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Roesser model representation of 2D periodic behaviors: the (2,2)-periodic SISO case

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Abstract

In this paper we consider 2D single-input/single-output behaviors described by linear partial difference equations with (2,2)-periodically varying coefficients, and present a method to obtain 2D *Roesser* state-space representations (or realizations) for such behaviors. Since these cannot be obtained by separately realizing each shift-invariant system resulting from "freezing" the varying coefficients, we propose a method based on the realization of an invariant input/output behavior obtained by suitably "lifting" the trajectories of the original periodic behavior.

I. Introduction

In spite of the difficulty in defining solid notions of *past*, *future* and *state* for systems evolving over multidimensional domains (where the independent variable has often no natural evolution direction), the construction of first order representations for multidimensional systems has deserved much attention over the years. In the 2D case, the Fornasini–Marchesini, [1], and the *Roesser* state-space models, [2], are the most well-known first-order descriptions of quarter-plane causal input/output 2D systems.

Here, we consider input/output 2D systems defined over $\mathbb{N}^2$, which are periodic in the sense that they are described by linear partial difference equations whose coefficients vary periodically in the two directions of the domain. Such systems may be of particular interest for the design of 2D digital filters, where the option of allowing the filter coefficients to vary periodically gives an extra degree of freedom that can be advantageous, [3]. The construction of periodically time-varying first order representations (realizations) for 2D periodic input/output systems, clearly plays an important role in this context.

The aim of the present paper is to give a preliminary contribution to the solution of the aforementioned realization problem. In particular, following the ideas of [4], we propose a method to obtain a 2D periodic *Roesser* model representation which consists in first determining an invariant input/output system associated with the original periodic one, then making (if possible) an invariant *Roesser* model realization, and, finally, obtaining a 2D periodic *Roesser* model from the invariant one. For the sake of simplicity, we shall focus on single-input/single-output (SISO) $(2,2)$–periodic 2D systems, i.e., systems whose coefficients periodically vary with period 2 both in the horizontal and in the vertical direction.

II. Preliminaries

We define a two–dimensional (2D) single input/single output (SISO) periodic behavior $\mathfrak{B}$ as a set of 2D input/output trajectories (u, y), defined over $\mathbb{N}^2$ and taking values in $\mathbb{R}\times\mathbb{R}$, that satisfy an equation of the type:

$$\big(p_{(i,j)}(\sigma_1,\sigma_2)y\big)(i,j)=\big(q_{(i,j)}(\sigma_1,\sigma_2)u\big)(i,j)\,,\quad (i,j)\in\mathbb{N}^2, \tag{1}$$

where σ_1 and σ_2 represent the usual 2D shifts (i.e., $(\sigma_1 v)(i,j) = v(i+1,j)$ and $(\sigma_2 v)(i,j) = v(i,j+1)$), and, for $(i,j)\in\mathbb{N}^2$, $p_{(i,j)}(z_1,z_2)\in\mathbb{R}[z_1,z_2]\backslash\{0\}$, $q_{(i,j)}(z_1,z_2)\in\mathbb{R}[z_1,z_2]$ are 2D polynomials. Moreover, $p_{(i,j)}$ and $q_{(i,j)}$ are such that

$$\begin{aligned} p_{(i,j)} &= p_{(i+P,j)} = p_{(i,j+Q)} \\ q_{(i,j)} &= q_{(i+P,j)} = q_{(i,j+Q)}, \end{aligned} \tag{2}$$

respectively, and where P and Q are the smallest integers for which all the equalities occur.

Thus, equation (1) is a 2D linear partial difference equation with periodically varying coefficients, of period (P,Q). We shall say that $\mathfrak{B}$ is a 2D (P,Q)–periodic behavior and call P and Q the horizontal and the vertical period, respectively. Note that a period equal to 1 means invariance of the coefficients in the corresponding (vertical or horizontal) direction.

The question to be studied is whether or not the behavior $\mathfrak{B}$ can be alternatively represented by means of a 2D *Roesser* model with (P,Q)–periodically varying coefficients:

$$\begin{aligned} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= A(i,j)\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + B(i,j)\,u(i,j) \\ y(i,j) &= C(i,j)\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + D(i,j)\,u(i,j) \end{aligned} \tag{3}$$

where $x^h(i,j)\in\mathbb{R}^{n_h}$ is the horizontal state vector, $x^v(i,j)\in\mathbb{R}^{n_v}$ is the vertical state vector, and $u(i,j)$, $y(i,j)$ are the input and the output, respectively. Moreover, the matrices A, B, C and D, suitably decomposed as follows

$$A(i,j)=\begin{bmatrix} A^{hh}(i,j) & A^{hv}(i,j) \\ A^{vh}(i,j) & A^{vv}(i,j) \end{bmatrix},\quad B(i,j)=\begin{bmatrix} B^h(i,j) \\ B^v(i,j) \end{bmatrix} \tag{4}$$

$$C(i,j)=\begin{bmatrix} C^h(i,j) & C^v(i,j) \end{bmatrix},$$

vary periodically with period (P,Q), meaning that, for all possible values of the horizontal and vertical discrete variables (i,j),

$$\begin{aligned} A(i,j) &= A(i+P,j) = A(i,j+Q) \\ B(i,j) &= B(i+P,j) = B(i,j+Q) \\ C(i,j) &= C(i+P,j) = C(i,j+Q) \\ D(i,j) &= D(i+P,j) = D(i,j+Q) \,. \end{aligned} \tag{5}$$

This model will be denoted by

$$\Sigma(\cdot,\cdot) = (A(\cdot,\cdot), B(\cdot,\cdot), C(\cdot,\cdot), D(\cdot,\cdot))\,.$$

We say that $\Sigma(\cdot,\cdot)$ represents (or is a *representation* or a *realization* of) $\mathfrak{B}$ if the set of all possible input/output trajectories of $\Sigma(\cdot,\cdot)$ coincides with $\mathfrak{B}$.

