ON TREES WITH THE SAME RESTRICTED U-POLYNOMIAL AND THE PROUHET-TARRY-ESCOTT PROBLEM

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Abstract. This paper focuses on the well-known problem due to Stanley of whether two non-isomorphic trees can have the same U-polynomial (or, equivalently, the same chromatic symmetric function). We consider the U_k-polynomial, which is a restricted version of U-polynomial, and construct with the help of solutions of the Prouhet-Tarry-Escott problem, non-isomorphic trees with the same U_k-polynomial for any given k. By doing so, we also find a new class of trees that are distinguished by the U-polynomial up to isomorphism.

1. Introduction

The aim of this paper is to contribute towards a solution of Stanley's question of whether there exist two non-isomorphic trees with the same chromatic symmetric function.

The U-polynomial (introduced by Noble and Welsh [1]) and the chromatic symmetric function X_G (introduced by Stanley [2]) of a graph G are powerful graph isomorphism invariants. They encode much of the combinatorics of the given graph. In particular, many other well-known invariants such as the Tutte polynomial and the chromatic polynomial can be obtained as evaluations of them.

The main problem about any graph invariant is to understand which classes of graphs it distinguishes. One way of approaching this problem is by finding non-isomorphic graphs with the same invariant. For the chromatic symmetric function, such examples already appear in [2]. For the U-polynomial, by results of Sarmiento [3], one checks that work of Brylawski [4] in the context of the polychromate provides such examples (see also [5]). More recently, Markstrom [6] has found the smallest pairs of non-isomorphic graphs with the same U-polynomial. However, the following question by Stanley [2] remains unsolved.

Question 1.1 (Stanley's question)**.** Do there exist non-isomorphic trees with the same chromatic symmetric function?

We note that it is well known that Stanley's question is equivalent to the similar question for the U-polynomial, since, when restricted to trees, the chromatic symmetric function and the U-polynomial determine each other (see [1, Theorem 6.1]). This means, in particular, that many statements related to the chromatic symmetric function of a tree can be rewritten in terms of the U-polynomial. In this article, we prefer to write everything in terms of the U-polynomial.

We give a brief review of what is known about Stanley's question in the literature. First, Russell [7] verified that the U-polynomial distinguishes trees with 25 or fewer vertices (see also [8]). Also, there are several special classes of trees where the restriction of Stanley's question has a known solution. In [9], Martin, Morin and Wagner showed that the U-polynomial distinguishes spider trees and a subclass of caterpillars (they also showed how to compute much of the combinatorics of a

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tree from the coefficients of its U-polynomial). In [10], the first and third author showed that the U-polynomial distinguishes the larger class of all proper caterpillars (a tree is *proper* if every non-leaf vertex is adjacent to a leaf). It is still unknown to the authors whether the U-polynomial distinguishes non-proper caterpillars. In a different direction, in [11] Bollobás and Riordan defined a rooted polychromate and showed that it distinguishes rooted trees. In a similar vein, Orellana and Scott showed [12] how to reconstruct a tree from a *labeled* version of the U_2-polynomial, provided the tree has a unique vertex as a centroid (the centroid and the U_2-polynomial are defined later on in this article; for the meaning of what labeled means in this context, we refer the reader directly to [12]). More recently, in [8], Smith, Smith and Tian have extended Orellana and Scott's results to show that a *labeled* version of the U_3-polynomial suffices to reconstruct any tree. Finally, Loebl and Sereni [13] have developed some techniques for constructing families of weighted graphs that are distinguished by the W-polynomial, which is the weighted version of the U-polynomial.

In this paper, we consider a restricted version of the U-polynomial, which we call the U_k-polynomial for any fixed integer k. Our main result is to exhibit examples of non-isomorphic trees with the same U_k-polynomial for every k. One of the motivations for this work comes from Orellana and Scott's results, in the sense that our examples could shed some light about possible obstructions for extending Orellana and Scott's results into a solution of Stanley's question. This is emphasized by the fact that our examples generalize some of the examples already found in [12] (see Figure 2). Let us note that Smith, Smith and Tian [8] have found, with the aid of the computer, the smallest pairs of non-isomorphic trees with the same U_k-polynomial for $k \in \{1, 2, 3, 4\}$.

