1}) \sqcap^+ \cdots \sqcap^+ \mathcal{B}(\mathcal{M}_{1,r_1})$*, where* $\mathcal{B}(\mathcal{M}_{1,1}), \dots, \mathcal{B}(\mathcal{M}_{1,r_1})$ *are the minimal elements of the poset* $\big(\mathcal{B}_u^+(\Lambda), \leqslant^+\big)$.

(2) $\Lambda = \mathcal{B}(\mathcal{M}_{2,1}) \sqcup^+ \cdots \sqcup^+ \mathcal{B}(\mathcal{M}_{2,r_2})$*, where* $\mathcal{B}(\mathcal{M}_{2,1}), \dots, \mathcal{B}(\mathcal{M}_{2,r_2})$ *are the maximal elements of the poset* $\big(\mathcal{B}_\ell^+(\Lambda), \leqslant^+\big)$.

(3) $\Lambda = \mathcal{B}(\mathcal{M}_{3,1}) \sqcap^- \cdots \sqcap^- \mathcal{B}(\mathcal{M}_{3,r_3})$*, where* $\mathcal{B}(\mathcal{M}_{3,1}), \dots, \mathcal{B}(\mathcal{M}_{3,r_3})$ *are the minimal elements of the poset* $\big(\mathcal{B}_u^-(\Lambda), \leqslant^-\big)$.

(4) $\Lambda = \mathcal{B}(\mathcal{M}_{4,1}) \sqcup^- \cdots \sqcup^- \mathcal{B}(\mathcal{M}_{4,r_4})$*, where* $\mathcal{B}(\mathcal{M}_{4,1}), \dots, \mathcal{B}(\mathcal{M}_{4,r_4})$ *are the maximal elements of the poset* $\big(\mathcal{B}_\ell^-(\Lambda), \leqslant^-\big)$.

In particular, the clutter Λ is a basis clutter if and only if $r_{i_0} = 1$ for some $i_0 \in \{1,2,3,4\}$, if and only if $r_i = 1$ for all $i \in \{1,2,3,4\}$.

Proof. The proof of the four statements follows by applying Theorems 3.1, 3.2, 3.4 and 3.5 to the case where Σ is the collection of all basis clutters on Ω. It is enough to show that for all subsets $X \subseteq \Omega$ the clutters $\{X\}$, Λ_X and $\big(\Lambda_X\big)^c$ are basis clutters, except for $\Lambda_\emptyset$ and $(\Lambda_\emptyset)^c$. Indeed, if X is a subset of size $|X| = r \geq 0$, then the clutter $\{X\}$ is the clutter of bases of the partition matroid $\Pi(X, \Omega \setminus X; r, 0)$, whereas for $r > 0$ the clutter $\Lambda_X = \{\{x\} : x \in X\}$ is the clutter of bases of the partition matroid $\Pi(X, \Omega \setminus X; 1, 0)$, and the clutter $\big(\Lambda_X\big)^c = \{\Omega \setminus \{x\} : x \in X\}$ is the basis clutter of the partition matroid $\Pi(X, \Omega \setminus X; r-1, n-r)$. $\square$

4.3. **Hyperplane completions and decompositions of clutters.** To conclude this section, let us consider the following four sets of *hyperplane completions* of a clutter Λ on a finite set Ω:

$$\begin{aligned}
\mathcal{H}_u^+(\Lambda) &= \{\mathcal{H}(\mathcal{M}) : \Lambda \leqslant^+ \mathcal{H}(\mathcal{M}) \text{ and } \mathcal{M} \in \mathrm{Mat}\,(\Omega)\},\\
\mathcal{H}_\ell^+(\Lambda) &= \{\mathcal{H}(\mathcal{M}) : \mathcal{H}(\mathcal{M}) \leqslant^+ \Lambda \text{ and } \mathcal{M} \in \mathrm{Mat}\,(\Omega)\},\\
\mathcal{H}_u^-(\Lambda) &= \{\mathcal{H}(\mathcal{M}) : \Lambda \leqslant^- \mathcal{H}(\mathcal{M}) \text{ and } \mathcal{M} \in \mathrm{Mat}\,(\Omega)\},\\
\mathcal{H}_\ell^-(\Lambda) &= \{\mathcal{H}(\mathcal{M}) : \mathcal{H}(\mathcal{M}) \leqslant^- \Lambda \text{ and } \mathcal{M} \in \mathrm{Mat}\,(\Omega)\}.
\end{aligned}$$