It is well-known that, in the invariant case, a 2D SISO input/output behavior described by:

$$p(\sigma_1,\sigma_2)\,y = q(\sigma_1,\sigma_2)\,u$$

is representable by a 2D *Roesser* model if and only if the corresponding transfer-function $g(z_1,z_2) = \frac{q(z_1,z_2)}{p(z_1,z_2)}$ is quarter-plane causal, see [2]. However, as shown in the following example, the representation of a 2D periodic behavior by means of a 2D periodic *Roesser* model cannot be obtained by individually realizing each invariant behavior obtained by “freezing” the periodically varying coefficients.

Example 2.1: Consider the $(2,1)$–periodic 2D input/output behavior $\mathfrak{B}$ described by:

$$\left(p_{(i,j)}(\sigma_1,\sigma_2)\,y\right)(i,j) = \left(q_{(i,j)}(\sigma_1,\sigma_2)\,u\right)(i,j)\,, \quad (i,j)\in\mathbb{N}^2$$

with

$$\begin{aligned} p_{(0,0)}(\sigma_1,\sigma_2) &= \sigma_1^2+\sigma_1+1\,, & q_{(0,0)}(\sigma_1,\sigma_2) &= 1 \\ p_{(1,0)}(\sigma_1,\sigma_2) &= \sigma_1^2\,, & q_{(1,0)}(\sigma_1,\sigma_2) &= \sigma_1+1\,, \end{aligned}$$

and, for $k=0,1$, denote by $\mathfrak{B}_k$ the invariant input/output behavior described by

$$\left(p_{(k,0)}(\sigma_1,\sigma_2)\,y\right)(i,j) = \left(q_{(k,0)}(\sigma_1,\sigma_2)\,u\right)(i,j)\,, \quad (i,j)\in\mathbb{N}^2\,.$$

Note that the 2D periodic behavior $\mathfrak{B}$ as well as the invariant behaviors $\mathfrak{B}_0$ and $\mathfrak{B}_1$ only have dynamics in the horizontal direction. Therefore each of them can be regarded as coupled 1D systems evolving along horizontal lines according to the same laws, but possibly with different initial conditions. It is not difficult to check that $\Sigma_0 = (A_0,B_0,C_0,D_0)$ with

$$A_0 = \begin{bmatrix} A_0^{hh} & A_0^{hv} \\ A_0^{vh} & A_0^{vv} \end{bmatrix}, \; B_0 = \begin{bmatrix} B_0^h \\ B_0^v \end{bmatrix}, \; C_0 = \begin{bmatrix} C_0^h & C_0^v \end{bmatrix},$$

$$A_0^{hh} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \; B_0^h = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_0^h = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D_0 = 0,$$

and where all the other matrices are void, is a 2D invariant *Roesser* model representation of $\mathfrak{B}_0$ (with empty vertical state). On the other hand, $\Sigma_1 = (A_1,B_1,C_1,D_1)$, with the matrices partitioned as above,

$$A_1^{hh} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B_1^h = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; C_1^h = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D_1 = 0,$$

and where all the other matrices are void, is a 2D invariant *Roesser* model (with empty vertical state) of $\mathfrak{B}_1$.

Consider now the 2D $(2,1)$–periodic *Roesser* model $\Sigma(\cdot,\cdot)$ defined by the matrices $A(i,j) = A_0$, $B(i,j) = B_0$, $C(i,j) = C_0$, $D(i,j) = D_0$, if $i = 2\ell$, $\ell\in\mathbb{N}$, and $A(i,j) = A_1$, $B(i,j) = B_1$, $C(i,j) = C_1$, $D(i,j) = D_1$, if $i = 2\ell+1$, $\ell\in\mathbb{N}$. Computing the output trajectories generated by $\Sigma(\cdot,\cdot)$ for the initial condition

$$x^h(0,j) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; j\in\mathbb{N},$$

and for the input $u(i,j)\equiv 1$, $(i,j)\in\mathbb{N}^2$, one obtains, for instance:

$$y(0,0) = C_0^h x^h(0,0) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\,,$$

$$\begin{aligned} y(1,0) &= C_1^h x^h(1,0) = C_1^h\left(A_0^{hh}x^h(0,0) + B_0^h u(0,0)\right) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cdot 1\right) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2\,, \end{aligned}$$

$$\begin{aligned} y(2,0) &= C_0^h x^h(2,0) = C_0^h\left(A_1^{hh}x^h(1,0) + B_1^h u(1,0)\right) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\cdot 1\right) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1\,. \end{aligned}$$