In order to construct our examples, we reduce the problem to finding solutions of an old problem in number theory known as the *Prouhet-Tarry-Escott problem* (PTE problem for short). Given a positive integer k, we ask whether there exist integer sequences $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, distinct up to permutation, such that

$$\sum_{i=1}^{n} a_i^\ell = \sum_{i=1}^{n} b_i^\ell \quad \text{for all } 1 \le \ell \le k. \tag{1}$$

If a and b are solutions for this problem for some $k \ge 1$, we denote this by $a =_k b$ for short. We call k the *degree* of the solution and the length of the sequences its *size* (note that some terms could be zero). The history of the PTE problem probably goes back to Euler and Goldbach (1750-51) who noted that

$$(a, b, c, a + b + c) =_2 (0, a + b, a + c, b + c).$$

Independently, Prouhet (1851) and Tarry and Escott (1910) showed that for every k there are solutions to the PTE-problem. For more history and results related to the PTE problem, we refer the reader to [14, 15]. We also note that once one solution for the PTE problem has been found, many other *equivalent* solutions can be easily constructed. This follows from the fact that, if $a =_k b$ and $f(t) = \alpha t + \beta$ is an affine transformation with integer coefficients, then it is easy to check that $f(a) =_k f(b)$. For convenience, we usually write $\alpha a + \beta$ instead of $f(a)$. For instance, if $a =_k b$, then $a + 1 =_k b + 1$ and $\alpha - a =_k \alpha - b$ for every integer α.

This paper is organized as follows. Background and statement of the main results are the content of Section 2, while Sections 3, 4 and 5 are devoted to the proofs.

2. Background and Results

In this section we recall the definition of the U-polynomial and then state our results. It will be convenient to first recall the definition of the W-polynomial for weighted graphs. Given a graph G, a *weight function* is a map ω from the vertices of G to the positive integers. A *weighted graph* is a graph G endowed with a weight function ω, denoted by (G, ω). Given a subset U of vertices of G, the *weight* of U, denoted by $\omega(U)$, is the sum of the weights of all vertices in U.

The W-polynomial of a weighted graph was introduced in [1] by means of a *deletion-contraction* formula. For our purposes, it is easier to use its states model representation (see [1, Theorem 4.3]) as a definition. We need some notation first. Let (G, ω) be a weighted graph with $G = (V, E)$. The number of connected components of G is denoted by $k(G)$. Given $A \subseteq E$, the restriction $G|_A$ of G to A is obtained by deleting every edge that is not contained in A (but keeping all the vertices). The *rank* of A, denoted by $r_G(A)$, is defined as

$$r_G(A) = |V| - k(G|_A).$$

The *partition* induced by A, denoted by $\lambda_G(A)$, is the partition of $\omega(V)$ determined by the total weight of the vertices in each connected component of $G|_A$. When G is clear from the context, we write $\lambda(A)$ and $r(A)$ instead of $\lambda_G(A)$ and $r_G(A)$. Let $\mathbf{x} = x_1, x_2, \ldots$ be an infinite set of commuting indeterminates. Given any partition λ, we encode it as the monomial $\mathbf{x}_\lambda := x_{\lambda_1} \cdots x_{\lambda_l}$. The *$W$-polynomial* of (G, ω) is defined as

$$W(G, \omega; \mathbf{x}, y) = \sum_{A \subseteq E} \mathbf{x}_{\lambda(A)} (y-1)^{|A| - r(A)}. \tag{2}$$

If G is a (unweighted) graph, then the *U-polynomial* of G is defined as the W-polynomial of $(G, 1_V)$, where 1_V is the weight function that gives weight 1 to each vertex of G. Here, $\lambda(A)$ reduces to the partition induced by the number of vertices on each connected component of $G|_A$.

In what follows, all graphs are assumed to be trees. In this case, it is easy to check that $r(A) = |A|$ for every $A \subseteq E$. It follows that the U-polynomial of a tree T can be rewritten as

$$U(T) = U(T; \mathbf{x}) = \sum_{A \subseteq E} \mathbf{x}_{\lambda(A)} = \sum_{A \subseteq E} \mathbf{x}_{\lambda(E \setminus A)}.$$

In this note, we focus on the following restricted-version of the U-polynomial. Given a positive integer k, let

$$U_k(T) = U_k(T; \mathbf{x}) = \sum_{A \subseteq E, |A| \leq k} \mathbf{x}_{\lambda(E \setminus A)},$$

that is, we restrict the cardinality of the edge sets appearing in the definition of the U-polynomial. Of course, if $k = |E|$, then $U(T; \mathbf{x}) = U_k(T; \mathbf{x})$.