By using uniform matroids we get that, except for one of the hyperplane completions of the clutter $\{\Omega\}$, all the sets of hyperplane completions are non-empty. Indeed, since $\Lambda = \{\}$ is a hyperplane clutter, $\{\}$ lies in its four hyperplane completions. Moreover, in general, if $\Lambda = \{A_1, \dots, A_k\}$ is a clutter on an n-element set and $s \geq 0$ and $S \leq n$ are the minimum and the maximum of the cardinalities of the sets A_i, respectively, then

$$\begin{aligned}
\mathcal{H}(\mathcal{U}_{r,n}) &\in \mathcal{H}_u^+(\Lambda) \text{ if } 0 \leq r \leq s+1;\\
\mathcal{H}(\mathcal{U}_{r,n}) &\in \mathcal{H}_\ell^+(\Lambda) \text{ if } r = 0;\\
\mathcal{H}(\mathcal{U}_{r,n}) &\in \mathcal{H}_u^-(\Lambda) \text{ if } S+1 \leq r \leq n;\\
\mathcal{H}(\mathcal{U}_{r,n}) &\in \mathcal{H}_\ell^-(\Lambda) \text{ if } 0 \leq r \leq 1.
\end{aligned}$$

Therefore all hyperplane completions are non-empty, except $\mathcal{H}_u^-(\{\Omega\})$. However, if Λ is a non-trivial and non-cotrivial clutter on Ω, then only three of these four non-empty sets of hyperplane completions provide a hyperplane decomposition of Λ.

Namely, there are decompositions associated to the sets of hyperplane completions $\mathcal{H}_\ell^+(\Lambda)$, $\mathcal{H}_u^-(\Lambda)$ and $\mathcal{H}_\ell^-(\Lambda)$ (see Theorem 4.9), but there is no decomposition of Λ associated to the set of hyperplane completions $\mathcal{H}_u^+(\Lambda)$ (see Remark 4.10).

Theorem 4.9. *Let Λ be a non-trivial and non-cotrivial clutter on Ω. The following statements hold:*

(1) $\Lambda = \mathcal{H}(\mathcal{M}_{1,1}) \sqcup^+ \cdots \sqcup^+ \mathcal{H}(\mathcal{M}_{1,r_1})$, *where $\mathcal{H}(\mathcal{M}_{1,1}), \ldots, \mathcal{H}(\mathcal{M}_{1,r_1})$ are the maximal elements of the poset $\left(\mathcal{H}_\ell^+(\Lambda), \leqslant^+\right)$.*
(2) $\Lambda = \mathcal{H}(\mathcal{M}_{2,1}) \sqcap^- \cdots \sqcap^- \mathcal{H}(\mathcal{M}_{2,r_2})$, *where $\mathcal{H}(\mathcal{M}_{2,1}), \ldots, \mathcal{H}(\mathcal{M}_{2,r_2})$ are the minimal elements of the poset $\left(\mathcal{H}_u^-(\Lambda), \leqslant^-\right)$.*
(3) $\Lambda = \mathcal{H}(\mathcal{M}_{3,1}) \sqcup^- \cdots \sqcup^- \mathcal{H}(\mathcal{M}_{3,r_3})$, *where $\mathcal{H}(\mathcal{M}_{3,1}), \ldots, \mathcal{H}(\mathcal{M}_{3,r_3})$ are the maximal elements of the poset $\left(\mathcal{H}_\ell^-(\Lambda), \leqslant^-\right)$.*

In particular, the clutter Λ is a hyperplane clutter if and only if $r_{i_0} = 1$ for some $i_0 \in \{1,2,3\}$, if and only if $r_i = 1$ for all $i \in \{1,2,3\}$.

