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Iranian Journal of Science and Technology, Transactions A: Science

ISSN 1028-6276

Iran J Sci Technol Trans Sci
DOI 10.1007/s40995-017-0176-2

Springer

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RESEARCH PAPER

A Hybrid Direct–Indirect Approach for Solving the Singular Optimal Control Problems of Finite and Infinite Order

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Received: 24 July 2016 / Accepted: 23 February 2017

Abstract This paper presents a hybrid approach to solve singular optimal control problems. It combines the direct Euler method with a modified indirect shooting method. The presented method circumvents the main difficulties and drawbacks of both the direct and indirect methods, when applied to the singular optimal control problems. This method does not require a priori knowledge of the switching structure of the solution and it can be applied to finite or infinite order singular optimal control problems. It provides not only an approximate optimal solution for the problem but, remarkably, it also produces the switching times. We illustrate the features of this new approach treating numerically through two optimal control problems, one of finite order and the other with infinite order.

Keywords Singular optimal control problem · Hybrid method · Direct Euler method · Indirect shooting method

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1 Introduction

A classic and challenging subject in the optimal control field is singular optimal control problem (SOCP). In these problems, Pontryagin's maximum principle fails to directly determine the optimal control over at least one interval. SOCPs arise in many areas, ranging from aerospace engineering (Goddard 1920; Powers and McDanell 1971) to robotic (Chen and Desrochers 1993), industrial chemistry (Oberle and Sothmann 1999; Luus and Okongwu 1999) and biological science (Ledzewicz et al. 2011; Ledzewicz and Schättler 2008).

In many practical optimal control problems and especially SOCPs, the analytical solution cannot be obtained and we must resort to numerical approximate solution. Numerical methods for optimal control are classified into the *indirect* and *direct* approaches. Indirect methods use necessary conditions of optimality. These conditions yield a Hamiltonian boundary value problem. Such problem is commonly solved numerically by the shooting or collocations methods. In a direct method, without using necessary optimal conditions, the optimal control problem is transcribed to a nonlinear programming problem which can be solved by well-developed algorithms and softwares.

As advantages of the indirect methods, we can refer to high accuracy in the solution and the assurance that the solution satisfies the optimality conditions. However, the indirect methods suffer from two major disadvantages. First, the need to a good initial guess, not only for the state trajectories but also for the costates. Second, the need to a priori knowledge of the control structure. On the other hand, the direct methods have much larger radii of convergence than the indirect methods and unlike the indirect methods and also for problems with path constraints, the

Published online: 04 April 2017

switching structure of control does not need to be known a priori. In the direct method, nevertheless, the obtained solution is less accurate than the one obtained by the indirect approaches.

Despite the great developments in the direct and indirect methods for numerical solution of optimal control problems with path constraint (Maurer and Osmolovskii 2013; Razmjooy and Ramezani 2016; Marzban and Hoseini 2015; Azhmyakov et al. 2015), the solution of SOCPs has remained a challenge. Accordingly, the simulation and numerical approximation of SOCPs have received considerable attention. For instance, we can refer to indirect multiple shooting method (Maurer 1976; Aronna et al. 2013), direct shooting method (Vossen 2010), iterative dynamic programming method (Luus 1992) and continuation approach (Bonnans et al. 2008). The main difficulties in the indirect solving SOCPs lie in determining the switching structure of optimal control function and extreme sensitivity to initial guess (Maurer 1976). On the other hand, in the direct solution of SOCPs, the accuracy of the obtained solution, especially in singular arcs, is not satisfactory. These technical difficulties have caused serious barriers for the solution of SOCPs, by both the direct and indirect methods.

In this paper, we show that these mentioned difficulties in solving SOCPs can be overcome by combining the direct and indirect methods. Such an approach is called *hybrid method*, which combines some of the best features of both the direct and indirect methods to develop a robust and accurate numerical method (von Stryk and Bulirsch 1992). The aim of this hybrid method is to obtain accurate results for SOCPs so that the user need not provide a good initial guess and need not know a priori the switching structure. The presented hybrid method combines the direct and indirect methods in two steps. In the first step, the direct Euler method is used (Betts and Huffman 1992). Based on our experience, the Euler method is a robust method that provides an approximate solution, which is sufficient for detecting the structure of optimal control. However, the obtained solution is not accurate and the position of switching points is not obtained accurately. To improve the low accuracy of the direct Euler method, in the second stage, we propose an adaptive shooting method which is initialized by the obtained information of Euler method in the first stage.

The paper is organized as follows. In Sect. 2, the formulation of SOCPs and some necessary definitions are reviewed. In Sect. 3, by combining the direct Euler method and a modified indirect shooting method, a hybrid method to solve SOCPs is presented. The proposed hybrid method is applied to two examples in Sect. 4. Finally, conclusions are given in Sect. 5.

2 Statement of the Problem and Preliminaries

Consider the following optimal control problems, in which the scalar control function is appeared linearly in dynamic system and cost functional is of Mayer type

$$\min\ \mathcal{J}(u,t_f) = g(\mathbf{x}(t_f),t_f), \tag{1a}$$

$$\text{s.t. } \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t),u(t),t) = \mathbf{f}_1(\mathbf{x}(t),t) + \mathbf{f}_2(\mathbf{x}(t),t)u(t), \tag{1b}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{1c}$$

$$\boldsymbol{\psi}(\mathbf{x}(t_f),t_f) = 0, \tag{1d}$$

$$u \in \mathcal{U} := \{u \mid u(\cdot) \in [u^{\min},u^{\max}] \text{ is piecewise continuous}\}. \tag{1e}$$

Here, t_f may be fixed or free, the state variable $\mathbf{x}(t) = [x_1(t),\ldots,x_p(t)]^T \in \mathbb{R}^p$ is a continuous vector function and $u(t) \in \mathbb{R}$ is a piecewise continuous function. Furthermore, the functions $g, \mathbf{f}_1, \mathbf{f}_2$ and $\boldsymbol{\psi}$ are sufficiently continuously differentiable in all arguments and defined by the following mappings:

$$g : \mathbb{R}^{p+1} \to \mathbb{R}, \quad \mathbf{f}_1, \mathbf{f}_2 : \mathbb{R}^{p+1} \to \mathbb{R}^p,$$
$$\boldsymbol{\psi} : \mathbb{R}^{p+1} \to \mathbb{R}^r,\ 0 \le r \le p.$$

The Hamiltonian function for the above problem is defined by:

$$\mathscr{H}(\mathbf{x},u,\boldsymbol{\lambda},t) := \boldsymbol{\lambda}^T\mathbf{f}_1(\mathbf{x},t) + \boldsymbol{\lambda}^T\mathbf{f}_2(\mathbf{x},t)u, \tag{2}$$

where $\boldsymbol{\lambda}(t) = [\lambda_1(t),\ldots,\lambda_p(t)]^T \in \mathbb{R}^p$ is the so-called adjoint or costate vector function.