However, using the initial input/output description for the 2D $(2,1)$–periodic behavior $\mathfrak{B}$, one has:

$$\left(p_{(0,0)}(\sigma_1,\sigma_2)\,y\right)(0,0) = \left(q_{(0,0)}(\sigma_1,\sigma_2)\,u\right)(0,0)$$

which yields:

$$\left[\left(\sigma_1^2+\sigma_1+1\right)y\right](0,0) = [1\cdot u](0,0)$$

or, equivalently:

$$y(2,0)+y(1,0)+y(0,0) = u(0,0)$$

i.e.,

$$y(2,0) = -y(1,0)-y(0,0)+u(0,0)\,.$$

Using the previously calculated values $y(0,0)=1$, $y(1,0)=2$ and the given value for u, $u(0,0)=1$, one obtains:

$$y(2,0) = -2-1+1 = -2 \neq -1.$$

$\diamond$

In this way, we conclude that a 2D periodic *Roesser* representation of a 2D periodic behavior cannot be derived by the naive procedure presented in the previous example.

Following the ideas of [4], an alternative procedure is to obtain an invariant formulation of the original 2D (P,Q)–periodic behavior, determine (if possible) a 2D invariant *Roesser* model representation of the obtained invariant behavior, and finally try to obtain a 2D (P,Q)–periodic *Roesser* model representation from the invariant one.

III. Invariant formulations

For the sake of simplicity we consider only the case $P=2=Q$. In this case, letting $t_i=0,1$, with $i=1,2$, the periodic input/output equations defining a $(2,2)$–periodic input/output behavior $\mathfrak{B}$ can be rewritten as

$$\begin{aligned}\big(p_{(t_1,t_2)}\,(\sigma_1,\sigma_2)\,y\big)\,(2k+t_1,2\ell+t_2)&\\ =\big(q_{(t_1,t_2)}\,(\sigma_1,\sigma_2)\,u\big)\,(2k+t_1,2\ell+t_2)\,,&\end{aligned}\tag{6}$$

with $k,\ell\in\mathbb{N}$. This is equivalent to

$$\begin{aligned}\underbrace{\begin{bmatrix} p_{(0,0)}\,(\sigma_1,\sigma_2)\\ p_{(1,0)}\,(\sigma_1,\sigma_2)\,\sigma_1\\ p_{(0,1)}\,(\sigma_1,\sigma_2)\,\sigma_2\\ p_{(1,1)}\,(\sigma_1,\sigma_2)\,\sigma_1\sigma_2\end{bmatrix}}_{\mathcal{P}(\sigma_1,\sigma_2)} y\,(2k,2\ell)&\\ =\underbrace{\begin{bmatrix} q_{(0,0)}\,(\sigma_1,\sigma_2)\\ q_{(1,0)}\,(\sigma_1,\sigma_2)\,\sigma_1\\ q_{(0,1)}\,(\sigma_1,\sigma_2)\,\sigma_2\\ q_{(1,1)}\,(\sigma_1,\sigma_2)\,\sigma_1\sigma_2\end{bmatrix}}_{\mathcal{Q}(\sigma_1,\sigma_2)} u\,(2k,2\ell)\,,&\end{aligned}\tag{7}$$

with $k,\ell\in\mathbb{N}$. Decomposing the polynomials columns $\mathcal{P}\,(z_1,z_2)$ and $\mathcal{Q}\,(z_1,z_2)$ as

$$\mathcal{P}\,(z_1,z_2)=\mathcal{P}^L\left(z_1^2,z_2^2\right)\begin{bmatrix}1\\ z_1\\ z_2\\ z_1z_2\end{bmatrix}\tag{8a}$$

and

$$\mathcal{Q}\,(z_1,z_2)=\mathcal{Q}^L\left(z_1^2,z_2^2\right)\begin{bmatrix}1\\ z_1\\ z_2\\ z_1z_2\end{bmatrix},\tag{8b}$$

(7) can be written as:

$$\big(\mathcal{P}^L(\sigma_1,\sigma_2)Y\big)(k,\ell)=\big(\mathcal{Q}^L(\sigma_1,\sigma_2)U\big)(k,\ell)\,,\ k,\ell\in\mathbb{N},\tag{9}$$

where

$$U(k,\ell)=\begin{bmatrix}u(2k,2\ell)\\ u(2k+1,2\ell)\\ u(2k,2\ell+1)\\ u(2k+1,2\ell+1)\end{bmatrix}\tag{10a}$$

and

$$Y(k,\ell)=\begin{bmatrix}y(2k,2\ell)\\ y(2k+1,2\ell)\\ y(2k,2\ell+1)\\ y(2k+1,2\ell+1)\end{bmatrix}\tag{10b}$$

are the *lifted* trajectories corresponding to u and y, respectively, (notice the replacement of the shifts σ_i^2 (i=1,2) by σ_i (i=1,2), due to the change of independent variable). This defines an invariant 2D input/output behavior $\mathfrak{B}^L$ which is called the *invariant formulation*, or the *lifted version*, of $\mathfrak{B}$. Clearly, an input/output trajectory (u,y) belongs to $\mathfrak{B}$ if and only if the corresponding lifted trajectory (U,Y) belongs to $\mathfrak{B}^L$.