Proof. Recall that $\mathcal{H}(\mathcal{M}) = \left(\mathcal{C}(\mathcal{M}^*)\right)^c$ (see Table 1). Therefore the three statements in this theorem follow by applying Theorem 4.3 to the non-trivial and non-cotrivial clutter Λ^c and, after this, by applying the complementary operator and Lemmas 2.4 and 2.6. □

Remark 4.10. Observe that if $\Sigma \subseteq \operatorname{Clutt}(\Omega)$ is the collection of all hyperplane clutters on Ω, then the condition "$\Lambda_X \in \Sigma$ *for all non-empty subsets X of Ω*" from Theorem 3.1 is not satisfied (indeed, if $X \neq \Omega$ then Λ_X is not the clutter of the hyperplanes of any matroid with ground set Ω). Therefore, we cannot apply Theorem 3.1 in order to obtain a decomposition with the minimal elements of the poset $\left(\Sigma_u^+(\Lambda), \leqslant^+\right) = \left(\mathcal{H}_u^+(\Lambda), \leqslant^+\right)$. In fact, for the set of hyperplane completions $\mathcal{H}_u^+(\Lambda)$ there does not exist a statement analogous to those in Theorem 4.9. More concretely:

(1) There are clutters Λ such that the poset $\left(\mathcal{H}_u^+(\Lambda), \leqslant^+\right)$ has a unique minimal element, but Λ is not a hyperplane clutter.
(2) There are clutters Λ such that the poset $\left(\mathcal{H}_u^+(\Lambda), \leqslant^+\right)$ has $r \geq 2$ minimal elements $\mathcal{H}(\mathcal{M}_1), \ldots, \mathcal{H}(\mathcal{M}_r)$, but $\Lambda \neq \mathcal{H}(\mathcal{M}_1) \sqcap^+ \cdots \sqcap^+ \mathcal{H}(\mathcal{M}_r)$.

Concrete examples are given by the complementaries of the clutters in Examples 4.5 and 4.6.

5. Matroidal transformations of clutters

In the previous section we have seen that for a non-trivial, non-cotrivial clutter Λ and a choice of $*_1 \in \{+,-\}$ and of $*_2 \in \{u,\ell\}$, the sets $\mathcal{C}_{*_2}^{*_1}(\Lambda)$, $\mathcal{B}_{*2}^{*1}(\Lambda)$ and $\mathcal{H}_{*2}^{*1}(\Lambda)$ are non-empty and provide decompositions of Λ, except for $\mathcal{C}_u^-(\Lambda)$ and $\mathcal{H}_u^+(\Lambda)$. Namely, the decomposition is achieved by considering the minimal (maximal) matroidal clutters in the upper (lower) circuit, basis and hyperplane completions. If we do not need to be specific, we will just speak of *optimal completions*.

In this section we give algorithms to compute all these optimal completions of a clutter.

5.1. **Optimal basis and hyperplane completions.**

We first express optimal basis and hyperplane completions in terms of optimal circuit completions. Our results are gathered in the following two lemmas and involve the blocker and complementary operations. Both lemmas follow easily from Lemma 2.4 and matroid duality (Table 1).

Lemma 5.1. *Let Λ be a clutter on Ω. The following statements hold:*

(1) $\min\left(\mathcal{B}_u^+(\Lambda), \leqslant^+\right) = \left\{b(\Lambda') \,:\, \Lambda' \in \max\left(\mathcal{C}_\ell^+(b(\Lambda)), \leqslant^+\right)\right\}$.
(2) $\max\left(\mathcal{B}_\ell^+(\Lambda), \leqslant^+\right) = \left\{b(\Lambda') \,:\, \Lambda' \in \min\left(\mathcal{C}_u^+(b(\Lambda)), \leqslant^+\right)\right\}$.

(3) $\min\left(\mathcal{B}_u^-(\Lambda), \leqslant^-\right) = \{(b(\Lambda'))^c \,:\, \Lambda' \in \max\left(\mathcal{C}_\ell^+(b(\Lambda^c), \leqslant^+\right)\}$.
(4) $\max\left(\mathcal{B}_\ell^-(\Lambda), \leqslant^+\right) = \{(b(\Lambda'))^c \,:\, \Lambda' \in \min\left(\mathcal{C}_u^+(b(\Lambda^c), \leqslant^+\right)\}$.

Lemma 5.2. *Let Λ be a clutter on Ω. The following statements hold:*

(1) $\max\left(\mathcal{H}_\ell^+(\Lambda), \leqslant^+\right) = \left\{(\Lambda')^c \,:\, \Lambda' \in \max\left(\mathcal{C}_\ell^-(\Lambda^c), \leqslant^-\right)\right\}$.
(2) $\min\left(\mathcal{H}_u^-(\Lambda), \leqslant^-\right) = \left\{(\Lambda')^c \,:\, \Lambda' \in \min\left(\mathcal{C}_u^+(\Lambda^c), \leqslant^+\right)\right\}$.
(3) $\max\left(\mathcal{H}_\ell^-(\Lambda), \leqslant^-\right) = \left\{(\Lambda')^c \,:\, \Lambda' \in \max\left(\mathcal{C}_\ell^+(\Lambda^c), \leqslant^+\right)\right\}$.