According to the Pontryagin's minimum principle Pontryagin et al. (1962), the solution of the problem (1) requires minimization of the Hamiltonian function (2) with respect to $u^* \in \mathcal{U}$ along the entire trajectories, which satisfy (1b), (1d) and the following costate equations

$$\dot{\boldsymbol{\lambda}}^*(t) = -\mathscr{H}_{\mathbf{x}}(\mathbf{x}^*(t),u^*(t),\boldsymbol{\lambda}^*,t), \tag{3}$$

and the following terminal conditions

$$\boldsymbol{\lambda}^*(t_f) = \ell_{\mathbf{x}_f}(u^*(t_f),t_f^*,\boldsymbol{\rho}), \tag{4a}$$

$$\mathscr{H}|_{t=t_f} = -\ell_{t_f}(u^*(t_f),t_f^*,\boldsymbol{\rho}), \text{if } t_f \text{ is free}, \tag{4b}$$

where

$$\ell(\mathbf{x}_f,t_f,\boldsymbol{\rho}) := g(\mathbf{x}_f,t_f) + \boldsymbol{\rho}^T\boldsymbol{\psi}(\mathbf{x}_f,t_f).$$

In the considered problem, u appears linearly in the dynamic equations. So, the Hamiltonian is linear in the control u as well. The factor u in the Hamiltonian is called switching function and denoted by:

$$\sigma(\mathbf{x},\boldsymbol{\lambda},t) := \boldsymbol{\lambda}^T\mathbf{f}_2(\mathbf{x},t).$$

As a result of Pontryagin's minimum principle, if in the time interval $[t_1,t_2] \in [t_0,t_f]$, the switching function $\sigma(t)$ be positive (negative), then $u(t)$ takes the smallest (largest) admissible control value $u^{\min}$ ($u^{\max}$). So, if $\sigma(t)$ in the time interval $[t_1, t_2] \in [t_0, t_f]$ has finite isolated zeros, then the optimal control $u^*(t)$ fulfills:

$$u^*(t) \in \{u^{\min}, u^{\max}\}, \qquad \forall t \in [t_1, t_2],$$

which, in this case, the u is called *bang-bang* in the interval $[t_1, t_2]$. However, if there is a time interval $[t_1, t_2] \in [t_0, t_f]$ in which the switching function $\sigma(t)$ vanishes, then Pontryagin's minimum principle provides no information about how to select $u^*(t)$. In this case, it is said that the problem is *singular* and the interval $[t_1, t_2]$ is called a *singular interval*, in addition, the control over a singular interval is referred to *singular arc*.

In summary, minimization of the Hamiltonian function leads to the following control law (Pontryagin et al. 1962; Kirk 2012):

$$u^*(t) = \begin{cases} u^{\min}, & \text{if} \quad \sigma(t) > 0, \\ u^{\max}, & \text{if} \quad \sigma(t) < 0, \\ u^{\sin}, & \text{if} \quad \sigma(t) = 0. \end{cases}$$

Accordingly, in general, singular optimal control contains both bang-bang and singular sub-arcs. Each point that is a transition between one bang-bang arc and another bang-bang or singular arc is called *switching point*.

2.1 Order of Singular Optimal Control Problems

Note that, $\frac{d}{dt}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t)$ is explicitly a function of $\mathbf{x}$, $\boldsymbol{\lambda}$, $\dot{\mathbf{x}}$, $\dot{\boldsymbol{\lambda}}$ and t. By substituting $\dot{\mathbf{x}}$ and $\dot{\boldsymbol{\lambda}}$ from (1b) and (3), $\frac{d}{dt}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t)$ can be expressed as a function of $\mathbf{x}$, $\boldsymbol{\lambda}$ and t. It is easy to show that the control function u does not appear in $\frac{d}{dt}\sigma$ (Lewis 1980). By repeating this manner, $\frac{d^j}{dt^j}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t)$ can be expressed as a function of $\mathbf{x}$, $\boldsymbol{\lambda}$, t and maybe u. Furthermore, if u appears in $\frac{d^j}{dt^j}\sigma$, then it appears linearly (Lewis 1980). It is possible that the control u does not appear in $\frac{d^j}{dt^j}\sigma$ for any j. However, if w be the first integer number, in which u appears in $\frac{d^w}{dt^w}\sigma$, then w is always even (Lewis 1980; Lamnabhi-Lagarrigue 1987). In the former case, the order of SOCP is defined to be infinite and in the latter case, the integer number $\kappa = \frac{w}{2}$ is called order of the singular problem.

Definition 1 [*order of singular problem* (Lamnabhi-Lagarrigue 1987)] The integer number κ is called order of the singular problem when 2κ is the lowest order derivative of switching function σ such that u appears explicitly. In other words

$$\frac{d^{2\kappa}}{dt^{2\kappa}}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t) \equiv e(\mathbf{x}, \boldsymbol{\lambda}, t) + d(\mathbf{x}, \boldsymbol{\lambda}, t)u, \qquad d \neq 0. \tag{5}$$

If u never appears explicitly in the differentiation process, then the optimal control problem is called an infinite-order singular problem.