Now, the equations of the $(2,2)$–periodic *Roesser* model can be rewritten as:

$$\begin{aligned}\begin{bmatrix}x^h(2k{+}t_1{+}1,2\ell{+}t_2)\\ x^v(2k{+}t_1,2\ell{+}t_2{+}1)\end{bmatrix}&=A(t_1,t_2)\begin{bmatrix}x^h(2k{+}t_1,2\ell{+}t_2)\\ x^v(2k{+}t_1,2\ell{+}t_2)\end{bmatrix}\\ &\quad+B(t_1,t_2)\,u(2k{+}t_1,2\ell{+}t_2)\\ y(2k{+}t_1,2\ell{+}t_2)&=C(t_1,t_2)\begin{bmatrix}x^h(2k{+}t_1,2\ell{+}t_2)\\ x^v(2k{+}t_1,2\ell{+}t_2)\end{bmatrix}\\ &\quad+D(t_1,t_2)\,u(2k{+}t_1,2\ell{+}t_2)\end{aligned}\tag{11}$$

where $k,\ell\in\mathbb{N}$, and $t_i=0,1$ $(i=1,2)$, and the matrices A, B, C and D are decomposed as in (4). Denote

$$\begin{aligned}A(0,0)&=:A_1\ ;\quad A(1,0)=:A_2\\ A(0,1)&=:A_3\ ;\quad A(1,1)=:A_4\end{aligned}\tag{12}$$

and likewise for all the other matrices.

With the purpose of obtaining an invariant formulation of (11), following the ideas of [5], we now define lifted versions of the horizontal and vertical states as:

$$X^h(k,\ell)=\begin{bmatrix}x^h(2k,2\ell)\\ x^h(2k,2\ell+1)\end{bmatrix}\tag{13a}$$

and

$$X^v(k,\ell)=\begin{bmatrix}x^v(2k,2\ell)\\ x^v(2k+1,2\ell)\end{bmatrix},\tag{13b}$$

respectively, and consider $U\,(k,\ell)$ and $Y\,(k,\ell)$ as previously defined in eqs. (10a) and (10b). This yields the following linear 2D shift-invariant *Roesser* model

$$\begin{aligned}\begin{bmatrix}X^h\,(k+1,\ell)\\ X^v\,(k,\ell+1)\end{bmatrix}&=F\begin{bmatrix}X^h\,(k,\ell)\\ X^v\,(k,\ell)\end{bmatrix}+GU\,(k,\ell)\ ,\\ Y\,(k,\ell)&=H\begin{bmatrix}X^h\,(k,\ell)\\ X^v\,(k,\ell)\end{bmatrix}+JU\,(k,\ell)\end{aligned}\tag{14}$$

where matrices F, G, H and J are constant and can be decomposed as follows

$$\begin{aligned}F&=\begin{bmatrix}F^{hh} & F^{hv}\\ F^{vh} & F^{vv}\end{bmatrix}\ ,\ G=\begin{bmatrix}G^h\\ G^v\end{bmatrix}\\ H&=\begin{bmatrix}H^h & H^v\end{bmatrix}\ ,\end{aligned}\tag{15}$$

with the size of the blocks is determined by the sizes of X^h and X^v, and, moreover:

$$F^{hh}=\begin{bmatrix} A_2^{hh}A_1^{hh} & 0 \\ A_4^{hh}A_3^{hv}A_1^{vh}+A_4^{hv}A_2^{vh}A_1^{hh} & A_4^{hh}A_3^{hh} \end{bmatrix}$$

$$F^{hv}=\begin{bmatrix} A_2^{hh}A_1^{hv} & A_2^{hv} \\ A_4^{hh}A_3^{hv}A_1^{vv}+A_4^{hv}A_2^{vh}A_1^{hv} & A_4^{hv}A_2^{vv} \end{bmatrix}$$

$$F^{vh}=\begin{bmatrix} A_3^{vv}A_1^{vh} & A_3^{vh} \\ A_4^{vh}A_3^{hv}A_1^{vh}+A_4^{vv}A_2^{vh}A_1^{hh} & A_4^{vh}A_3^{hh} \end{bmatrix}$$

$$F^{vv}=\begin{bmatrix} A_3^{vv}A_1^{vv} & 0 \\ A_4^{vh}A_3^{hv}A_1^{vv}+A_4^{vv}A_2^{vh}A_1^{hv} & A_4^{vv}A_2^{vv} \end{bmatrix}$$

$$G^{h}=\begin{bmatrix} A_2^{hh}B_1^{h} & B_2^h & 0 & 0 \\ A_4^{hh}A_3^{hv}B_1^{v}+A_4^{hv}A_2^{vh}B_1^{h} & A_4^{hv}B_2^v & A_4^{hh}B_3^h & B_4^h \end{bmatrix}$$

$$G^{v}=\begin{bmatrix} A_3^{vv}B_1^{v} & 0 & B_3^v & 0 \\ A_4^{vh}A_3^{hv}B_1^{v}+A_4^{vv}A_2^{vh}B_1^{h} & A_4^{vv}B_2^v & A_4^{vh}B_3^h & B_4^v \end{bmatrix} \qquad (16)$$

$$H^{h}=\begin{bmatrix} C_1^h & 0 \\ C_2^hA_1^{hh} & 0 \\ C_3^hA_1^{vh} & C_3^h \\ C_4^hA_3^{hv}A_1^{vh}+C_4^vA_2^{vh}A_1^{hh} & C_4^hA_3^{hh} \end{bmatrix}$$

$$H^{v}=\begin{bmatrix} C_1^v & 0 \\ C_2^hA_1^{hv} & C_2^v \\ C_3^vA_1^{vv} & 0 \\ C_4^hA_3^{hv}A_1^{vv}+C_4^vA_2^{vh}A_1^{hv} & C_4^vA_2^{vv} \end{bmatrix}$$

and

$$J=\begin{bmatrix} D_1 & 0 & 0 & 0 \\ C_2^hB_1^h & D_2 & 0 & 0 \\ C_3^vB_1^v & 0 & D_3 & 0 \\ C_4^hA_3^{hv}B_1^v+C_4^vA_2^{vh}B_1^h & C_4^vB_2^v & C_4^hB_3^h & D_4 \end{bmatrix}$$