Thus, from these lemmas we conclude that it is enough to know the optimal completions for the three kinds of circuit completions to determine all basis and hyperplane optimal completions; that is, knowing an algorithmic procedure to obtain the optimal circuit completions $\min\left(\mathcal{C}_u^+(\Lambda), \leqslant^+\right)$, $\max\left(\mathcal{C}_\ell^+(\Lambda), \leqslant^+\right)$ and $\max\left(\mathcal{C}_\ell^-(\Lambda), \leqslant^-\right)$ is theoretically enough to determine optimal basis and hyperplane completions.

Remark 5.3. Getting optimal hyperplane completions from optimal circuit completions is straightforward, as it only requires applying the complementary operation; however, to get optimal basis completions we need to compute blockers, and this in principle is not a trivial task. For instance, it is not known whether the problem of computing the blocker of a clutter is output-polynomial time, that is, polynomial in both the size of the input and the output (see [4] for more details and references). We do not know of an algorithm for obtaining optimal basis completions directly, avoiding the use of the blocker operator.

5.2. **Optimal circuit completions.** The remainder of this section is devoted to computing the optimal circuit completions $\min\left(\mathcal{C}_u^+(\Lambda), \leqslant^+\right)$, $\max\left(\mathcal{C}_\ell^+(\Lambda), \leqslant^+\right)$ and $\max\left(\mathcal{C}_\ell^-(\Lambda), \leqslant^-\right)$. Actually, the first case was already dealt with in [5]. We first recall it for completeness (Subsection 5.2.1), and then show how to apply the same spirit to the other two cases (Subsections 5.2.2 and 5.2.3).

Remark 5.4. We have not studied in detail the complexity of the algorithms we propose, although trials with rather small clutters suggest a behaviour far from polynomial. More generally, we do not know which is the actual complexity of the problem of finding the optimal circuit completions of a clutter for any of the three cases.

We repeatedly use the following characterization of circuit clutters, which follows easily from the circuit elimination property. For a clutter Λ on Ω and $B \subseteq \Omega$, define $I_\Lambda(B) = \bigcap_{A\in\Lambda, A\subseteq B} A$.

Lemma 5.5. *Let $\Lambda \neq \{\emptyset\}$ be a clutter on Ω. Then, Λ is a circuit clutter if and only if $I_\Lambda(A_1 \cup A_2) = \emptyset$ for all $A_1, A_2 \in \Lambda$ with $A_1 \neq A_2$.*

From now on we assume that $\Lambda \neq \{\emptyset\}$ to avoid degenerate cases in several definitions. In any case, the optimal circuit completions of $\{\emptyset\}$, whenever they exist, are straightforward to find.

5.2.1. *Minimal upper circuit completions with respect to $\leqslant^+$.* The main idea from [5] is to introduce three transformations on clutters such that if Λ is not a circuit clutter, then at least one of the transformations produces a clutter above Λ with respect to the order $\leqslant^+$. Then one shows that all minimal elements of $\mathcal{C}_u^+(\Lambda)$ can be obtained by successively applying the transformations, starting with Λ. These transformations were originally called $\mathcal{I}$-, $\mathcal{T}^{(1)}$- and $\mathcal{T}^{(2)}$-transformations, but we rename them to avoid confusion with independent sets in a matroid, and also to make notation more uniform throughout this section.

Let Λ be a clutter on Ω. For $A_1, A_2 \in \Lambda$ we define the α_1-transformation of Λ as follows:

$$\alpha_1(\Lambda; A_1, A_2) = \begin{cases} \min\big(\Lambda \cup \{A_1 \cap A_2\}\big) & \text{if } I_\Lambda(A_1 \cup A_2) \neq \emptyset, \\ \Lambda & \text{otherwise;} \end{cases}$$

whereas the α_2- and α_3-transformations of the clutter Λ are defined as the clutters:

$$\alpha_2(\Lambda) = \min\big(\Lambda \cup \{(A_1 \cup A_2) \setminus \{x\} : A_1, A_2 \in \Lambda, A_1 \neq A_2, x \in A_1 \cap A_2\}\big),$$
$$\alpha_3(\Lambda) = \min\big(\Lambda \cup \{(A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2) : A_1, A_2 \in \Lambda, A_1 \neq A_2\}\big).$$

It is easy to check using Lemma 5.5 that Λ is a circuit clutter if and only if $\alpha_1(\Lambda; A_1, A_2) = \Lambda$ for all $A_1, A_2 \in \Lambda$; and that Λ is a circuit clutter if and only if $\alpha_2(\Lambda) = \Lambda$, equivalently if and only if $\alpha_3(\Lambda) = \Lambda$. So that none of the three transformations actually modifies clutters that are already circuit clutters.