Let the problem (1) be a singular problem of order κ and $[t_1, t_2]$ be the singular interval. So, the control u appears explicitly and linearly in the 2κ-th derivative of the switching function σ with respect to t, as Eq. (5). Therefore, by noting that $\sigma = 0$ for $t \in [t_1, t_2]$, we conclude

$$\frac{d^{2\kappa}}{dt^{2\kappa}}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t) = 0 = e(\mathbf{x}, \boldsymbol{\lambda}, t) + d(\mathbf{x}, \boldsymbol{\lambda}, t)u, \qquad d \neq 0. \tag{6}$$

Now, by solving the Eq. (6) for u, we get

$$u = u(\mathbf{x}, \boldsymbol{\lambda}, t) = -\frac{d(\mathbf{x}, \boldsymbol{\lambda}, t)}{e(\mathbf{x}, \boldsymbol{\lambda}, t)}, \quad t \in [t_1, t_2]. \tag{7}$$

In summary, if the singularity order of the problem be finite, then by successive differentiation of the switching function, the control function u can be expressed as a function of $\mathbf{x}$, $\boldsymbol{\lambda}$ and t.

3 Proposed Hybrid Method for Solving SOCPs

In this section, a hybrid method to solve SOCPs which contains two stages is proposed. In the first stage, we use a robust method which can detect the structure of optimal control. Basically, for this purpose, the direct methods are more suitable than the indirect ones. Apart from various direct transcription methods developed for solving optimal control problems, we select Euler transcription method for the first stage. Of course, there are more sophisticated direct methods, such as Simpson (Betts and Huffman 1992) and Pseudo-spectral methods (Elnagar et al. 1995). However, based upon our experience, in singular problems, Euler method is more robust than other methods. Indeed, Euler method has the advantage of directly finding appropriate approximate solutions, which is sufficient for detecting the structure of optimal control. But, in Pseudo-spectral and Simpson methods, some oscillations are appeared in the obtained solution, which cause trouble in detecting the structure of optimal control.

The obtained solution by Euler method offers information on the structure of optimal control. It also provides estimation for the costate variables. However, the accuracy of the obtained solution is not satisfactory. Thus, we have to improve the obtained results of the first stage by another one. In the second stage, we use a modified shooting method which leads to very accurate results. It goes without saying that the obtained information and estimations of the first stage are used to initialize this method.

3.1 Stage I: Direct Euler Method

The basic approach to solve the optimal control problem by Euler transcription has been presented in details in (Betts 1994; Betts and Huffman 1993). However, in what follows, we shortly recall Euler method for solving SOCP (1).

At first, the domain $[0, t_f]$ is mapped to [0, 1], via the affine transformation $t \longmapsto \frac{t}{t_f}$. Consequently, the optimal control problem given in (1) is converted to the following problem

$$\min \mathcal{J}(u, t_f) = g(\mathbf{x}(1), t_f), \tag{8a}$$

$$s.t.\ \dot{\mathbf{x}} = t_f \mathbf{f}(\mathbf{x}(t), u(t), t), \quad 0 < t < 1, \tag{8b}$$

$$\mathbf{x}(0) = \mathbf{x}_0, \tag{8c}$$

$$\boldsymbol{\psi}(\mathbf{x}(1), t_f) = 0, \tag{8d}$$

$$u \in \mathcal{U}. \tag{8e}$$

It is noting that, by applying the transformation, the symbols of variables will change and new symbols should be used instead. However, For simplicity, we will retain the symbols already used.

Now, we divide the interval [0, 1] into the n equal parts by the following mesh points

$$\tau_k = kh, \ \ k = 0, \ldots, n,$$

where $h = \frac{1}{n}$. We note that, for the sake of simplicity, the above uniform mesh is taken, although Euler method can be extended to variable meshes. Utilization of the Euler discretization scheme to the problem (8) results the following NLP problem:

$$\min \mathcal{J}(u, t_f) = g(\mathbf{x}_n, t_f), \tag{9a}$$

$$s.t.\ \mathbf{x}_{k+1} = \mathbf{x}_k + ht_f \mathbf{f}(\mathbf{x}_k, u_k, \tau_k), \quad k = 0, \ldots, n-1, \tag{9b}$$

$$\boldsymbol{\psi}(\mathbf{x}_n, t_f) = 0, \tag{9c}$$

where $\mathbf{x}_k$, u_k, $\mathbf{f}_k$ and $\boldsymbol{\psi}_k$ stand for $\mathbf{x}(\tau_k)$, $u(\tau_k)$, $\mathbf{f}(\tau_k, \mathbf{x}(\tau_k), u(\tau_k))$ and $\boldsymbol{\psi}(\tau_k)$ respectivly. Note that, the variables of NLP (9) are $\mathbf{x}_k,\ k = 1, \ldots, n,\ u_k,\ k = 0, \ldots, n$ and maybe t_f. Using an optimization solver, we can solve the NLP (9) and the approximations of state and control functions are obtained in the mesh points $\tau_k, k = 0, \ldots, n$.

As we see, Euler method does not concern with costate variables. Thus, it seems that by Euler method we cannot provide any information about costate variables. Nevertheless, it is demonstrated that the Lagrange multipliers(dual variables) of NLP (9) are related to the costate variables Betts (2010) and these multipliers can be used to estimate the costate variables in the mesh points. See the following theorem.

Theorem 1 (Betts 2010) *Let $\boldsymbol{\mu}_k \in \mathbb{R}^p$ be the Lagrange multiplier associated with the constraints $\mathbf{x}_{k+1} = \mathbf{x}_k + h\mathbf{f}(\mathbf{x}_k, u_k, \tau_k)$ in NLP (9), then $\boldsymbol{\mu}_k$ is a first order estimation for the costate variable at the grid points, i.e.*

$$\|\boldsymbol{\mu}_k - \boldsymbol{\lambda}(\tau_k)\| = \mathcal{O}(h), \quad k = 0, \ldots, n.$$

According to the above theorem, one can use the Lagrange multipliers obtained from the solution NLP (9) to estimate the costate variables.