We shall denote this invariant lifted model by $\Sigma^L = (F,G,H,J)$, and say that Σ^L is *induced by* the original model $\Sigma(\cdot,\cdot)$, or equivalently, that $\Sigma(\cdot,\cdot)$ *induces* Σ^L.

IV. $(2,2)$–PERIODIC *Roesser* REPRESENTATIONS

In this section we investigate the questions of determining whether a given 2D invariant *Roesser* model is or not induced by a SISO $(2,2)$–periodic one, and of obtaining a corresponding inducing $(2,2)$–periodic *Roesser* model in the case the answer to the previous question is positive.

For this purpose, consider the $(2,2)$–periodic *Roesser* model (11), with horizontal and vertical states of sizes n_h and n_v, respectively. Consider also the corresponding (induced) invariant representation (14) (with horizontal and vertical states of sizes $2n_h$ and $2n_v$, respectively) and define the (n_h+n_v+1)–square matrix

$$\mathcal{M} := \begin{bmatrix} F_{21}^{hh} & F_{21}^{hv} & G_{21}^{h} \\ F_{21}^{vh} & F_{21}^{vv} & G_{21}^{v} \\ H_{41}^{h} & H_{41}^{v} & J_{41} \end{bmatrix}, \qquad (17)$$

where F_{21}^{hh}, F_{21}^{hv}, F_{21}^{vh}, F_{21}^{vv}, G_{21}^{h}, G_{21}^{v}, H_{41}^{h}, H_{41}^{v}, and J_{41} are defined in the obvious way by the block–divisions in (16), more concretely:

$$\begin{bmatrix} F_{11}^{\bullet\star} & F_{12}^{\bullet\star} \\ F_{21}^{\bullet\star} & F_{22}^{\bullet\star} \end{bmatrix} =: F^{\bullet\star} \quad \begin{bmatrix} G_{11}^{\star} & G_{12}^{\star} & G_{13}^{\star} & G_{14}^{\star} \\ G_{21}^{\star} & G_{22}^{\star} & G_{23}^{\star} & G_{24}^{\star} \end{bmatrix} =: G^{\star}$$
$$\begin{bmatrix} H_{11}^{\star} & H_{12}^{\star} \\ H_{21}^{\star} & H_{22}^{\star} \\ H_{31}^{\star} & H_{32}^{\star} \\ H_{41}^{\star} & H_{42}^{\star} \end{bmatrix} =: H^{\star} \quad \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \\ J_{41} & J_{42} & J_{43} & J_{44} \end{bmatrix} =: J \qquad (18)$$

where each of the symbols $\bullet$ and $\star$ represents either h or v. Note that $\mathcal{M}$ can be factored as

$$\mathcal{M} = \underbrace{\left[\begin{array}{c|c} A_4^{hv}A_2^{vh} & A_4^{hh}A_3^{hv} \\ A_4^{vv}A_2^{vh} & A_4^{vh}A_3^{hv} \\ C_4^{v}A_2^{vh} & C_4^{h}A_3^{hv} \end{array}\right]}_{n_h+n_v \text{ columns}} \left.\left[\begin{array}{cc|c} A_1^{hh} & A_1^{hv} & B_1^h \\ \hline A_1^{vh} & A_1^{vv} & B_1^v \end{array}\right]\right\} n_h+n_v \text{ rows}$$

$$=: \left[\begin{array}{c|c} \underbrace{L_1}_{\substack{n_h \\ \text{columns}}} & \underbrace{L_2}_{\substack{n_v \\ \text{columns}}} \end{array}\right] \begin{array}{l} \left[\begin{array}{c} R_1 \\ \hline R_2 \end{array}\right] \end{array} \begin{array}{l} \}\, n_h \text{ rows} \\ \}\, n_v \text{ rows} \end{array}$$

$$=: \left[\begin{array}{c|c} L_1 & L_2 \end{array}\right] \left[\begin{array}{c|c} R_{1,1} & R_{1,2} \\ \hline R_{2,1} & R_{2,2} \end{array}\right]$$

implying that

$$\operatorname{rank} \mathcal{M} \leqslant n_h + n_v \ .$$

Consider now matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ defined as follows:

$$\mathcal{M}_1 := \left[\begin{array}{ccc} F_{11}^{hh} & F_{11}^{hv} & G_{11}^{h} \\ H_{21}^{h} & H_{21}^{v} & J_{21} \\ \hline & L_1R_1 & \end{array}\right] \qquad (19a)$$

