MTNS 2016 - extended abstract

On periodically time-varying convolutional codes

Ricardo Pereira[1] and Paula Rocha[2]

I. Extended Abstract

Convolutional codes [4] are an important type of error correcting codes that can be represented as a time-invariant discrete linear system over a finite field [8]. They are used to achieve reliable data transfer, for instance, in mobile communications, digital video and satellite communications [9]. In particular, periodically time-varying convolutional codes have attracted the attention of several researchers [2], [3]. One of the advantages of this type of codes is that they can have better distance properties than the best time-invariant convolutional code of the same rate and total encoder memory [7].

In this work we consider convolutional codes $\mathcal{C}$ with P-periodic encoders, i.e.:

$$\begin{aligned} \mathcal{C} := \{w : w(Pl+t) = (G^t(D)u)(Pl+t);\\ t = 0, ..., P-1; l = 0, 1, ...\}, \end{aligned} \tag{1}$$

where each $G^t(D)$ is a $n \times k$ polynomial matrix over a finite field $\mathbb{F}$, i.e., $G^t(D) \in \mathbb{F}^{n\times k}[D]$, D represents the shift $Du(l) = u(l-1)$, and u is an information sequence in $\mathbb{F}^k$.

Inspired by the ideas developed in [6] and [1] for the case of behaviors, considering the linear map

$$L_p : (\mathbb{R}^n)^{\mathbb{Z}} \to \left(\mathbb{R}^{Pn}\right)^{\mathbb{Z}}$$

defined by

$$(L_p w)(l) := \begin{bmatrix} w(Pl) \\ w(Pl+1) \\ \vdots \\ w(Pl+P-1) \end{bmatrix}, \quad P \in \mathbb{N}$$

we associate with $\mathcal{C}$ a time-invariant convolutional code $\mathcal{C}^L$, the "lifted" version of $\mathcal{C}$, defined as

$$\mathcal{C}^L := \left\{ \widetilde{w} \in \left(\mathbb{R}^{Pn}\right)^{\mathbb{Z}} | \widetilde{w} = L_p w, \ w \in \mathcal{C} \right\}.$$

Note that, since

$$(G^t(D)u)(Pl+t) = \left(\left(D^{-t}G^t(D)\right)u\right)(Pl),$$

Equation (1) can also be written as

$$(\Omega_{P,n}(D)w)(Pl) = (G(D)u)(Pl), l = 0, 1, \ldots,$$

with

$$\Omega_{P,n}(D) = \begin{bmatrix} I_n \\ D^{-1}I_n \\ \vdots \\ D^{-P+1}I_n \end{bmatrix}$$

and

$$G(D) = \begin{bmatrix} G^0(D) \\ D^{-1}G^1(D) \\ \vdots \\ D^{-P+1}G^{P-1}(D) \end{bmatrix} \in \mathbb{R}^{Pn\times k}.$$

Moreover, by decomposing the matrix G as

$$G(D) = G^L\left(D^P\right)\Omega_{P,n}(D)$$

with

$$G^L(D) = \left[G^{L_0}(D) \quad G^{L_1}(D) \quad \cdots \quad G^{L_{P-1}}(D)\right],$$

the lifted code can be represented as

$$\mathcal{C}^L := \left\{\widetilde{w} : \widetilde{w}(l) = (G^L(D)\widetilde{u})(l), l = 0, 1, ...\right\},$$

where $\widetilde{w} = L_P w$ and $\widetilde{u} = L_P u$.

In the sequel, we consider convolutional codes $\mathcal{C}$ with 2-periodic encoders, i.e., such $P = 2$ and

$$G^t(D) = G_0^t + G_1^t D + \cdots + G_N^t D^N \in \mathbb{F}^{n\times k}[D], \ t = 0, 1.$$

Moreover we assume that the matrices G_0^0, G_0^1, G_N^0 and G_N^1 are full column rank. This implies that the matrices $G^t(D)$ are column reduced (see [5] for a definition), $t = 0, 1$. Assuming also that $G^t(D)$ are right-prime, $t = 0, 1$, we have that $G^t(D)$ are minimal encoders, $t = 0, 1$ [4]. Minimal encoders are particularly important since their McMillan degrees correspond to the code degree, which is a measure of its complexity.