3.2 Stage II: Adaptive Indirect Shooting Method

By Stage I, the structure of optimal control, i.e. the sequence and type of subarcs are determined. However, the accurate positions of switching points are not attained, and moreover, singular control function does not compute precisely. These drawbacks are resolved in Stage II. For this purpose, we consider an indirect scenario in which, by utilizing Pontryagin's maximum principle and the obtained information of Stage I, a Multi-Domain Boundary Value Problem(MDBVP) is obtained. Then, using shooting method, this MDBVP is solved and accurate solution of the singular optimal control problem is obtained. In addition, the accurate positions of the switching points are captured.

3.2.1 *Converting the Singular Optimal Control Problem to a Multi-domain Boundary Value Problem*

At first, based upon the results of Stage I, let s be the number of switching points. We consider the decision variables $t_1, \ldots, t_s$ as switching points, where

$$t_0 \leq t_1 \leq \ldots \leq t_s \leq t_{s+1} = t_f. \tag{10}$$

Therefore, the control function can be expressed as:

$$u(t) = \begin{cases} u^{[0]}(t), & t \in [t_0, t_1], \\ u^{[1]}(t), & t \in [t_1, t_2], \\ \quad\vdots & \\ u^{[s]}(t), & t \in [t_s, t_{s+1}]. \end{cases} \tag{11}$$

Here, the control function in the k-th subinterval $[t_k, t_{k+1}]$ is denoted by $u^{[k]}(t)$. Note that, according to the obtained structure in Stage I, we know that in each sub-domain $[t_k, t_{k+1}]$, the control function $u^{[k]}(t)$ is singular or takes its maximum value (i.e. $u^{\max}$) or minimum value (i.e. $u^{\min}$).

In the cases that the singular problem has a finite order, as mentioned in Sect. 2, the control in the singular arcs can be expressed based on state and costate functions. In this way, if the control in $[t_k, t_{k+1}]$ be singular, then the control function in this interval can be expressed as:

$$u^{[k]}(t) = u(t; \mathbf{x}^{[k]}, \boldsymbol{\lambda}^{[k]}, t_k, t_{k+1}). \tag{12}$$

On the contrary, in the cases that the order of the singular optimal control problem be infinite, we cannot express the control function by state and costate functions. In this case, if the control in kth interval be singular, then we approximate $u^{[k]}(t)$ by the following expansion

$$u^{[k]}(t) \simeq \sum_{i=0}^{m} \alpha_i P_i\left(2\frac{t-t_k}{t_{k+1}-t_k}-1\right), \tag{13}$$

where, $P_i(\cdot)$ is the well-known Legendre function of degree i and $\alpha_i,\ i=0,\ldots,n$ are unknown coefficients.

Based on (10) and (11), the state and costate equations (1b) and (3), in the optimality conditions, are reformulated to the following equations:

$$\begin{cases} \dot{\mathbf{x}}^{[k]}(t) = \mathscr{H}_{\boldsymbol{\lambda}}(t,\mathbf{x}^{[k]},u^{[k]},\boldsymbol{\lambda}^{[k]}), \\ \dot{\boldsymbol{\lambda}}^{[k]}(t) = -\mathscr{H}_{\mathbf{x}}(t,\mathbf{x}^{[k]},u^{[k]},\boldsymbol{\lambda}^{[k]}), \end{cases} t\in[t_k,t_{k+1}], \quad k=1,\ldots,s, \tag{14}$$

where, $\mathbf{x}^{[k]}(t)=[x_1^{[k]}(t),\ldots,x_p^{[k]}(t)]^T$ and $\boldsymbol{\lambda}^{[k]}(t)=[\lambda_1^{[k]}(t),\ldots,\lambda_p^{[k]}(t)]^T$ are the state function $\mathbf{x}(t)$ and costate function $\boldsymbol{\lambda}(t)$ in the k-th subinterval $[t_k,t_{k+1}]$, respectively. It is noted that, in (14), the control $u^{[k]}$ is defined by (12) or (13).

Now, we associate the following boundary conditions with the above system of differential equations

$$\mathbf{x}^{[0]}(t_0)=\mathbf{x}_0, \tag{15a}$$

$$\mathbf{x}^{[k]}(t_{k+1})=\mathbf{x}^{[k+1]}(t_{k+1}), \quad k=0,\ldots,s-1, \tag{15b}$$

$$\boldsymbol{\lambda}^{[k]}(t_{k+1})=\boldsymbol{\lambda}^{[k+1]}(t_{k+1}), \quad k=0,\ldots,s-1, \tag{15c}$$

$$\sigma(\mathbf{x}^{[k]}(t_k),\boldsymbol{\lambda}^{[k]}(t_k),t_k)=0, \quad k=1,\ldots,s, \tag{15d}$$

$$\boldsymbol{\psi}(\mathbf{x}^{[s]}(t_f),t_f)=0, \tag{15e}$$

$$\boldsymbol{\lambda}^{[s]}(t_f)=\ell_{\mathbf{x}_f}(\mathbf{x}^{[s]}(t_f),t_f,\boldsymbol{\rho}), \tag{15f}$$

$$\begin{aligned}&\mathscr{H}(\mathbf{x}^{[s]}(t_f),u^{[s]}(t_f),\boldsymbol{\lambda}^{[s]}(t_f),t_f)\\ &= -\ell_{t_f}(\mathbf{x}^{[s]}(t_f),t_f,\boldsymbol{\rho}), \text{if } t_f \text{ is free.}\end{aligned} \tag{15g}$$

The Eqs. (15b) and (15c) are considered to guarantee the continuity of state and costate functions, respectively. Moreover, as we know from the optimality conditions, the value of switching function σ at switching points $t_k, k=1,\ldots,s$ must be vanished. Hence, the conditions (15d) were considered.