(of size $(2n_h+n_v+2)\times(n_h+n_v+1)$), and

$$\mathcal{M}_2 := \left[\begin{array}{ccc} F_{11}^{vh} & F_{11}^{vv} & G_{11}^{v} \\ H_{31}^{h} & H_{31}^{v} & J_{31} \\ \hline & L_2R_2 & \end{array}\right] \qquad (19b)$$

(of size $(2n_v+n_h+2) \times (n_h+n_v+1)$). It is not difficult to see that these matrices can be factored as

$$\mathcal{M}_1 = \underbrace{\begin{bmatrix} A_2^{hh} \\ C_2^h \\ \hline L_1 \end{bmatrix}}_{n_h \text{ columns}} \left[\begin{array}{c|c} R_{1,1} & R_{1,2} \end{array}\right] \tag{20a}$$

and

$$\mathcal{M}_2 = \underbrace{\begin{bmatrix} A_3^{vv} \\ C_3^v \\ \hline L_2 \end{bmatrix}}_{n_v \text{ columns}} \left[\begin{array}{c|c} R_{2,1} & R_{2,2} \end{array}\right], \tag{20b}$$

allowing us to conclude that

$$\operatorname{rank} \mathcal{M}_1 \leqslant n_h \quad \text{and} \quad \operatorname{rank} \mathcal{M}_2 \leqslant n_v \ .$$

Finally, consider the $(n_h + n_v + 1)$–square matrices

$$\mathcal{M}_3 := \left[\begin{array}{c|c|c} F_{22}^{hh} & & G_{23}^h \\ F_{22}^{vh} & L_1 & G_{23}^v \\ H_{42}^h & & J_{43} \end{array}\right] \tag{21a}$$

and

$$\mathcal{M}_4 := \left[\begin{array}{c|cc} & F_{22}^{hv} & G_{22}^h \\ L_2 & F_{22}^{vv} & G_{22}^v \\ & H_{42}^v & J_{42} \end{array}\right]. \tag{21b}$$

These matrices can be factored as

$$\mathcal{M}_3 = \underbrace{\begin{bmatrix} A_4^{hh} \\ A_4^{vh} \\ \hline C_4^h \end{bmatrix}}_{n_h \text{ columns}} \left.\left[\begin{array}{cc|c} A_3^{hh} & A_3^{hv} & B_3^h \end{array}\right]\right\} n_h \text{ rows} \tag{22a}$$

and

$$\mathcal{M}_4 = \underbrace{\begin{bmatrix} A_4^{hv} \\ A_4^{vv} \\ \hline C_4^v \end{bmatrix}}_{n_v \text{ columns}} \left.\left[\begin{array}{cc|c} A_2^{vh} & A_2^{vv} & B_2^v \end{array}\right]\right\} n_v \text{ rows} \ , \tag{22b}$$

respectively, implying that

$$\operatorname{rank} \mathcal{M}_3 \leqslant n_h \quad \text{and} \quad \operatorname{rank} \mathcal{M}_4 \leqslant n_v \ .$$

Conversely, let now $\mathfrak{B}$ be a $(2,2)$–periodic SISO 2D behavior, let $\mathfrak{B}^L$ be the corresponding lifted invariant behavior and assume that $\Sigma^L = (F, G, H, J)$ is a 2D invariant *Roesser* model representation of $\mathfrak{B}^L$, where the horizontal state vector X^h and the vertical state vector X^v have both an even number of components, say $2n_h$ and $2n_v$, respectively, and where the input U and the output Y have both 4 components. Furthermore, decompose matrices F, G, H and J as in (18). Define the matrix $\widetilde{\mathcal{M}}$ as in equation (17), and assume that

$$\operatorname{rank} \widetilde{\mathcal{M}} \leqslant n_h + n_v.$$

Decompose this matrix as

$$\widetilde{\mathcal{M}} =: \left[\begin{array}{c|c} \widetilde{L}_1 & \widetilde{L}_2 \end{array}\right] \begin{bmatrix} \widetilde{R}_1 \\ \hline \widetilde{R}_2 \end{bmatrix}, \tag{23}$$

where the block–matrices $\widetilde{L}_1$ and $\widetilde{L}_2$ have n_h and n_v columns, respectively, while block-matrices $\widetilde{R}_1$ and $\widetilde{R}_2$ have n_h and n_v rows, respectively.

Now, define matrices $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ similarly to what is done for $\mathcal{M}_1$ and $\mathcal{M}_2$ in equations eqs. (19a) and (19b), but using the matrices $\widetilde{L}_1$, $\widetilde{L}_2$, $\widetilde{R}_1$ and $\widetilde{R}_2$ obtained in (23) instead of L_1, L_2, R_1 and R_2, respectively. Assume that

$$\operatorname{rank} \widetilde{\mathcal{M}}_1 \leqslant n_h \quad \text{and} \quad \operatorname{rank} \widetilde{\mathcal{M}}_2 \leqslant n_v$$

and decompose

$$\widetilde{\mathcal{M}}_1 =: \underbrace{\begin{bmatrix} \widetilde{\mathcal{M}}_{1,1}^\ell \\ \widetilde{\mathcal{M}}_{1,2}^\ell \\ \hline \widetilde{\mathcal{M}}_{1,3}^\ell \end{bmatrix}}_{n_h \text{ columns}} \left[\begin{array}{c|c} \widetilde{\mathcal{M}}_{1,1}^r & \widetilde{\mathcal{M}}_{1,2}^r \end{array}\right] \tag{24a}$$