The parameters of the encoders G^t, $t = 0, 1$, are (n, k, δ), where $\delta = kN$ is the degree of the code $\mathcal{C}^t$ generated by $G^t(D)$.

The generalized Singleton bound for each code $\mathcal{C}^t$ is

$$\begin{aligned} & (n-k)\left(\left\lfloor \tfrac{\delta}{k} \right\rfloor + 1\right) + \delta + 1 \\ = \ & (n-k)\left(\left\lfloor \tfrac{kN}{k} \right\rfloor + 1\right) + kN + 1 \\ = \ & nN + n - k + 1. \end{aligned} \tag{2}$$

By definition,

$$\begin{aligned} G^0(D) &= G_0^0 + G_1^0 D + \cdots + G_N^0 D^N + 0D^{N+1} \\ DG^1(D) &= 0 + G_0^1 D + G_1^1 D^2 + \cdots + \\ & \quad G_{N-1}^1 D^N + G_N^1 D^{N+1} \end{aligned}$$

[1]CIDMA - Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Aveiro, Portugal ricardopereira@ua.pt
[2]SYSTEC, Faculty of Engineering, University of Porto, Portugal mprocha@fe.up.pt

which can be decomposed as

$$\begin{bmatrix} G^0(D) \\ DG^1(D) \end{bmatrix} = \left(\begin{bmatrix} G_0^0 \\ 0 \end{bmatrix} + \begin{bmatrix} G_2^0 \\ G_1^1 \end{bmatrix} D^2 + \dots \right) + \left(\begin{bmatrix} G_1^0 \\ G_0^1 \end{bmatrix} + \begin{bmatrix} G_3^0 \\ G_2^1 \end{bmatrix} D^2 + \dots \right) D$$

The matrix G^L can be written as

$$G^L(D) = \begin{bmatrix} G_0^0 & G_1^0 \\ 0 & G_0^1 \end{bmatrix} + \begin{bmatrix} G_2^0 & G_3^0 \\ G_1^1 & G_2^1 \end{bmatrix} D + \dots + \begin{bmatrix} G_N^0 & 0 \\ G_{N-1}^1 & G_N^1 \end{bmatrix} D^{\lceil \frac{N}{2} \rceil} \quad (3)$$

We will prove the following result.

Theorem 1: $G^L(D)$ is a minimal time-invariant encoder.

Proof: Since, by hypothesis, G_0^0, G_0^1, G_N^0 and G_N^1 are full column rank, we have that $G^L(D)$ is column reduced. We prove now that $G^L(D)$ is right-prime. By hypothesis, $\begin{bmatrix} G^0(D) \\ DG^1(D) \end{bmatrix}$ is right-prime and

$$\begin{bmatrix} G^0(D) \\ DG^1(D) \end{bmatrix} = G^L\left(D^2\right) \begin{bmatrix} I_n \\ DI_n \end{bmatrix} \quad (4)$$

If $G^L\left(D^2\right) = \left[G^{L_0}\left(D^2\right) \quad G^{L_1}\left(D^2\right)\right]$ is not right-prime, then $G^{L_0}\left(D^2\right)$ and $G^{L_1}\left(D^2\right)$ have a squared non unimodular common factor, $F(D) \in \mathbb{F}^{k \times k}[D]$, i.e.,

$$G^{L_0}\left(D^2\right) = \widetilde{G}^{L_0}(D)F(D) \text{ and } G^{L_1}\left(D^2\right) = \widetilde{G}^{L_1}(D)F(D)$$