3.2.2 Shooting Method for Solving the Resulted MDBVP

Associate with MDBVP (14)–(15), the following initial value problem is considered

$$\begin{cases} \dot{\mathbf{x}}^{[k]}(t)=\mathscr{H}_{\boldsymbol{\lambda}}(t,\mathbf{x}^{[k]},u^{[k]},\boldsymbol{\lambda}^{[k]}), & t\in[t_k,t_{k+1}], \quad k=1,\ldots,s,\\ \dot{\boldsymbol{\lambda}}^{[k]}(t)=-\mathscr{H}_{\mathbf{x}}(t,\mathbf{x}^{[\mathbf{k}]},u^{[k]},\boldsymbol{\lambda}^{[k]}), & t\in[t_k,t_{k+1}], \quad k=1,\ldots,s,\\ \mathbf{x}^{[k]}(t_k)=\boldsymbol{\theta}_k, & k=0,1,\ldots,s\\ \boldsymbol{\lambda}^{[k]}(t_k)=\boldsymbol{\zeta}_k, & k=0,1,\ldots,s, \end{cases} \tag{16}$$

where $\boldsymbol{\theta}_0=\mathbf{x}_0$. Let $\mathbf{z}$ be the vector of initial values and unknown parameters in the above initial value problem. Note that, in the case of finite order, the unknown parameters are the switching point t_i and in the case of infinite order, the coefficients are the switching points t_i and coefficients α_j, which are appeared in the singular arc as (13). As such, if the problem be of singular order, we set

$$\mathbf{z}=[\boldsymbol{\zeta}_0,\boldsymbol{\theta}_1,\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\theta}_s,\boldsymbol{\zeta}_s \mid t_1,t_2,\ldots,t_s,t_f],$$

and if the problem be of finite order, the vector $\mathbf{z}$ is set as

$$\mathbf{z}=[\boldsymbol{\zeta}_0,\boldsymbol{\theta}_1,\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\theta}_s,\boldsymbol{\zeta}_s \mid t_1,t_2,\ldots,t_s,t_f \mid \alpha_0,\ldots,\alpha_n].$$

It is clear that the solution of (16) depends not only on t but also on the vector $\mathbf{z}$, and to emphasize this dependence, we denote the solution of (16) by $\mathbf{x}^{[k]}(t;\mathbf{z}),\ \boldsymbol{\lambda}^{[k]}(t;\mathbf{z}),\ k=0,1,\ldots,s$. Now, we must find $\mathbf{z}$ such that the following equations might be satisfied

$$\mathbf{x}^{[k]}(t_{k+1};\mathbf{z})=\boldsymbol{\theta}_{k+1}, \quad k=0,\ldots,s-1, \tag{17a}$$

$$\boldsymbol{\lambda}^{[k]}(t_{k+1};\mathbf{z})=\boldsymbol{\zeta}_{k+1}, \quad k=0,\ldots,s-1, \tag{17b}$$

$$\sigma(\mathbf{x}^{[k]}(t_k;\mathbf{z}),\boldsymbol{\lambda}^{[k]}(t_k;\mathbf{z}),t_k)=0, \quad k=1,\ldots,s. \tag{17c}$$

$$\boldsymbol{\psi}(\mathbf{x}^{[s]}(t_f;\mathbf{z}),t_f)=0, \tag{17d}$$

$$\boldsymbol{\lambda}^{[s]}(t_f;\mathbf{z})=\ell_{\mathbf{x}_f}(\mathbf{x}^{[s]}(t_f;\mathbf{z}),t_f,\boldsymbol{\rho}), \tag{17e}$$

$$\mathscr{H}(\mathbf{x}^{[s]}(t_f;\mathbf{z}),u^{[s]}(t_f),\boldsymbol{\lambda}^{[s]}(t_f;\mathbf{z}),t_f)=-\ell_{t_f}(\mathbf{x}^{[s]}(t_f;\mathbf{z}),t_f,\boldsymbol{\rho}). \tag{17f}$$

The above equations form a system of nonlinear equations and by solving it a solution $\mathbf{z}^*$ is obtained. Then, by solving the initial value problem (16) with $\mathbf{z}=\mathbf{z}^*$, an approximation is obtained for MDBVP (14)–(15).

4 Illustrative Examples

This section has been devoted to numerical experiments. We have implemented the proposed method in Sect. 3 using MATLAB in a personal computer and to solve the NLP (9), the Interior-Point Optimization Solver IPOPT (Wächter and Biegler 2006) is used. In addition, the MATLAB function `ode45` is utilized for solving IVP (16). It is noted that `ode45` controls the error by two parameters `RelTol` and `AbsTol`. By these parameters, we can adjust the relative and absolute error tolerances. Moreover, using the MATLAB function `fsolve`, the system of equations (17) is solved.

4.1 Example 1 (A Finite Order Singular Problem)

We consider the following control problem

$$
\begin{aligned}
&\min \mathcal{J}(u) = \frac{1}{2}(\mathbf{x}(t_f) - \mathbf{x}_f)^T(\mathbf{x}(t_f) - \mathbf{x}_f),\\
&\dot{x}_1 = x_2,\\
&\dot{x}_2 = F_2,\\
&\dot{x}_3 = x_4,\\
&\dot{x}_4 = F_4,\\
&\mathbf{x}(0) = [0,0,0,0],\\
&-1 \le u \le 1,
\end{aligned}
$$

where

$$
F_2 = -\frac{\epsilon x_2^2 c + s + cu}{D}, \qquad F_4 = -\frac{\epsilon s(x_2^2 + c) + u}{D},
$$

and

$$
c = \cos(x_1), \qquad s = \sin(x_1), \qquad D = 1 - \epsilon c^2,\\
\epsilon = 0.5, \qquad t_f = 4.012.
$$

The Hamiltonian function of the above optimal control problem is:

$$
\mathcal{H}(\mathbf{x}, u, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 x_2 + \lambda_2\left(-\frac{\epsilon x_2^2 c + s + cu}{D}\right) + \lambda_3(x_4) + \lambda_4\left(-\frac{\epsilon s(x_2^2 + c) + u}{D}\right).
$$