The main goal of this note is, for any given k, to exhibit examples of non-isomorphic trees with the same U_k-polynomial. To do so, we first introduce a new class of trees encoded by non-negative integer sequences. As a convention, the term *n-star* will refer to a star $K_{1,n-1}$ with n vertices, and the term *n-path* will refer to a path with n vertices.

Given two non-negative integers p and s, the tree $B(p, s)$ is the tree formed by taking p disjoint copies of a 4-path and s disjoint copies of a 4-star, and then identifying one leaf-vertex of each copy into a common vertex v. The vertex v is considered as the root of $B(p, s)$. Note that $B(0, 0)$ consists of an isolated vertex v (see Figure 1 for an example).

Next, given two sequences $p = (p_1, \ldots, p_n)$ and $s = (s_1, \ldots, s_n)$ of non-negative integers with length $n \geq 2$, the tree $T(p, s)$ is constructed as follows. Take the disjoint union of $B(p_i, s_i)$ for all $1 \leq i \leq n$ and denote their respective roots by v_i. Next, join each vertex v_i to a central vertex c. This yields $T(p, s)$. The subgraph induced on the vertices $\{c, v_1, \ldots, v_n\}$ will be referred to as the *core* of $T(p, s)$. It

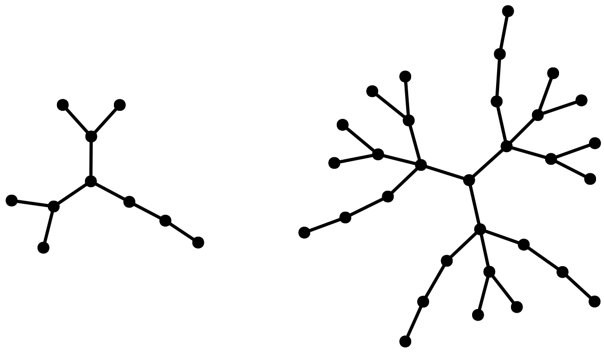

FIGURE 1. The graph on the left is $B(1,2)$, the graph on the right is $T(1\ 1\ 2, 2\ 2\ 1) = T_3(1\ 1\ 2)$.

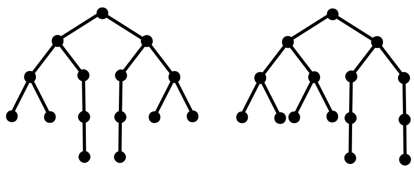

FIGURE 2. The graphs $T_2(1\ 1)$ and $T_2(2\ 0)$ have the same U_2-polynomial but distinct U_3-polynomial. Note that $1+1 = 2+0$ but $1^2+1^2 \neq 2^2+0^2$.

is easy to see that the core is isomorphic to a $(n+1)$-star (see also Figure 1 for an example).

Finally, we consider a special case of the last construction. Given a positive integer α, we say that a sequence of non-negative integers $p = (p_1, \dots, p_n)$ is α-compatible if $n \geq 2$ and $\max_i p_i \leq \alpha$. If p is an α-compatible sequence, then $\alpha - p$ denotes the sequence $(\alpha - p_1, \dots, \alpha - p_n)$. Define $T_\alpha(p) := T(p, \alpha - p)$. We say that a tree T is a *Prouhet-Tarry-Escott tree* (for short PTE-tree) if there exist α and p such that T is isomorphic to $T_\alpha(p)$.

Our first result, to be proved in Sections 3 and 5 (see Theorem 3.5 and Proposition 5.1), is the following:

Theorem 2.1. *Two non-isomorphic PTE-trees have the same U_{k+1}-polynomial if and only if their associated sequences are solutions for the PTE-problem with degree k.*

A simple example of application of Theorem 2.1 can be seen in Figure 2. To construct our examples, by Theorem 2.1, we need to find solutions for the PTE-problem for any degree. As noted in the introduction, the existence of such solutions was already known to Prouhet. In [16], Wright proved the following stronger result:

Theorem 2.2. *For $k \geq 1$, $j \geq 2$ there exist sequences $a_1, a_2, a_3, \dots, a_j$ of length $n \leq (k^2+k+2)/2$, distinct up to permutation, such that*

$$a_1 =_k a_2 =_k \cdots =_k a_j.$$

The latter theorem allows us to obtain sets of any given cardinality of non-isomorphic trees with same U_k-polynomial.