Clearly $\Lambda \leqslant^+ \alpha_1(\Lambda; A_1, A_2)$, $\Lambda \leqslant^+ \alpha_2(\Lambda)$ and $\Lambda \leqslant^+ \alpha_3(\Lambda)$. Since for a non-circuit clutter Λ these inequalities are strict, and the number of possible clutters on Ω is finite, if we start from Λ and repeatedly apply the transformations α_1, α_2, and α_3 in any order, we will eventually obtain a circuit clutter, and thus a completion from $\mathcal{C}_u^+(\Lambda)$. Let us say that such a completion is an *extremal α-transformation* of Λ and let us denote by $\mathcal{T}_\alpha(\Lambda)$ the set of all such extremal α-transformations. In other words, $\mathcal{T}_\alpha(\Lambda)$ is the set of circuit clutters obtained in the following way: start with Λ and generate all of its α_1-, α_2- and α_3-transformations; for all the clutters among these, generate again all their α_1-, α_2- and α_3-transformations; repeat this process as many times as necessary until no more new clutters appear. All resulting clutters will be upper circuit completions of Λ and, in general, not all such completions will appear in the process, that is, the inclusion $\mathcal{T}_\alpha(\Lambda) \subseteq \mathcal{C}_u^+(\Lambda)$ is strict. However, it turns out that $\mathcal{T}_\alpha(\Lambda)$ contains all minimal circuit upper completions of Λ, which are the ones needed in the decomposition of Theorem 4.3. The following result was proved in [5, Thm. 13].

Theorem 5.6. *Let $\Lambda \neq \{\emptyset\}$ be a clutter on Ω. The minimal circuit upper completions of Λ with respect to $\leqslant^+$ are its minimal extremal α-transformations; that is,*

$$\min\big(\mathcal{C}_u^+(\Lambda), \leqslant^+\big) = \min\big(\mathcal{T}_\alpha(\Lambda), \leqslant^+\big).$$

We refer to [5] for examples, including some that show that all three transformations are necessary.

5.2.2. *Maximal lower circuit completions with respect to $\leqslant^+$.* We follow the same approach as in the previous subsection; in this case, one single transformation on clutters is enough.

Let A be an element of Λ such that there is $A' \in \Lambda$ with $A \neq A'$ and $I_\Lambda(A \cup A') \neq \emptyset$. In such a case, we define

$$\beta(\Lambda; A) = \min\big((\Lambda \setminus \{A\}) \cup \{A \cup \{x\} : x \in \Omega \setminus A\}\big).$$

Otherwise we set $\beta(\Lambda; A) = \Lambda$. Observe that $\beta(\Lambda; A) \leqslant^+ \Lambda$, with the inequality being strict if the first case of the definition applies and, in particular, if Λ is not a circuit clutter. Moreover, from the definition and by using Lemma 5.5 it is not hard to prove that the clutter Λ is a circuit clutter if and only if $\beta(\Lambda; A) = \Lambda$ for all $A \in \Lambda$.

Again by the finiteness of the number of clutters on Ω, we have an algorithmic procedure to obtain circuit lower completions with respect to the partial order $\leqslant^+$: starting from Λ, we repeatedly apply β-transformations until no more new clutters appear. We refer to the resulting clutters as *extremal β-transformations* of Λ, denoted $\mathcal{T}_\beta(\Lambda)$. They are circuit lower completions of Λ with respect to $\leqslant^+$; that is, $\mathcal{T}_\beta(\Lambda) \subseteq \mathcal{C}_\ell^+(\Lambda)$.

We have the analogous of Theorem 5.6 for β-transformations and maximal elements of $\mathcal{C}_\ell^+(\Lambda)$.