Applying Pontryagin's maximum principle leads to the following costate equations:

$$
\begin{aligned}
&\dot{\lambda}_1^*(t) = -\mathcal{H}_{x_1} = A\lambda_2 + B\lambda_4,\\
&\dot{\lambda}_2^*(t) = -\mathcal{H}_{x_2} = \dot{\lambda}_2^*(t) = -\lambda_1 + E(c\lambda_2 - \lambda_4),\\
&\dot{\lambda}_3^*(t) = -\mathcal{H}_{x_3} = 0,\\
&\dot{\lambda}_4^*(t) = -\mathcal{H}_{x_4} = -\lambda_3,
\end{aligned}
$$

such that

$$
A = \frac{\epsilon(SF_2 + Cx_2^2) + c - su}{D}, \quad B = \frac{\epsilon(SF_4 - cx_2^2 - C)}{D},\\
E = \frac{2\epsilon s x_2}{D}, \quad S = \sin(2x_1), \quad C = \cos(2x_1).
$$

The factor of u in the Hamiltonian function is $\frac{-c\lambda_2 - \lambda_4}{D}$, so the switching function is given by $\sigma(\mathbf{x}, \boldsymbol{\lambda}, t) = \frac{-c\lambda_2 - \lambda_4}{D}$. Now, we have

$$
\begin{aligned}
&\frac{d}{dt}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t) = s x_2 \lambda_2 + c\lambda_1 - \lambda_3\\
&\frac{d^2}{dt^2}\sigma(\mathbf{x}, \boldsymbol{\lambda}, t) = (c(x_2^2 + A + cB) + sF_2)\lambda_2 - 2 s x_2 \lambda_1.
\end{aligned}
$$

It can be seen that the control u appears in the second derivative of σ, therefore, the order of the problem is $\kappa = 1$. Moreover, by extracting v from $\frac{d^2}{dt^2}\sigma = 0$, the control function on the singular interval is obtained as:

$$
u^{\text{sing}}(\mathbf{x}, \boldsymbol{\lambda}, t) = \frac{(2 - 2\epsilon - D)cx_2^2 - 2s^2 + D}{S} - \frac{Dx_2\lambda_1}{c\lambda_2}. \tag{18}
$$

Now, we apply the proposed method to this problem. At first, we apply the Euler method in Stage I. The obtaining control and state functions for $n = 1000$ are plotted in Fig. 1. In addition, the estimated costate functions $\lambda_1, \ldots, \lambda_4$ are plotted in Fig. 2.

As we see in Fig. 1, the structure of control function is detected as

$$
u(t) = \begin{cases}
u^{\max}, & \text{if} \quad 0 \le t \le t_1,\\
u^{\sin}, & \text{if} \quad t_1 \le t \le t_2,\\
u^{\min}, & \text{if} \quad t_2 \le t \le t_3,\\
u^{\max}, & \text{if} \quad t_3 \le t \le t_4,\\
u^{\min}, & \text{if} \quad t_4 \le t \le t_f.
\end{cases}
$$

Moreover, the approximations of the switching points are obtained as:

$$
t_1 = 0.69, \quad t_2 = 1.17, \quad t_3 = 2.72, \quad t_4 = 4.32.
$$

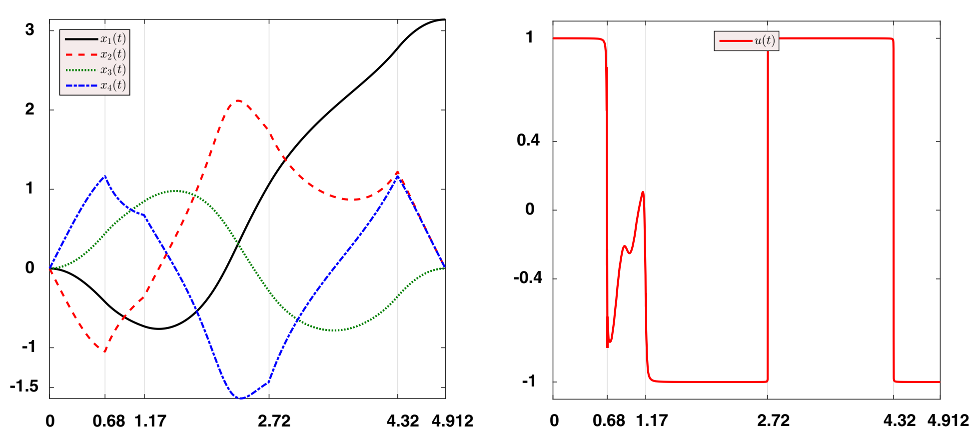


Fig. 1 (Example 1: Finite order singular Problem) Stag I: State and control histories obtained by the Euler method with $n = 1000$

Fig. 2 (Example 1: Finite order singular Problem) Stage I: The obtained costate function with $n = 1000$

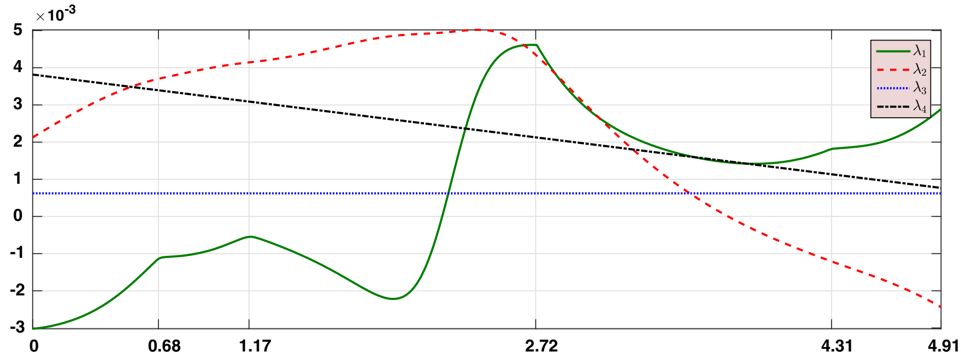


Table 1 (Example 1: Finite order singular Problem) The obtained values of switching times and performance index for various values of `RealTol`

`RealTol`	t_1	t_2	t_3	t_f	$\mathcal{J}$
1e−10	**0.688283063**87	**1.176405501**.76	**2.7239316480**.9	**4.320071249**80	**3.5205972**513-06
1e−11	**0.688283063**79	**1.1764055018**6	**2.723931648**03	**4.32007124977**	**3.5205972**51e-06
1e−12	**0.68828306378**	**1.17640550187**	**2.72393164802**	**4.32007124977**	**3.520597249**e-06
1e−13	**0.68828306378**	**1.17640550187**	**2.72393164802**	**4.32007124977**	**3.520597249**e-06