and

$$\widetilde{\mathcal{M}}_2 =: \underbrace{\begin{bmatrix} \widetilde{\mathcal{M}}_{2,1}^\ell \\ \widetilde{\mathcal{M}}_{2,2}^\ell \\ \hline \widetilde{\mathcal{M}}_{2,3}^\ell \end{bmatrix}}_{n_v \text{ columns}} \left[\begin{array}{c|c} \widetilde{\mathcal{M}}_{2,1}^r & \widetilde{\mathcal{M}}_{2,2}^r \end{array}\right], \tag{24b}$$

where $\widetilde{\mathcal{M}}_{1,1}^\ell$ is a (n_h)–square matrix, $\widetilde{\mathcal{M}}_{2,1}^\ell$ is a (n_v)–square matrix while $\widetilde{\mathcal{M}}_{1,2}^\ell$ and $\widetilde{\mathcal{M}}_{1,2}^\ell$ are row matrices and $\widetilde{\mathcal{M}}_{1,2}^r$ and $\widetilde{\mathcal{M}}_{2,2}^r$ are column matrices.

Finally, define matrices $\widetilde{\mathcal{M}}_3$ and $\widetilde{\mathcal{M}}_4$ similarly to what is done for $\mathcal{M}_3$ and $\mathcal{M}_4$ in equations eqs. (21a) and (21b) but using the matrices $\widetilde{\mathcal{M}}_{1,3}^\ell$ and $\widetilde{\mathcal{M}}_{2,3}^\ell$ instead of L_1 and L_2, respectively. Assume that

$$\operatorname{rank} \widetilde{\mathcal{M}}_3 \leqslant n_h \quad \text{and} \quad \operatorname{rank} \widetilde{\mathcal{M}}_4 \leqslant n_v$$

and decompose

$$\widetilde{\mathcal{M}}_3 =: \underbrace{\begin{bmatrix} \widetilde{\mathcal{M}}_{3,1}^\ell \\ \hline \widetilde{\mathcal{M}}_{3,2}^\ell \end{bmatrix}}_{n_h \text{ columns}} \left[\begin{array}{c|c} \widetilde{\mathcal{M}}_{3,1}^r & \widetilde{\mathcal{M}}_{3,2}^r \end{array}\right] \tag{25a}$$

and

$$\widetilde{\mathcal{M}}_4 =: \underbrace{\begin{bmatrix} \widetilde{\mathcal{M}}_{4,1}^\ell \\ \hline \widetilde{\mathcal{M}}_{4,2}^\ell \end{bmatrix}}_{n_v \text{ columns}} \left[\begin{array}{c|c} \widetilde{\mathcal{M}}_{4,1}^r & \widetilde{\mathcal{M}}_{4,2}^r \end{array}\right], \tag{25b}$$

where $\widetilde{\mathcal{M}}^{\ell}_{3,2}$ and $\widetilde{\mathcal{M}}^{\ell}_{4,2}$ are row matrices while $\widetilde{\mathcal{M}}^{r}_{3,2}$ and $\widetilde{\mathcal{M}}^{r}_{4,2}$ are column matrices.

Now, assume that, in decomposition (18), the blocks F^{hh}_{12}, F^{vv}_{12}, G^{h}_{13}, G^{h}_{14}, G^{v}_{12}, G^{v}_{14}, H^{h}_{12}, H^{h}_{22}, H^{v}_{12}, H^{v}_{32}, J_{12}, J_{13}, J_{14}, J_{23}, J_{24}, J_{32} and J_{34} are null, and define a $(2,2)$–periodic SISO *Roesser* model of dimension $(n_h + n_v)$ $\Sigma(\cdot,\cdot) = (A(\cdot,\cdot), B(\cdot,\cdot), C(\cdot,\cdot), D(\cdot,\cdot))$, where the matrices in eqs. (3) and (4) are given by:

$$
\begin{aligned}
\left[\begin{array}{c|c} A(0,0) & B(0,0) \end{array}\right] &= \left[\begin{array}{c|c} \widetilde{\mathcal{M}}^{r}_{1,1} & \widetilde{\mathcal{M}}^{r}_{1,2} \\ \widetilde{\mathcal{M}}^{r}_{2,1} & \widetilde{\mathcal{M}}^{r}_{2,2} \end{array}\right] \\
\left[\begin{array}{c|c} C(0,0) & D(0,0) \end{array}\right] &= \left[\begin{array}{cc|c} H^{h}_{11} & H^{v}_{11} & J_{11} \end{array}\right] \\
\left[\begin{array}{c|c} A(1,0) & B(1,0) \end{array}\right] &= \left[\begin{array}{cc|c} \widetilde{\mathcal{M}}^{\ell}_{1,1} & F^{hv}_{12} & G^{h}_{12} \\ \hdashline \multicolumn{2}{c|}{\widetilde{\mathcal{M}}^{r}_{4,1}} & \widetilde{\mathcal{M}}^{r}_{4,2} \end{array}\right] \\
\left[\begin{array}{c|c} C(1,0) & D(1,0) \end{array}\right] &= \left[\begin{array}{cc|c} \widetilde{\mathcal{M}}^{\ell}_{1,2} & H^{v}_{22} & J_{22} \end{array}\right] \\
\left[\begin{array}{c|c} A(0,1) & B(0,1) \end{array}\right] &= \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\widetilde{\mathcal{M}}^{r}_{3,1}} & \widetilde{\mathcal{M}}^{r}_{3,2} \\ \hdashline F^{vh}_{12} & \widetilde{\mathcal{M}}^{\ell}_{2,1} & G^{v}_{13} \end{array}\right] \\
\left[\begin{array}{c|c} C(0,1) & D(0,1) \end{array}\right] &= \left[\begin{array}{cc|c} H^{h}_{32} & \widetilde{\mathcal{M}}^{\ell}_{2,2} & J_{33} \end{array}\right] \\
\left[\begin{array}{c|c} A(1,1) & B(1,1) \end{array}\right] &= \left[\begin{array}{cc|c} \widetilde{\mathcal{M}}^{\ell}_{3,1} & \widetilde{\mathcal{M}}^{\ell}_{4,1} & \begin{array}{c} G^{h}_{24} \\ G^{v}_{24} \end{array} \end{array}\right]
\end{aligned}
\tag{26}
$$