Theorem 5.7. *Let $\Lambda \neq \{\emptyset\}$ be a clutter on Ω. The maximal circuit lower completions of Λ with respect to $\leqslant^+$ are its maximal extremal β-transformations; that is,*

$$\max\left(\mathcal{C}_\ell^+(\Lambda), \leqslant^+\right) = \max\left(\mathcal{T}_\beta(\Lambda), \leqslant^+\right).$$

Proof. Assume that Λ is not a circuit clutter, otherwise there is nothing to prove. We show that if Λ_* is a maximal element of $\mathcal{C}_\ell^+(\Lambda)$, then there is a finite sequence of pairwise different clutters $\Lambda_0, \Lambda_1, \dots, \Lambda_k$ such that $\Lambda_* = \Lambda_k \leqslant^+ \Lambda_{k-1} \leqslant^+ \cdots \leqslant^+ \Lambda_0 = \Lambda$ and, for each i with $1 \leq i \leq k$, there is $A^{(i)} \in \Lambda_{i-1}$ such that $\Lambda_i = \beta(\Lambda_{i-1}; A^{(i)})$.

To do this, it is enough to prove that if Λ' is an element of $\mathcal{C}_\ell^+(\Lambda)$, then there is some clutter $\widetilde{\Lambda}$ (perhaps Λ' itself) that is a β-transformation of Λ and such that $\Lambda' \leqslant^+ \widetilde{\Lambda} \leqslant^+ \Lambda$.

As Λ is not a circuit clutter, from Lemma 5.5 there are $A_1 \neq A_2 \in \Lambda$ such that $I_\Lambda(A_1 \cup A_2) \neq \emptyset$. The relation $\Lambda' \leqslant^+ \Lambda$ means that for all $C \in \Lambda'$ there is some $A \in \Lambda$ such that $A \subseteq C$. If either A_1 or A_2 does not belong to Λ', then we can take $\widetilde{\Lambda}$ as the transformation $\beta(\Lambda; A_1)$ or $\beta(\Lambda; A_2)$, respectively. We conclude the proof by showing that the case $A_1, A_2 \in \Lambda'$ is impossible. Assume that this were the case and, in particular, that A_1 and A_2 belong to the clutter of circuits of some matroid.

Let us take $\xi \in I_\Lambda(A_1 \cup A_2)$; in particular, $\xi \in A_1 \cap A_2$. By the circuit-elimination property, there is $C \in \Lambda'$ such that $C \subseteq (A_1 \cup A_2) \setminus \{\xi\}$. As $\Lambda' \leqslant^+ \Lambda$, there is $A_3 \in \Lambda$ such that $A_3 \subseteq C$. But then $A_3 \subseteq A_1 \cup A_2$ and thus $I_\Lambda(A_1 \cup A_2) \subseteq A_3$, which is a contradiction since $\xi \notin A_3$. □

Example 5.8. Let $\Omega = \{1, 2, 3, 4, 5\}$ and $\Lambda = \{123, 124, 345\}$ (for ease of reading, in this section we omit the braces and the commas in writing subsets of Ω). The only pair A_1, A_2 with $I_\Lambda(A_1 \cup A_2) \neq \emptyset$ is $A_1 = 123$ and $A_2 = 124$. We have

$$\beta(\Lambda; 123) = \{124, 345, 1235\},$$
$$\beta(\Lambda; 124) = \{123, 345, 1245\}.$$

They are both circuit clutters, so they are the maximal lower completions of Λ with respect to the order $\leqslant^+$. Observe that now, from Theorem 4.3 we have the following circuit decomposition of the non-circuit clutter Λ

$$\{123, 124, 345\} = \{124, 345, 1235\} \sqcup^+ \{123, 345, 1245\}.$$

5.2.3. *Maximal lower circuit completions with respect to $\leqslant^-$.* In this case it is also enough to consider one transformation, but it needs to be applied in two situations.

Let Λ be a clutter and let $A \in \Lambda$. If A satisfies at least one of the following two conditions:

(a) there is $A' \in \Lambda$ with $A \neq A'$ and $I_\Lambda(A \cup A') \neq \emptyset$,
(b) there are $A', A'' \in \Lambda$ with $A \cap (A' \setminus A'') \neq \emptyset$, $A \cap (A'' \setminus A') \neq \emptyset$ and $I_\Lambda(A' \cup A'') \neq \emptyset$,

then we define the γ-transformation of Λ as the clutter

$$\gamma(\Lambda; A) = \max\left(\Lambda \setminus \{A\} \cup \{A \setminus \{x\} : x \in A\}\right);$$

otherwise we set $\gamma(\Lambda; A) = \Lambda$. Clearly $\gamma(\Lambda; A) \leqslant^- \Lambda$ and moreover, from the definition and by using Lemma 5.5, the clutter Λ is a circuit clutter if and only if $\gamma(\Lambda; A) = \Lambda$ for all $A \in \Lambda$.