Corollary 2.3. *For every pair of positive integers k and j, there is a set of j non-isomorphic trees with the same U_k-polynomial.*

It is well-known that if two sequences of length n satisfy $p =_k p'$, then $n > k$ (see [17, Prop. 2]). Thus, as another consequence of Theorem 2.1 we get

Corollary 2.4. *Let T and T' be two PTE-trees. Then T and T' have the same U-polynomial if and only if they are isomorphic.*

The last corollary suggests the following question: Does the U-polynomial distinguishPTE-trees from non-PTE trees? It turns out that it suffices to consider the U_1-polynomial to answer this question.

Proposition 2.5. *The U_1-polynomial recognizes whether a tree is a PTE-tree or not.*

The proof of this proposition uses some techniques from [12] and it is given in Section 4. As a direct corollary of this proposition and Theorem 2.1 we get that all PTE-trees are distinguished up to isomorphism by the U-polynomial:

Corollary 2.6. *If T is a PTE-tree, and T' is another tree such that $U(T) = U(T')$, then T' is isomorphic to T.*

3. Solutions to the PTE problem of degree k induce trees with the same U_{k+1}-polynomial

In this section, we will prove one of the directions of Theorem 2.1. Let T be a tree and q and t two non-negative sequences of the same length. If S is a subtree of T isomorphic to $T(q,t)$, we say that S is of type (q,t). We denote by $S_{q,t}(T)$ the set of all subtrees of T of type (q,t). If T is a PTE-tree associated with a sequence of length n, then the sequences q, t are also assumed to be of length n.

Lemma 3.1. *Let α be a non-negative integer and q, t be two non-negative integer sequences of length n. Then, there is a symmetric polynomial $P_{\alpha,q,t}(x_1, x_2, \ldots, x_n)$ of degree at most $\sum_i (q_i + t_i)$ such that, for every α-compatible sequence p of length n, we have*

$$|S_{q,t}(T_\alpha(p))| = P_{\alpha,q,t}(p_1, p_2, \ldots, p_n).$$

Proof. For each $1 \le i \le n$, denote by B_i be the subtree of $T_\alpha(p)$ rooted at v_i (note that B_i is isomorphic to $B(p_i, \alpha - p_i)$). Let π be a permutation of $[n]$. Let $S^\pi_{q,t}(T_\alpha(p))$ be the collection of subtrees S of $T_\alpha(p)$ of type (q,t) such that, for all $1 \le i \le n$, the subtree of S induced on the vertices of B_i is isomorphic to $B(q_{\pi(i)}, t_{\pi(i)})$. Clearly, each S in $S^\pi_{q,t}(T_\alpha(p))$ corresponds to a choice of $q_{\pi(i)}$ 4-paths and $t_{\pi(i)}$ 4-stars from B_i for each $1 \le i \le n$. Thus,

$$|S^\pi_{q,t}(T_\alpha(p))| = \prod_{i=1}^{n} \binom{p_i}{q_{\pi(i)}} \binom{\alpha - p_i}{t_{\pi(i)}}.$$

On the other hand,

$$S_{q,t}(T_\alpha(p)) = \bigcup_{\pi \in S_n} S^\pi_{q,t}(T_\alpha(p)).$$

We claim that any two sets in the family $\{S^\pi_{q,t}(T_\alpha(p))\}_{\pi \in S_n}$ are either disjoint or equal. Indeed, let us suppose that there is a subtree S belonging to $S^\pi_{q,t}(T_\alpha(p)) \cap S^{\pi'}_{q,t}(T_\alpha(p))$ for some π, π' in S_n. By definition, for each $1 \le i \le n$, the subtree of S induced on B_i must be isomorphic to both $B(q_{\pi(i)}, t_{\pi(i)})$ and $B(q_{\pi'(i)}, t_{\pi'(i)})$. Clearly, this can only happen when $q_{\pi(i)} = q_{\pi'(i)}$ and $t_{\pi(i)} = t_{\pi'(i)}$ for all $1 \le i \le n$ and in this case it is easy to conclude that $S^\pi_{q,t}(T_\alpha(p)) = S^{\pi'}_{q,t}(T_\alpha(p))$.