Then, by equation (4),

$$\begin{aligned} \begin{bmatrix} G^0(D) \\ DG^1(D) \end{bmatrix} &= \left[\widetilde{G}^{L_0}(D)F(D) \quad \widetilde{G}^{L_1}(D)F(D)\right] \begin{bmatrix} I_n \\ DI_n \end{bmatrix} \\ &= \widetilde{G}^{L_0}(D)F(D) + \widetilde{G}^{L_1}(D)F(D)D \\ &= \widetilde{G}^{L_0}(D)F(D) + \widetilde{G}^{L_1}(D)DF(D) \\ &= \left(\widetilde{G}^{L_0}(D) + \widetilde{G}^{L_1}(D)D\right)F(D) \end{aligned}$$

which is a contradiction because $\begin{bmatrix} G^0(D) \\ DG^1(D) \end{bmatrix}$ is right-prime. Hence $G^L\left(D^2\right)$ is right-prime and therefore also $G^L(D)$ is right-prime. ■

The parameters of the encoder $G^L(D)$ are $(2n, 2k, kN)$ since it can be shown, by equation (3), that if N is even the degree is $\delta = 2k\left\lceil \frac{N}{2} \right\rceil = 2k\frac{N}{2} = kN$ and if N is odd the degree is $\delta = k\left\lceil \frac{N}{2} \right\rceil + k\left\lfloor \frac{N}{2} \right\rfloor = kN$. Then the generalized Singleton bound is

$$(2n - 2k)\left(\left\lfloor \frac{kN}{2k} \right\rfloor + 1\right) + kN + 1$$

$$= \begin{cases} nN + 2n - 2k + 1, & \text{if} N \text{ even} \\ nN + n - k + 1, & \text{if } N \text{ odd} \end{cases}$$

which, is equal to the bound of each periodic encoder (2) when N is odd, but has an increase of $n-k$ when N is even.

This result is similar to the one derived in [2] using a different reasoning.

Obtaining a larger bound for the odd case is encouraging from the point of view of achieving a larger distance for the periodic case. However, the question whether the obtained bound can be reached is still the subject of current investigation.

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References

[1] J.C. Aleixo, J.W. Polderman and P. Rocha, State space representation of SISO periodic behaviors, *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, Orlando, FL, USA, 12-15 Dec. 2011, pp. 1545-1550.

[2] J-J Climent, V. Herranz, C. Perea and V. Tomás, A construction of periodically time-varying convolutional codes, *Proceedings of the International Symposium on the Mathematical Theory of Networks and Systems (MTNS 2008), Virginia Institute of Technology, Blacksburg, USA*, 2008.

[3] Q. Hu and L.C. Perez, Some periodic time-varying convolutional codes with free distance achieving the Heller bound, *Proceedings of the IEEE International Symposium on Information Theory*, pp. 247, 2001.

[4] R. Johannesson and K.S. Zigangirov, Fundamentals of convolutional coding, IEEE press, New York, 1999.

[5] T. Kailath, Linear Systems, Prentice-Hall, 1980.

[6] M. Kuijper and J. C. Willems, A behavioral framework for periodically time-varying systems, *Proceedings of the 36th IEEE Conference on Decision & Control - CDC'97*, Vol. 3, San Diego, California USA, 10-12 Dec. 1997, pp. 2013-2016.

[7] M. Mooser, Some periodic convolutional codes better than any fixed code, *IEEE Transactions on Information Theory*, Vol. 29, No 5, pp. 750-751, 1983.

[8] J. Rosenthal, Connections between linear systems and convolutional codes, *Codes, Systems, and Graphical Models*, B. Marcus and J. Rosenthal, eds, Springer New York, Vol. 123, pp. 39-66, 2001.

[9] A.J. Viterbi, Convolutional codes and their performance in communication systems, *IEEE Transactions on Communication Technology*, Vol. 19, No 5, pp. 751-772, 1971.