Now, as in the two previous cases, γ-transformations give an algorithmic procedure to obtain circuit lower completions with respect to $\leqslant^-$: starting from Λ, we repeatedly apply γ-transformations until no new clutters arise. The clutters obtained in this way are the *extremal γ-transformations* of Λ, denoted $\mathcal{T}_\gamma(\Lambda)$. They clearly are circuit lower completions of Λ with respect to $\leqslant^-$; that is, $\mathcal{T}_\gamma(\Lambda) \subseteq \mathcal{C}_\ell^-(\Lambda)$.

As before, the extremal γ-transformations of Λ are enough to obtain all maximal completions in $\mathcal{C}_\ell^-(\Lambda)$. Namely we have the following theorem which is the analogous of Theorems 5.6 and 5.7.

Theorem 5.9. *Let $\Lambda \neq \{\emptyset\}$ be a clutter on Ω. The maximal circuit lower completions of Λ with respect to $\leqslant^-$ are its maximal extremal γ-transformations; that is,*

$$\max\left(\mathcal{C}_\ell^-(\Lambda), \leqslant^-\right) = \max\left(\mathcal{T}_\gamma(\Lambda), \leqslant^-\right).$$

Proof. Assume Λ is not a circuit clutter, as otherwise there is nothing to prove. Let Λ_* be a maximal element of $\mathcal{C}_\ell^-(\Lambda)$. We show that there is a finite sequence of pairwise different clutters $\Lambda_0, \Lambda_1, \ldots, \Lambda_k$ such that $\Lambda_* = \Lambda_k \leqslant^- \Lambda_{k-1} \leqslant^- \cdots \leqslant^- \Lambda_0 = \Lambda$ and for each i with $1 \leq i \leq k$ there is $A^{(i)} \in \Lambda_{i-1}$ such that $\Lambda_i = \gamma(\Lambda_{i-1}; A^{(i)})$. In order to do this, it is enough to prove that for any $\Lambda' \in \mathcal{C}_\ell^-(\Lambda)$ there is some clutter $\widetilde{\Lambda}$ (perhaps Λ' itself) that is a γ-transformation of Λ and such that $\Lambda' \leqslant^- \widetilde{\Lambda} \leqslant^- \Lambda$.

Let $\Lambda' = \mathcal{C}(\mathcal{M})$. For each circuit $C \in \mathcal{C}(\mathcal{M})$, let $C_\Lambda = \{A \in \Lambda : C \subseteq A\}$; this set is non-empty as $\mathcal{C}(\mathcal{M}) = \Lambda' \leqslant^- \Lambda$.

As Λ is not a circuit clutter, from Lemma 5.5 there are $A_1, A_2 \in \Lambda$, distinct, such that $I_\Lambda(A_1 \cup A_2) \neq \emptyset$. If for every $C \in \mathcal{C}(\mathcal{M})$ we have $C_\Lambda \neq \{A_1\}$ then we take as $\widetilde{\Lambda}$ the transformation $\gamma(\Lambda; A_1)$ of Λ. Similarly, if for every $C \in \mathcal{C}(\mathcal{M})$ we have $C_\Lambda \neq \{A_2\}$, we take as $\widetilde{\Lambda}$ the transformation $\gamma(\Lambda; A_2)$ of Λ.

We can thus assume that there are $C_1, C_2 \in \mathcal{C}(\mathcal{M})$ such that $(C_1)_\Lambda = \{A_1\}$ and $(C_2)_\Lambda = \{A_2\}$. In fact, we claim that we can choose C_1 and C_2 such that $C_1 = A_1$ and $C_2 = A_2$. Indeed, otherwise we could take again $\widetilde{\Lambda}$ to be $\gamma(\Lambda; A_1)$ or $\gamma(\Lambda; A_2)$.