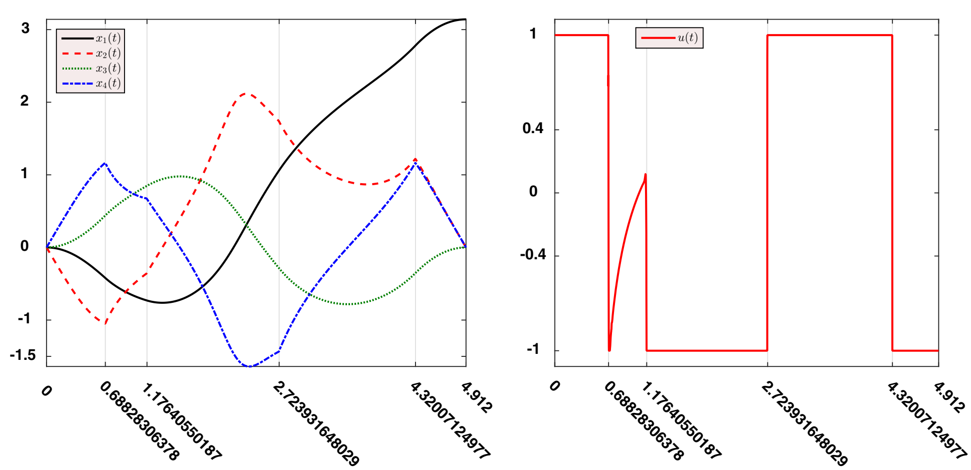
Correct decimal places of the approximations are highlighted in bold

Table 2 (Example 1: Finite order singular Problem) The obtained values of costate functions in initial point for various values of `RealTol`

`RealTol`	$\lambda_1(t_0)$	$\lambda_2(t_0)$	$\lambda_3(t_0)$	$\lambda_4(t_0)$
1e−10	**0.002054574**7762	**−0.001456427**.5310	**−0.000424399**.5824	**−0.00260070**97869
1e−11	**0.002054574**6434	**−0.00145642740**85	**−0.00042439947**59	**−0.00260070964**61
1e−12	**0.00205457464**05	**−0.00145642740**65	**−0.00042439947**53	**−0.0026007096425**
1e−13	**0.0020545746402**	**−0.0014564274063**	**−0.0004243994752**	**−0.0026007096421**
1e−14	**0.0020545746402**	**−0.0014564274063**	**−0.0004243994752**	**−0.0026007096421**

Correct decimal places of the approximations are highlighted in bold

Fig. 3 (Example 1: Finite order singular Problem) Stage II: The obtained State and control histories with `RealTol=1e-14`



Furthermore, using Theorem 1, we can find estimation of the costate functions in the above approximation for the switching points.

Now, we apply the method of Stage II for this problem. In this step, the unknowns vector is $\mathbf{z} = [\zeta_0, \boldsymbol{\theta}_1, \zeta_1, \boldsymbol{\theta}_2, \zeta_2, \boldsymbol{\theta}_3, \zeta_3, \boldsymbol{\theta}_4, \zeta_4 | t_1, t_2, t_3, t_4]$, where is initialized by the results obtained from Stage I. In this way, the values of the switching points and costates in $t = 0$ for various values of `RealTol` are obtained and reported in Tables 1 and 2.

The obtained control and state functions for `RealTol=1e-14` are plotted in Fig. 3. To show the accuracy of

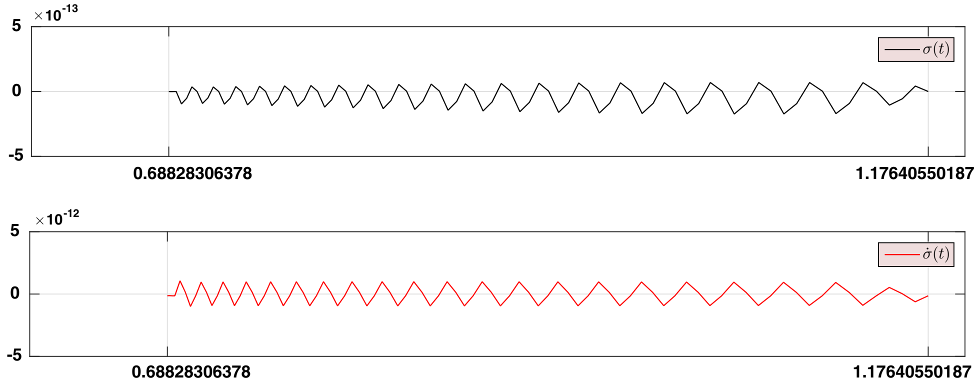


Fig. 4 (Example 1: Finite order singular Problem) Switching function and derivative of switching function in singular arc obtained by the Stage II method with `RealTol=1e-14`

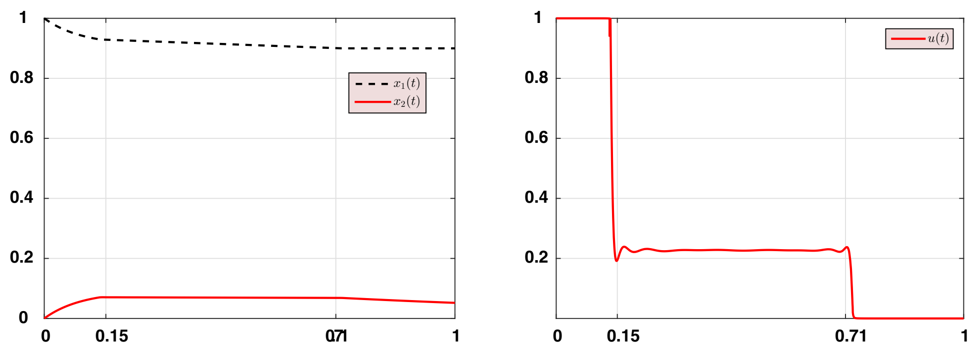


Fig. 5 (Example 2: Infinite order singular Problem) Stage I: State and control histories obtained by the Euler method with $n = 1000$

the method, the switching function and derivative of switching function in the singular arc are plotted in Fig. 4. We note that the switching function and its derivatives must be zero in the singular arc.