With π fixed, the number of permutations π' satisfying $q_{\pi(i)} = q_{\pi'(i)}$ and $t_{\pi(i)} = t_{\pi'(i)}$ for all $1 \le i \le n$ depends only on the symmetries of (q,t) (and not on π).

More concretely, it equals the number of permutations $\sigma \in S_n$ such that $q_{\sigma(i)} = q_i$ and $t_{\sigma(i)} = t_i$ for all $1 \leq i \leq n$. Let us denote this number by $N_{q,t}$ and define

$$P_{\alpha,q,t}(x_1, x_2, \dots, x_n) := \frac{1}{N_{q,t}} \sum_{\pi \in S_n} \prod_{i=1}^{n} \binom{x_i}{q_{\pi(i)}} \binom{\alpha - x_i}{t_{\pi(i)}}. \tag{3}$$

It is clear that $P_{\alpha,q,t}$ is a symmetric polynomial of degree at most $\sum_i (q_i + t_i)$, and by the discussion above $P_{\alpha,q,t}(p_1, p_2, \dots, p_n)$ is the required number of subtrees. □

Corollary 3.2. *Let α be a positive integer. Suppose that p and p' are two α-compatible sequences such that $p =_k p'$. Then, for every (q, t) such that $\sum_i (q_i + t_i) \leq k$ we have*

$$|S_{q,t}(T_\alpha(p))| = |S_{q,t}(T_\alpha(p'))|.$$

Proof. By Lemma 3.1, there exists a symmetric polynomial $P(x_1, x_2, \dots, x_n)$ of degree less or equal than k such that $P(p_1, p_2, \dots, p_n) = |S_{q,t}(T_\alpha(p))|$ and $P(p'_1, p'_2, \dots, p'_n) = |S_{q,t}(T_\alpha(p'))|$. By [18, Corollary 7.7.2], this polynomial can be written as a linear combination of the power sum symmetric polynomials of degree less or equal than k. Since $p =_k p'$, the conclusion follows. □

Let S be a subgraph of T. Then, the contraction $S_{\omega,T}$ of S in T is the weighted tree obtained by contracting all the edges not in S and adding weights along contracted edges. If F is a subset of edges of T, define

$$U_F(T) = \sum_{A \subseteq F} \mathbf{x}_{\lambda(E \setminus A)}.$$

The proof of the lemma below follows directly from the definitions.

Lemma 3.3. *Let $T = (V, E)$ be a tree and $S = (W, F)$ be a subgraph of T. For every $A \subseteq F$, we have*

$$\lambda_T(E \setminus A) = \lambda_{S_{\omega,T}}(F \setminus A). \tag{4}$$

Moreover, $U_F(T) = W(S_{\omega,T})$.

Proposition 3.4. *Let $T = T_\alpha(p)$ and $T' = T_\alpha(p')$ for two sequences p and p' such that $p =_1 p'$. If K and K' denote, respectively, the core of T and T', then*

$$U_K(T) = U_{K'}(T') \quad \text{and} \quad U_{E \setminus K}(T) = U_{E \setminus K'}(T').$$

Proof. Both assertions follow from Lemma 3.3 after observing that $K_{\omega,T}$ and $K'_{\omega,T'}$ are isomorphic weighted graphs and, also, $(E \setminus K)_{\omega,T}$ and $(E' \setminus K')_{\omega,T'}$ are isomorphic weighted graphs because $p =_1 p'$ and $\alpha - p =_1 \alpha - p'$. □

Theorem 3.5. *Let $T = T_\alpha(p)$ and $T' = T_\alpha(p')$ for two α-compatible sequences p and p' such that $p =_k p'$. Then*

$$U_{k+1}(T) = U_{k+1}(T').$$

Proof. We give a bijection φ between edge-subsets of T and T' of size at most $k+1$ such that the complements of A and $\varphi(A)$ induce the same partition in T and T', respectively. By Proposition 3.4, it suffices to define φ on subsets of edges that intersect both the core of T and its complement.