Now let $\xi \in I_\Lambda(A_1 \cup A_2)$. By the circuit elimination property, there is $C_3 \in \mathcal{C}(\mathcal{M})$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{\xi\}$. As $\mathcal{C}(\mathcal{M}) \leqslant^- \Lambda$, there is $A_3 \in \Lambda$ with $C_3 \subseteq A_3$. Observe that $C_3 \neq A_3$ because otherwise we would have $A_3 \subseteq C_1 \cup C_2 = A_1 \cup A_2$ that implies $I_\Lambda(A_1 \cup A_2) \subseteq A_3$, but ξ does not belong to $C_3 = A_3$. Note also that for any other $C \in \mathcal{C}(\mathcal{M})$ such that $C_\Lambda = \{A_3\}$ it is also the case that $C \neq A_3$, since circuits are incomparable.

In this case we claim that we can take as $\widetilde{\Lambda}$ the transformation $\gamma(\Lambda; A_3)$ of Λ. By the remarks above it is clear that this clutter is above $\mathcal{C}(\mathcal{M})$ with respect to $\leqslant^-$. It only remains to check that $A_3 \cap (A_1 \setminus A_2) \neq \emptyset$ and $A_3 \cap (A_2 \setminus A_1) \neq \emptyset$. For the first claim it is enough to show that C_3 contains some element of A_1 that does not belong to A_2, and this is clear since otherwise we would have the circuit inclusion $C_3 \subseteq A_1 \cap A_2 \subseteq A_1 = C_1$. An analogous argument gives $A_3 \cap (A_2 \setminus A_1) \neq \emptyset$, and this completes the proof. □

Example 5.10. On the finite set $\Omega = \{1, 2, 3, 4\}$ let $\Lambda = \{123, 124\}$, which is not a circuit clutter. The γ-transformations yield

$$\Lambda_1 = \gamma(\Lambda; 123) = \{124, 13, 23\},$$
$$\Lambda_2 = \gamma(\Lambda; 124) = \{123, 14, 24\}.$$

As Λ_2 is the result of permuting 3 and 4 in Λ_1, we focus on Λ_1 only. We have $I_{\Lambda_1}(13 \cup 23) = 3$, $124 \cap (13 \setminus 23) \neq \emptyset$ and $124 \cap (23 \setminus 13) \neq \emptyset$, so we can apply the

γ-transformation with all three sets. This gives

$$\Lambda_3 = \gamma(\Lambda_1; 13) = \{124, 23\},$$
$$\Lambda_4 = \gamma(\Lambda_1; 23) = \{124, 13\},$$
$$\Lambda_5 = \gamma(\Lambda_1; 124) = \{13, 23, 12, 14, 24\}.$$

The clutters Λ_3 and Λ_4 are equal up to the permutation $1 \leftrightarrow 2$, so we just treat one of them. The γ-transformations of Λ_3 are

$$\Lambda_6 = \gamma(\Lambda_3; 124) = \{23, 12, 14, 24\},$$
$$\Lambda_7 = \gamma(\Lambda_3; 23) = \{124, 3\}.$$

The clutter Λ_7 is already a circuit clutter. Now, all elements of the clutters Λ_5 and Λ_6 have size 2; thus, by applying γ-transformations we will eventually reach circuit clutters whose elements have sizes 1 or 2; also, none of these circuits can be 34. It is easy to check that the maximal such clutters are $\Lambda_8 = \{12, 13, 23, 4\}, \Lambda_9 = \{12, 14, 24, 3\}, \Lambda_{10} = \{13, 24\}$ and $\Lambda_{11} = \{14, 23\}$. Thus, the clutters in $\max\left(\mathcal{C}_\ell^-(\Lambda), \leqslant^-\right)$ are the maximal ones among $\Lambda_7, \ldots, \Lambda_{11}$ and the ones obtained from them by applying the permutations $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$; notice that this only yields one new clutter $\{123, 4\}$. Hence, the maximal circuit completions of $\{123, 124\}$ with respect to the order $\leqslant^-$ are $\{123, 4\}, \{124, 3\}, \{13, 24\}$ and $\{14, 23\}$. Observe that now, from Theorem 4.3, we have the decomposition

$$\{123, 124\} = \{123, 4\} \sqcup^- \{124, 3\} \sqcup^- \{13, 24\} \sqcup^- \{14, 23\}$$

of the non-circuit clutter Λ into circuit clutters. Notice though that the first two clutters are enough to obtain a decomposition; that is, here we have that $\{123, 124\} = \{123, 4\} \sqcup^- \{124, 3\}$.

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