4.2 Example 2 (An Infinite Order Singular Problem)

We consider the following *Catalyst Mixing* optimal control problem

$$\min \mathcal{J}(u) = -1 + x_1(1) + x_2(1), \tag{19a}$$

$$\dot{x}_1 = (10x_2 - x_1)u, \tag{19b}$$

$$\dot{x}_2 = (x_1 - 10x_2)u - (1-u)x_2, \tag{19c}$$

$$\mathbf{x}(0) = [1, 0], \tag{19d}$$

$$0 \le u \le 1. \tag{19e}$$

In a similar manner, we can obtain the switching function as

$$\sigma(\mathbf{x}, \boldsymbol{\lambda}, t) = \lambda_1(10x_2 - x_1) + \lambda_2(x_1 - 9x_2).$$

Now, for $i = 1, 2, \ldots$ we can get

$$0 = \frac{d^i}{dt^i}\mathcal{H}_u = -10\lambda_1 x_2 + \lambda_2 x_1.$$

It can be seen that the control u does not appear in the derivative of σ, therefore, the order of the problem is infinite. Now, we apply the proposed method for this problem. At first, we apply the Euler method in Stage I. The obtaining control and state functions for $n = 1000$ are plotted in Fig. 5.

As we see in Fig. 5, The structure of control function is obtained as

$$u(t) = \begin{cases} u^{\max}, & \text{if} \quad 0 \le t \le t_1, \\ u^{\sin}, & \text{if} \quad t_1 \le t \le t_2, \\ u^{\min}, & \text{if} \quad t_2 \le t \le t_f. \end{cases}$$

Moreover, the estimation of switching points, state and costate functions at the switching points can be obtained. By applying Stage II with $m = 15$, when the results of Stage I serve as the initial guess, the values of the obtained switching points for various choices of `RealTol` are reported in Table 3. The obtained control and state functions with `RealTol=1e-12` are plotted in Fig. 6. To show the accuracy of the method, the switching function and derivative of switching function in the singular arc are plotted in Fig. 7.

5 Conclusion

This paper presents an approach for the efficient and accurate solution of singular optimal control problems with finite or infinite order. The employed method is of the

Table 3 (Example 2: Infinite order singular Problem) The obtained values of switching times and performance index for various values of `RealTol`

N	t_1	t_2	$\mathcal{J}$
5	**0.1362990**41521257	**0.7252301**12252465	**−0.0480556**87663202
8	**0.136299037**933642	**0.725230107**925190	**−0.048055679**322507
10	**0.1362990375**95471	**0.7252301075**83132	**−0.0480556792**27743
12	**0.136299037593**173	**0.725230107592**331	**−0.0480556792**48854
14	**0.136299037593275**	**0.725230107592795**	**−0.0480556795**48613
16	**0.136299037593275**	**0.725230107592795**	**−0.0480556795**88677

Correct decimal places of the approximations are highlighted in bold

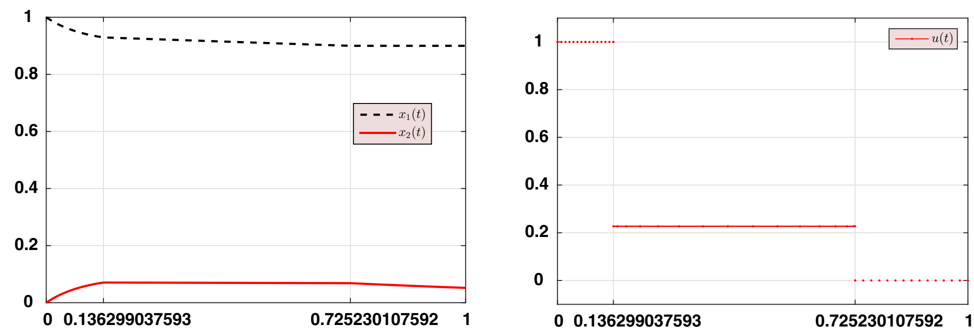


Fig. 6 (Example 2: Infinite order singular Problem) State and control histories obtained by the Stage II method with `RealTol`=1e-12 and $N = 15$

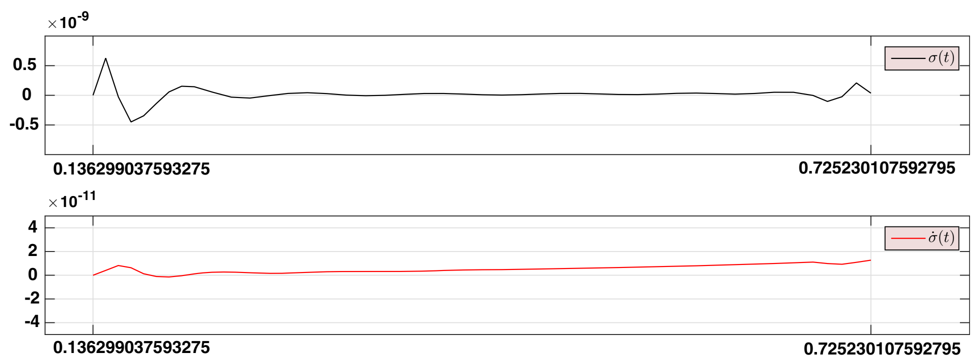


Fig. 7 (Example 2: Infinite order singular Problem) Switching function and derivative of switching function in singular arc obtained by the Stage II method with `RealTol`=1e-14 and $N = 15$

hybrid type, which combines the best features of the direct Euler method and indirect shooting techniques. The presented hybrid method is illustrated in two test problems and the results verify that the method detects the structure of optimal control without a priori information and can accurately capture the switching points. By means of these test problems, we see that the presented method converges and is stable for the singular optimal control problems with finite or infinite order. However, obtaining some theoretical estimates for the approximation errors would be desirable. This work is currently in progress.

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