Let

$$\mathcal{S}_T = \bigcup_{\substack{(q,t) \\ \sum_i (q_i + t_i) \leq k}} S_{q,t}(T),$$

and define $\mathcal{S}_{T'}$ analogously. By Corollary 3.2, there is a type-preserving bijection $\Phi : \mathcal{S}_T \to \mathcal{S}_{T'}$. Also, for each $S \in \mathcal{S}_T$ we fix an isomorphism between S and

$\Phi(S)$; this isomorphism is also a weighted-graph isomorphism between $S_{w,T}$ and $(\Phi(S))_{w,T'}$.

Let A be a subset of edges of T with $|A| \leq k+1$ that intersects the core of T and its complement. Since A has at most k edges in the complement of the core, it is contained in some subtree belonging to $\mathcal{S}_T$. Let S_A be the smallest such subtree, which is well defined since the intersection of elements of $\mathcal{S}_T$ is again in $\mathcal{S}_T$. Let $\varphi(A)$ be the edge-subset of T' to which A is mapped under the fixed bijection between S_A and $\Phi(S_A)$. By construction, A and $\varphi(A)$ contribute the same term to $U_{k+1}(T)$ and $U_{k+1}(T')$, respectively.

To finish the proof it is enough to notice that $\Phi(S_A)$ is the smallest subtree in $\mathcal{S}_{T'}$ that contains $\varphi(A)$, and thus A can be recovered from $\varphi(A)$ and φ is a bijection. □

4. Recognizing PTE-trees with the U-polynomial

In this section we show that the U_1-polynomial recognizes whether a tree is PTE or not. To see this, we recall some techniques introduced by Orellana and Scott [12]. Let T be a tree. Given any vertex v in T, a *branch* of v is any subtree of T having v as a leaf-vertex. Then, the *branch-weight* of v is the maximum number of edges in any branch of v. The *centroid* is defined as the set of vertices with minimum branch-weight. It is known that the centroid contains either one or two vertices (in which case they are connected by an edge). We say that two edges e and f *attract* if the unique path joining e and the centroid passes through f or the unique path joining f with the centroid passes through e. Otherwise, we say that e and f *repel*. When T has a unique centroid c, we set c as the root of T and label the edges of T as follows. For each edge e in T, its label θ_e is the unique positive integer satisfying $\lambda_T(E \setminus \{e\}) = (N-\theta_e, \theta_e)$ with $N - \theta_e \geq \theta_e$, where N denotes the number of vertices of T. We denote by $\mathcal{M}_T$ the multiset of labels of T. The following lemma summarizes the tools from [12] that we need here.

Lemma 4.1. *Let T be a tree labeled as above. Then the labels along any path starting at the centroid are strictly decreasing. Moreover, for each edge e, its label is the the sum of the labels of its child-edges plus one. In particular, edges with the same labels always repel.*

The following result is a more precise statement for Proposition 2.5.

Proposition 4.2. *Let α be a positive integer, p be an α-compatible sequence of length n and $T = T_\alpha(p)$ the associated PTE-tree. Let $N = (3\alpha+1)n+1$ be the total number of vertices of T and $\beta = \sum_i p_i$. Then, the following assertions hold:*

i) We have $\mathcal{M}_T = \{1^{2n\alpha-\beta}, 2^\beta, 3^{n\alpha}, (3\alpha+1)^n\}$.
ii) If T' is another tree with $U_1(T) = U_1(T')$, then T' is a PTE-tree $T_\alpha(p')$ with $\sum_i p'_i = \beta$.

Proof. The first assertion follows directly from the construction of $T_\alpha(p)$.

For the second assertion, first recall that a tree has a unique centroid if and only if there is no edge with label $N/2$, where N is the number of vertices. That is, the property of having a unique centroid is determined by the U_1-polynomial of the tree. In particular, since T has a unique centroid, so has T'. Since edges with the same label repel, it is clear that the n edges with label $3\alpha+1$ must be incident to the centroid. Since $N = (3\alpha+1)n+1$, it follows from Lemma 4.1 that no other edge may be incident to the centroid. Thus, each edge with label 3 must be attached to one edge of label $3\alpha+1$. Since we have $n\alpha$ such edges and n edges of label $3\alpha+1$, again by Lemma 4.1, each edge with label $3\alpha+1$ has exactly α child-edges with label 3 and cannot have any other child-edges. By now, edges with label 2 and 1 can only be incident with edges of label 3.