and

$$\left[\begin{array}{c|c} C(1,1) & D(1,1) \end{array}\right] = \left[\begin{array}{cc|c} \widetilde{\mathcal{M}}^{\ell}_{3,2} & \widetilde{\mathcal{M}}^{\ell}_{4,2} & J_{44} \end{array}\right],$$

(where the matrices are suitably partitioned according to the sizes of the horizontal state (n_h), the vertical state (n_v), the input (1) and the output (1)). It is not difficult to check that the obtained $(2,2)$–periodic *Roesser* model $\Sigma(\cdot,\cdot)$ induces the invariant *Roesser* representation Σ^L of $\mathfrak{B}^L$.

This leads to the following result.

Theorem 4.1: Let $\Sigma^L = (F, G, H, J)$ be a 2D invariant *Roesser* model. Then Σ^L is induced by a 2D $(2,2)$–periodic SISO *Roesser* model if and only if the following conditions are satisfied:

1) In Σ^L, the horizontal state has size $2n_h$ (for some $n_h \in \mathbb{N}$), the vertical state has size $2n_v$ (for some $n_v \in \mathbb{N}$); moreover the number of inputs and the number of outputs are equal to 4.
2) Considering the previously defined notations:
   2.1) $\operatorname{rank} \widetilde{\mathcal{M}} \leqslant n_h + n_v$
   2.2) $\operatorname{rank} \widetilde{\mathcal{M}}_1 \leqslant n_h$ and $\operatorname{rank} \widetilde{\mathcal{M}}_2 \leqslant n_v$
   2.3) $\operatorname{rank} \widetilde{\mathcal{M}}_3 \leqslant n_h$ and $\operatorname{rank} \widetilde{\mathcal{M}}_4 \leqslant n_v$
   2.4) F^{hh}_{12}, F^{vv}_{12}, G^{h}_{13}, G^{h}_{14}, G^{v}_{12}, G^{v}_{14}, H^{h}_{12}, H^{h}_{22}, H^{v}_{12}, H^{v}_{32}, J_{12}, J_{13}, J_{14}, J_{23}, J_{24}, J_{32} and J_{34} are null matrices. $\diamond$

When the conditions of Theorem 4.1 are satisfied, a 2D $(2,2)$–periodic *Roesser* model $\Sigma(\cdot,\cdot)$ that induces Σ^L can be determined as explained in the considerations preceding the theorem. Now, if $\mathfrak{B}$ is a 2D $(2,2)$–periodic behavior whose lifted version $\mathfrak{B}^L$ is represented by Σ^L, it is clear that the set of input/output trajectories generated by $\Sigma(\cdot,\cdot)$ coincides with the $(2,2)$–periodic behavior. In other words, $\Sigma(\cdot,\cdot)$ is a $(2,2)$–periodic *Roesser* model representation of $\mathfrak{B}$, and our goal of realizing $\mathfrak{B}$ has been achieved.

V. Conclusion

This paper is a first step for the development of a realization procedure of periodic 2D behaviors by means of periodic 2D *Roesser* models. Although we have considered the simpler SISO $(2,2)$–periodic case, the presented procedure can be easily generalized for MIMO systems. The generalization to arbitrary periods (P, Q) is also possible, although much more involved.

Even in the simple considered case, several questions were left open. The first one is to determine conditions on the periodic input/output behavior $\mathfrak{B}$ under which the invariant formulation $\mathfrak{B}^L$ is a quarter-plane causal input/output system with inputs u^L and outputs y^L (and hence admits a *Roesser* model realization with such inputs and outputs). Another question has to do with what happens when the invariant *Roesser* realization Σ^L of the lifted version $\mathfrak{B}^L$ turns out not to be induced by any periodic *Roesser* realization. This problem has been solved in [4] for the 1D case, but, although we conjecture that a similar procedure can be used here, its implementation in the 2D case will certainly be far more complicated. An equally interesting open problem is related to the question of minimality: will minimal invariant realizations Σ^L originate minimal periodic realizations $\Sigma(\cdot,\cdot)$? Due to the difficulty in characterizing minimality for general *Roesser* models, this question should be easier to answer in the particular case where Σ^L is a *Roesser* model of separable type (for which minimality is easier understood).

These open questions are currently being studied and will be reported in future work.

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