It is easy to see that each edge with label 3 is either incident to two edges of label 1 or to one edge of label 2, which in turn is incident to an edge of label 1. For each e_i with label $3\alpha+1$ set p'_i be the number of edges of label 3 of the second kind that are incident to e_i. It is clear that $\sum_i p'_i = \beta$ and that $T' = T_\alpha(p')$ and the proof is finished. □

5. Non-isomorphic PTE trees have different restricted U-polynomials

In this section we complete the proof of Theorem 2.1by showing that a pair of non-isomorphic PTE trees have different U_{k+2}-polynomials for some integer k.

Proposition 5.1. *Let $T = T_\alpha(p)$ and $T' = T_\alpha(p')$ for two sequences p and p' such that*

$$p =_k p' \quad \textit{but} \quad \sum_i p_i^{k+1} \neq \sum_i (p'_i)^{k+1}.$$

Then

$$U_{k+2}(T) \neq U_{k+2}(T').$$

Proof. We start by showing that $k+1 \leq \alpha$. It is easy to see that $p =_k p'$ implies that $(x-1)^{k+1}$ divides $\sum_i (x^{p_i} - x^{p'_i})$ (see for instance [17, Prop. 1]). This forces $k+1 \leq \max\{\max_i p_i, \max_i p'_i\} \leq \alpha$.

Let $N = n(3\alpha+1)+1$ be the number of vertices of T and T', $a = 3\alpha+1-2(k+1)$ and $b = N-(3\alpha+1)$. It follows from $k+1 \leq \alpha$ and $n \geq 2$, that $b > 3\alpha+1$ and $b > a \geq 3$. We will show that the coefficients of $x_2^{k+1}x_a x_b$ in $U(T)$ and $U(T')$ are different.

First note that a monomial of the form $x_2^{k+1}x_{a'}x_{b'}$, with $2 < a' < b'$, can only appear if, for a given edge set A with $k+2$ edges, there are $k+1$ isolated edges after removing the edges of A. An edge will be isolated if and only if A contains all the edges incident to it. Observing that edges with label 3 or $3\alpha+1$ are incident to at least $\alpha+1 \geq k+2$ edges, and that removing edges with label 1 would leave isolated vertices, we deduce that A must contain exactly $k+1$ edges with label 2 and one edge e with label $\phi \in \{3, 3\alpha+1\}$. Now, we find all the possible values that $\{a', b'\}$ can take and compare them with $\{a, b\}$ (note that $a'+b' = a+b$). First, if $\phi = 3\alpha+1$, then $e = cv_i$ for some fixed $1 \leq i \leq n$ and $\{a', b'\} = \{3\alpha+1-2l, N-(3\alpha+1)-2(k+1-l)\}$, where $l \leq k+1$ is the number of edges with label 2 in A that lie in the subtree B_i of T rooted at v_i. It is easy to see that $b \neq 3\alpha+1-2l$, and hence $\{a', b'\} = \{a, b\}$ if and only if $l = k+1$, i.e., all edges with label 2 in A lie in B_i. Second, if $\phi = 3$, then $a' = 3$, and e can be any of the edges with label 3 that is not adjacent to any of the edges of A with label 2. In this case, $a' = a$ if and only if $a = 3$ (which can only arise when $\alpha = 2$ and $k = 1$). From this, it is easy to see that

$$[x_2^{k+1}x_a x_b]U(T) = \delta_{a,3}(n\alpha-(k+1))\binom{\sum_i p_i}{k+1} + \sum_{i=1}^{n}\binom{p_i}{k+1},$$

where $\delta_{a,3} = 1$ if $a = 3$ and 0 otherwise. By applying the same arguments to T' we get a similar expression. Substracting we obtain

$$[x_2^{k+1}x_a x_b]U(T) - [x_2^{k+1}x_a x_b]U(T') = \sum_{i=1}^{n}\binom{p_i}{k+1} - \sum_{i=1}^{n}\binom{p'_i}{k+1}.$$

The difference $\sum_{i=1}^{n}\binom{p_i}{k+1} - \sum_{i=1}^{n}\binom{p'_i}{k+1}$ can be expressed as a linear combination of the power-sum symmetric polynomials, with the highest degree term being $\sum_i (p_i^{k+1} - (p'_i)^{k+1})/(k+1)!$. Thus, it is clear that both coefficients are different. □

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